18.212  Problem Set 2

due Friday, April 9, 2021

Solve 5 (or more) problems.

(1) Prove Erdős-Szekeres Theorem:
Fix \( m, n \geq 1 \). Any permutation of size \( \geq m \cdot n + 1 \) either has an increasing subsequence with \( m+1 \) elements or a decreasing subsequence with \( n+1 \) elements.

(In class, we explained that Greene's Thm easily implies Erdős-Szekeres Thm. Here you need to find a direct proof of Erdős-Szekeres Thm without using Greene's Thm or Schensted correspondence.)

(2) For \( n \geq 1 \) and \( 0 \leq k \leq \frac{n}{2} \), construct a bijection \( \varphi \) between \( k \)-element subsets in \([n] := \{1, 2, \ldots, n\}\) and \((n-k)\)-element subsets in \([n]\) such that \( \varphi(I) \supseteq I \), for any \( k \)-element subset \( I \subset [n] \).
(3) Let $T_n$ be the poset of all set partitions of $\{1, \ldots, n\}$ ordered by refinement. Prove that $T_n$ is a lattice.

(4) Give a complete proof of the Fundamental Theorem on Finite Distributive Lattices.

(5) Let $n_1, \ldots, n_k \geq 1$ and $n = n_1 + \cdots + n_k$. For a permutation $w = w_1, \ldots, w_n$ of the multiset

\[ \{1^{n_1}, 2^{n_2}, \ldots, k^{n_k}\} \]

define the inversion number

\[ \text{inv}(w) = \# \{ 1 \leq i < j \leq n \mid w_i > w_j \} \]

Prove that the $q$-multinomial coefficient

\[ \left[ \frac{n}{n_1, \ldots, n_k} \right]_q := \frac{[n]_q!}{[n_1]_q! \cdots [n_k]_q!} \]

equals sum over all permutations $w$ of the multiset $\{1^{n_1}, 2^{n_2}, \ldots, k^{n_k}\}$
(b) Let \( w \) be a permutation.
Let \( \lambda \) be the Schensted shape of \( \lambda \) (i.e. \( \lambda \) is the shape of the \( P \)-tableau and \( Q \)-tableau corresponding to \( \lambda \)).

(A) Prove that \( \lambda_1 \) is the size of a longest increasing subsequence in \( w \).

(B) Prove that \( \lambda'_1 \) is the size of a longest decreasing subsequence in \( w \).

(c) (A) Give a non-recursive construction of the Fibonacci lattice \( F \).

(B) Prove that the Fibonacci lattice \( F \) is a lattice.
8) Let \( \lambda = (\lambda_1, \ldots, \lambda_n) \leq n \times n \) be a Young diagram that fits inside the \( n \times n \) square. Let \( \lambda' = (\lambda'_1, \ldots, \lambda'_n) \) be its conjugate.

Prove that the multiset of numbers \( \lambda_n, \lambda_{n-1}-1, \lambda_{n-2}-2, \ldots, \lambda_1-n+1 \) coincides with the multiset \( \lambda'_n, \lambda'_{n-1}-1, \lambda'_{n-2}-2, \ldots, \lambda'_1-n+1 \).

(In class, we showed that \( \lambda_n \cdot (\lambda_{n-1}-1) \cdot \ldots \cdot (\lambda_1-n+1) \) equals the number of placements of \( n \) non-attacking rooks inside the Young diagram \( \lambda \).)

9) Let \( f_\lambda \) be the number of SYT's of shape \( \lambda \).

Find an explicit formula for

\[ \sum_{\lambda \leq n} f_\lambda. \]

(The formula will involve a sum of a certain closed expression.)
(10) Construct a bijection between perfect matchings in the complete graph $K_{2n}$ and the set of oscillatory tableaux $(\lambda(0), \ldots, \lambda(2n))$ such that

- $\lambda(0), \ldots, \lambda(2n)$ are Young diagrams.
- $\lambda(i)$ and $\lambda(i+1)$ are obtained from each other by adding or removing a single box, for any $i$.
- $\lambda(0) = \lambda(2n) = \emptyset$.

(11) (A) Prove that the number of oscillatory tableaux $(\lambda(0), \ldots, \lambda(2n))$ such that

- $\lambda(0), \ldots, \lambda(2n)$ are Young diagrams.
- $\lambda(i)$ and $\lambda(i+1)$ differ by a single box, $\forall i$.
- $\lambda(0) = \emptyset$

equals the number of involutions in $S_{2n}$ with all 2-cycles colored in 2 colors.

(B) Give an explicit formula (involving a summation) for this number.
12) Construct a bijection between partitions of $n$ with odd parts and partitions of $n$ with distinct parts.

13) Prove the following $q$-binomial formula:

$$(1 + x)(1 + xq)(1 + xq^2) \cdots (1 + xq^{n-1}) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_q \ q^{k(k-1)/2} \ x^k.$$

14) Prove the following identity:

$$\left[ \begin{array}{c} 2n \\ n \end{array} \right]_q = \sum_{k=0}^{n} q^{k^2} \left( \left[ \begin{array}{c} n \\ k \end{array} \right]_q \right)^2.$$
(15) Prove the identity

\[
\frac{1}{1 - q^x - q^{2x} - q^{3x} - \cdots} = \sum_{n \geq 0} q^{-\frac{n(n+1)}{2}} \left[ \frac{2n+1}{n} \right]_q x^n.
\]

(The L.H.S. is called the Ramanujan continued fraction.)
16) Prove Lemma on page 12 in Lecture 17.

17) Let $b_r = b_{r, n}$ be the number of shifted Young diagrams with $r$ boxes that fit inside the "shifted staircase" shape:

Prove the unimodality

$b_0 \leq b_1 \leq b_2 \leq \ldots \leq b_e = b_{e,n} \geq \ldots \geq b_M$,

where $M = n(n+1)/2$ and $e = \left\lfloor \frac{M}{2} \right\rfloor$. 