Tutte polynomial

Last time: Deletion-contraction
\[ G \rightarrow G \setminus e \quad (\text{deletion of edge } e) \]
\[ G / e \quad (\text{contraction of edge } e) \]

\[ G = e \]
\[ G = e \]
\[ G / e = \]

There are several invariants* of graphs that satisfy (a version of) deletion-contraction recurrence:

\[ \ell(G) = \ell(G \setminus e) + \ell(G / e) \]

- # acyclic orientations of \( G \):
  \[ AO(G) = AO(G \setminus e) + AO(G / e) \]

- chromatic polynomial \( \chi_G(t) \):
  \[ \chi_G(t) = \chi_{G \setminus e}(t) - \chi_{G / e}(t) \]

- # spanning trees \( ST(G) \):
  \[ ST(G) = ST(G \setminus e) + ST(G / e) \]

* Here the word "invariant" means that all these numbers and polynomials don’t depend on a choice of ordering of the vertices of \( G \).

Is there the most general graphical invariant that satisfies the deletion-contraction recurrence?
The Tutte polynomial $T_G(x,y)$

$G$ - an undirected graph (we allow multiple edges and loops).

An edge $e$ in $G$ is called a bridge (aka a isthmus) if $G / e$ has more connected components than $G$.

Example

\[ T_K_3 = x^2 + T_0 = x^2 + x + y. \]

**Theorem (Tutte)** There exists a unique polynomial $T_G(x,y)$, with positive integer coefficients, such that

- For any edge $e$ in $G$, if we 'cut' a loop or a bridge, we have $T_G = T_G - e + T_G/e$.
- If $G$ has $a$ bridges and $b$ loops (and no other edges), then $T_G = x^a y^b$.

Example. $G = K_3 = \begin{array}{c}
\text{Example} \\
\text{William Thomas Tutte} \\
(1917-2002)
\end{array}$

All other graphical invariants we discussed last time ($\chi(G)$, $\alpha(G)$, $ST(G)$) are special values of the Tutte polynomial.
Remark. The claim about uniqueness of $T_G(x,y)$ is clear. The deletion-contraction recurrence allows us to express $T_G(x,y)$ in terms of Tutte polynomials of graphs that consist only of bridges & loops, which are $x^\#\text{bridges} - y^\#\text{loops}$.

$$T_G = x^{10}y^{14}$$

a graph that consists only of bridges & loops.

da forest with some loops

But in order to prove existence of $T_G(x,y)$, we need to show that, if we do deletion-contraction in a different way, we obtain the same polynomial $T_G(x,y)$.

Possible approaches:
- induction on $|E|$
- give a non-recursive formula for $T_G(x,y)$, and show that it satisfies deletion-contraction
If we have a function $f(x)$, the derivative $f'(x)$ represents the slope of the tangent line at a point $x$. The second derivative $f''(x)$ gives information about the concavity of $f(x)$.

For example, if $f(x) = x^2$, then $f'(x) = 2x$ and $f''(x) = 2$, which indicates that the function is concave up everywhere.

In terms of optimization, the second derivative test can be used to determine whether a critical point is a local minimum, local maximum, or neither.

In conclusion, understanding derivatives and their applications is crucial in many areas of mathematics and science.
Theorem ( Tutte )

\[ T(G, y) = \sum_{\text{spanning trees } T} y^{\#(T)} \]

is a spanning tree of G.

Proof. Clearly \(\delta(T)\) and \(\pi(T)\) depend on a cycle of total order \(\pi(G)\). But the polynomial \(T(G, y)\) does not depend on the choice of chain.

Example: \( G = \begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array} \)

Sketch of proof. We need to show that

\[ T(G, y) = \sum_{\text{spanning trees } T} y^{\#(T)} \]

satisfies the deletion-contraction recurrence

\[ T(G, y) = T(E - e, y) + T(G - e, y) \]

(when inserted conditions).

It is hard to check this for any edge \(e\). But it is easier to prove on a case-by-case basis because it is an edge in \(T\). But it is already enough to check this on \(E\) and the rest of the graph.

