18.212 Problem Set 1 (due Monday, March 04, 2018)

Turn in as many problems as you want. (You don’t need to turn in all problems to get a perfect grade in the class. Around 6 problems should be enough.)

**Problem 1.** In class, we sketched a proof of the formula for the Catalan number $C_n = \frac{1}{2n+1} \binom{2n+1}{n}$ using cyclic shifts of sequences of $\pm 1$’s. The proof is based on the following two claims. Prove these claims.

Let $(e_1, \ldots, e_{2n+1})$ be a sequence such that such that $e_i \in \{1, -1\}$, 
$\# \{i \mid e_i = 1\} = n$, and $\# \{i \mid e_i = -1\} = n + 1$.

1. All $2n+1$ cyclic shifts $(e_i, \ldots, e_{2n+1}, e_1, \ldots, e_{i-1})$, for $i = 1, \ldots, 2n+1$, are different from each other.

2. Exactly one cyclic shift $(e'_1, \ldots, e'_{2n+1})$ among these $2n+1$ shifts satisfies $e'_1 + \cdots + e'_j \geq 0$, for $j = 1, \ldots, 2n$.

**Problem 2.** Consider the random walk of a man on the integer line $\mathbb{Z}$ such that, at each step, that the probability to go from position $i$ to position $i + 1$ is $p$, and the probability to go from $i$ to $i - 1$ is $1 - p$. The man “falls off the cliff” if he reaches the position 0. Suppose that the man starts at the initial position $i_0 \geq 1$. Find the probability that he falls off the cliff.

**Problem 3.** The same setup as in the previous problem. Find the probability that the man starting at position $i_0$ falls off the cliff after exactly $m$ steps. (Hint: Use the reflection principle.)

**Problem 4.** Prove that a permutation is queue-sortable if and only if it is 321-avoiding.

**Problem 5.** Prove that a permutation is stack-sortable if and only if it is 231-avoiding.

**Problem 6.** Find a bijection between 321-avoiding permutations of size $n$ and 231-avoiding permutations of size $n$. 
**Problem 7.** Find an expression for the number of permutations $w$ of size $n$ such that $w$ is both 321-avoiding and 3412-avoiding.

(Hint: Calculate the number of such permutations for small values of $n$, then guess the answer.)

**Problem 8.** Find an expression for the number of permutations $w$ of size $n$ such that $w$ is both 231-avoiding and 4321-avoiding.

**Problem 9.** In class, we proved part (1) of Schensted’s theorem. Prove part (2) of this theorem:

If the Schensted correspondence maps a permutation $w$ to a pair $(P,Q)$ of standard Young tableaux of the same shape $\lambda$, then the size of a largest decreasing subsequence in $w$ equals the number of nonzero parts in partition $\lambda$ (i.e., the number of rows of its Young diagram).

**Problem 10.** Fix two positive integers $m$ and $n$. Let $w$ be a permutation of size $m \cdot n + 1$. Prove that $w$ either has an increasing subsequence of size $m + 1$ or a decreasing subsequence of size $n + 1$.

(Hint: You can use properties of Schensted correspondence. There is also a direct proof based on the pigeonhole principle.)

**Problem 11.** Find an explicit expression for the number of permutations $w$ of size $m \cdot n$ such that $w$ does not have an increasing subsequence of size $m + 1$ nor a decreasing subsequence of size $n + 1$.

**Problem 12.** Prove the “baby hook-length formula”:

The number of linear extensions of the poset whose Hasse diagram is a rooted tree $T$ on $n$ vertices equals $n! / \prod_{v \in T} h(v)$, where the product is over all vertices $v$ of the tree, and the “hook-length” $h(v)$ equals the size of the branch of $T$ growing from vertex $v$.

**Problem 13.** For positive integers $n_1, \ldots, n_m$ and $n = n_1 + \cdots + n_m$, the $q$-multinomial coefficient is defined as

$$\left[ \begin{array}{c} n \\ n_1, \ldots, n_m \end{array} \right]_q := \frac{[n]_q!}{[n_1]_q! \cdots [n_m]_q!}.$$
Show that
\[
\binom{n}{n_1, \ldots, n_m}_q = \sum_w q^{\text{inv}(w)},
\]
where the sum is over all permutations \(w\) of the multiset \(\{1^{n_1}, 2^{n_2}, \ldots, m^{n_m}\}\), and \(\text{inv}(w)\) is the number of inversions in \(w\). Here \(i^n\) denotes \(i, \ldots, i\) (repeated \(n\) times).

**Problem 14.** Prove the identity for \(q\)-binomial coefficients
\[
\binom{2n}{n}_q = \sum_{k=0}^{n} q^{k^2} \binom{n}{k}_q \binom{n}{k}_q
\]
(Hint: Use the interpretation of \(q\)-binomial coefficients in terms of Young diagrams, and try to subdivide a Young diagram into several pieces to prove the identity.)

**Problem 15.** Prove the following noncommutative version of binomial theorem.

Let \(q\) be a parameter, and let \(x, y\) be two noncommuting variables that satisfy the relation
\[
yx = qxy.
\]
We assume that \(q\) commutes with both \(x\) and \(y\), i.e., \(qx = xq\) and \(qy = yq\). Show that
\[
(x + y)^n = \sum_{k=0}^{n} \binom{n}{k}_q x^k y^{n-k}.
\]

**Bonus Problems**

**Problem 16.** Show that the two statistics \(\text{inv}(w)\) (the number of inversions) and \(\text{maj}(w)\) (the major index) on permutations \(w \in S_n\) are equidistributed.

**Problem 17.** An **exceedance** in a permutation \(w \in S_n\) is an index \(i \in \{1, \ldots, n\}\) such that \(w(i) > i\). Similarly, a **weak exceedance** in a permutation \(w \in S_n\) is an index \(i \in \{1, \ldots, n\}\) such that \(w(i) \geq i\). Let \(\text{exc}(w)\) be the number of exceedances and \(\text{wexc}(w)\) be the number of weak exceedances in a permutation \(w\). Prove that the statistics \(\text{exc}(w)\) and \(\text{wexc}(w) - 1\) on permutations \(w \in S_n\) (for \(n \geq 1\)) are equidistributed.
Problem 18. Prove that the number of set-partitions $\pi$ of the set $[n] := \{1, \ldots, n\}$ such that, for any $i = 1, \ldots, n - 1$, the consecutive numbers $i$ and $i + 1$ do not belong to the same block of $\pi$ equals the number of set-partitions of the set $[n - 1]$.

Problem 19. For $1 \leq k \leq n/2$, find a bijection $f$ between $k$-element subsets of $\{1, \ldots, n\}$ and $(n - k)$-element subsets of $\{1, \ldots, n\}$ such that $f(I) \supseteq I$, for any $k$-element subset $I$.

Problem 20. We say that a pair $(i, j)$, $1 \leq i < j \leq n$, is an odd-length inversion of a permutation $w \in S_n$ if $w_i > w_j$ and $j - i$ is odd. Let $\text{inv}(w)$ be the number of all inversions in $w$ and $\text{oinv}(w)$ be the number of odd-length inversions in $w$. Prove the identity

$$\sum_{w \in S_n} (-1)^{\text{inv}(w)} x^{\text{oinv}(w)} = \prod_{i=2}^{n} (1 + (-1)^{i-1} x^{\lfloor i/2 \rfloor})$$