Problem Set 1

due Friday, March 12, 2021

It is enough to solve 5 problems,

1. Consider the random walk on the segment \([0, N] = \{0, 1, \ldots, N\}\) of the integer line such that
   - we can go from the position \(i\) either to \(i-1\) or to \(i+1\) with the probabilities
     \[
     \text{Prob}(i, i-1) = p \quad \text{and} \quad \text{Prob}(i, i+1) = 1-p,
     \]
   where \(i \in \{1, 2, \ldots, N-1\}\) and \(p\) is some fixed constant \(0 < p < 1\).
   - A random walk stops when it reaches either 0 (cliff) or \(N\) (house).

\[
\begin{array}{c}
0 \Rightarrow 1 \Rightarrow 2 \Rightarrow \cdots \Rightarrow i-1 \Rightarrow i \Rightarrow i+1 \Rightarrow \cdots \Rightarrow N \\
\text{cliff} \quad \text{house}
\end{array}
\]

Find the probability \(p_i\) that a random walk starting in the position \(i\) reaches the position \(N\) (before reaching 0).
2) Find an explicit formula for the number of lattice paths $P$ (similar to Dyck paths) such that

- $P$ starts at $(0,0)$ and ends at $(a,b)$
- $P$ has steps given by the vectors $(1,1)$ ("up" steps) and $(1,-1)$ ("down" steps)
- $P$ always stays in the upper half plane $\{ (x,y) \in \mathbb{R}^2 \mid y \geq 0 \}$

Here $a, b$ are any non-negative integers such that $a \geq b$ and $a+b$ is even.

For example, if $(a,b) = (2n,0)$ then such paths $P$ are exactly Dyck paths with $2n$ steps.
3. Prove the lemma from Lecture 2 about cyclic shifts.

4. Construct a bijection between Dyck paths with $2n$ steps and valid parenthesizations of $n+1$ letters.

5. Prove that the stack-sortable permutations are exactly the 231-avoiding permutations.

6. Prove that the queue-sortable permutations are exactly the 321-avoiding permutations.
(7) Prove that the Catalan number \( C_n \) equals:
(a) the number of 231-avoiding permutations in \( S_n \)
(b) the number of 231-avoiding permutations in \( S_n \)

(8) Find the number of permutations in \( S_n \) which are both 231-avoiding and 321-avoiding.

(9) Let \( T \) be a binary tree with \( n \) vertices.
Define a "standard tableau" of shape \( T \) as labelling of the vertices of \( T \) by numbers 1, 2, ..., \( n \) such that the label of any vertex \( S \) is greater than the label of any child of \( T \).

Let \( f_T \) be the number of such "standard tableaux" of shape \( T \).

![Binary Tree Diagram]

For a vertex \( S \) of \( T \), define its "hook length" \( h(S) \) as the number of all descendants of \( S \), including the vertex \( S \) itself.

Prove the following version of hook length formula:

\[
f_T = \frac{n!}{\prod_{S \text{ vertex of } T} h(S)}
\]
Let $f_T$ be the number defined in problem 9.

Find a closed formula for the sum

$$\sum_T \frac{f_T}{T}$$

over all binary trees with $n$ vertices.

For example, for $n=3$, we have

- has 2 standard tableaux:
  - $\begin{array}{c}
  \circ \\
  \circ & \circ
  \end{array}$
  - $\begin{array}{c}
  \circ \\
  \circ & \circ
  \end{array}$

and

- each of these binary trees has 1 standard tableau.

In total we get $2 + 1 + 1 + 1 + 1 = 6$. 
Recall that the Bell number $B(n)$ is the total number of set partitions of $[n]$. Show that $B(n)$ equals the number of set partitions $T_1$ of $[n+1]$ satisfying the condition: there is no $i$ such that both $i$ and $i+1$ belong to the same block of $T_1$.

$n = 3$

All set partitions of $[3]$

Set partitions of $[4]$ whose blocks are not allowed to contain adjacent entries $i$ & $i+1$. 

\[
egin{align*}
(1 \ 2 \ 3) & \quad (1 \ 3 \ 2 \ 4) \\
(1 \ 2 \ 3) & \quad (1 \ 3 \ 2 \ 4) \\
(1 \ 3 \ 2) & \quad (1 \ 4 \ 2 \ 3) \\
(2 \ 3 \ 1) & \quad (1 \ 2 \ 4 \ 3) \\
(1 \ 2 \ 3) & \quad (1 \ 2 \ 3 \ 4)
\end{align*}
\]
(12) Prove that the number of non-crossing set partitions of $[n]$ equals the Catalan number $C_n$.

(13) Prove that the number of non-crossing set partitions of $[n]$ with exactly $k$ blocks equals the Narayana number $N(n,k)$.

(In class, we proved a similar claim about non-nesting set partitions.)

Example, for $n=4$, $k=2$:

\[ N(4,2) = 6 \] non-crossing set partitions of $[4]$ with 2 blocks.

Recall that we defined the Narayana number $N(n,k)$ as the number of Dyck paths with $2n$ steps and $k$ peaks.
4) Prove the following symmetry of the Narayana numbers:

\[ N(n, k) = N(n, n-k+1) \]

5) Let \( M_n \) be the number of perfect matchings on \( n \) labelled vertices.

(A perfect matching is a simple graph such that degrees of all vertices equal to 1.)

For example, \( M_4 = 3 \)

\[ 
\begin{array}{ccc}
1 & 2 & 1 \\
\hline
4 & 3 & 4
\end{array}
\]

3 perfect matchings on 4 labelled vertices

(A) Find the exponential generating function

\[ \sum_{n=0}^{\infty} \frac{M_n x^n}{n!} \]

(B) Find a closed expression for \( M_n \).
16. Show that \( \operatorname{maj} \) (the major index) and \( \operatorname{inv} \) (the number of inversions) are equidistributed statistics on \( S_n \).

17. Show that \( \operatorname{exc}(w) \) (the number of exceedances) and \( \operatorname{wexc}(w) - 1 \) (the number of weak exceedances minus 1) are equidistributed statistics on \( S_n \).

18. Prove the following recurrence relations for the signless Stirling numbers of the first kind \( c(n, k) \), the Stirling numbers of the second kind \( S(n, k) \), and the Eulerian numbers \( A(n, k) \):

- (A) \( c(n, k) = c(n-1, k-1) + (k-1)c(n-1, k) \)
- (B) \( S(n, k) = S(n-1, k-1) + kS(n-1, k) \)
- (C) \( A(n, k) = (n-k)A(n-1, k-1) + (k+1)A(n-1, k) \)