The edge \(e\) is in order to derive that \(T(G, e) = 1\).

By induction on \(|E|\).
Special Values of $T_G(x,y)$

* $T_G(1,1) = \#$ spanning trees of $G$
  (if $G$ is connected)

* $X_G(t) = (-1)^{n-k} \sum_{k=0}^{r} \frac{T_G(1+k, 0)}{k!} t^k$
  $k = \#$ connected components of $G$.

* $T_G(2,0) = \#$ acyclic orientations of $G$.

* $T_G(2,1) = \#$ forests in $G$.

* $T_G(2,2) = 2^{\mid E \mid}$

* ... Many other graphical invariants are expressed as specializations of $T_G(x,y)$. 
The Tutte polynomial

Recall, \( I_n(y) := \)
\[
= \sum_{T \text{ spanning tree of } K_{n+1}} y^{|\text{inv}(T)|}
\]

**Proposition**

\[ I_n(y) = T_{K_{n+1}}(1, y) \]

**Proof.** There is a way to order the set of edges of \( K_{n+1} \)
\( s.t. \) \( \text{ext}(T) = \text{inv}(T) \). \( \Box \)

Also recall that \( I_n(-1) \) is the number \( A_n \) of alternating
permutations of size \( n \).

So the value \( T_G(1, -1) \)
is a generalization of the number \( A_n \) to any

The Tutte polynomial

appears in many different areas of math & physics,
for example:

- **Statistical Physics**
  
  (Ising & Potts model)

- **Knot theory**
  
  (Jones & HOMFLY
  polynomials)

- etc.
Domino Tilings

(Last week we discussed rhombus tilings.)

Def. A domino tiling is a way to subdivide some region on the plane (typically, an \( m \times n \) rectangle) into dominos (1 \( \times 2 \) or 2 \( \times 1 \) rectangles).

Example

\[ m=3, \ n=4 \]

\[ \begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 \\
\end{array} \]

A domino tiling of a \( 3 \times 4 \) rectangle

Clearly, we can tile an \( m \times n \) rectangle by dominos if and only if \( m \times n \) is even.
What if both \( m \) & \( n \) are odd?

Example: \( m=n=5 \)

We cannot subdivide the \( 5 \times 5 \) square into dominos, because

\[ 5 \times 5 = 25 \]

it has the odd number \( 25 \) of boxes.

How about the region obtained by removing a single box, e.g.

\[ 5^2 - 1 = 24 \] boxes

Can we tile this region by dominos?

Answer: No

Let's color all boxes in black & white like a chessboard:

- \( 11 \) black boxes
- \( 13 \) white boxes

But any domino \( (\square \blacksquare) \) should contain exactly one black box and exactly one white box.

So we cannot tile this region by dominos?

In order to have the same numbers of black & white boxes, the color of the removed box should be the same as the color of a corner box.

Example: We can tile the \( 5 \times 5 \) square without a corner box by dominos.

Can we find the number dominos tilings?
Theorem (Kasteleyn 1961)
Assume that \( n \) is even.

\[
\frac{n/2}{\prod_{k=1}^{n/2-1} \prod_{j=1}^{k} \left( 4 \cos^2 \frac{\pi k}{n+1} + 4 \cos^2 \frac{\pi j}{n+1} \right)}
\]

The proof is based on a clever way to express the permanent of a certain matrix as the determinant of another matrix.

Theorem (Temperley, 1974)
Suppose that \( m \) & \( n \) are both odd \( m = 2k+1 \), \( n = 2l+1 \).

Consider a region \( R \) obtained from a \( m \times n \) rectangle by removing a single box \( b \) such that:

- \( b \) is on the boundary of the rectangle
- \( b \) has the same color as corners of the rectangle

Then \# domino tilings equals \# spanning trees of the \( k \times l \) grid graph.
Proof: Let's construct a bijection between domino tilings & spanning trees.

Example

a domino tiling

Rule for edges:
Connect the black dots like this:

Claim: This construction gives a bijection between domino tilings & spanning trees.

Observation: If we remove any other box $b$ of the same color on the boundary of the main rectangle, we get the same number of domino tilings.