Chromatic polynomial

\[ G = (V, E) \] an undirected graph on a finite vertex set \( V \).
(We'll usually assume \( V = \{1, 2, \ldots, n\} \))

**Definition.**

For \( k \in \mathbb{Z}^+ \), a function \( c : V \to \{1, 2, \ldots, k\} \) is called a (proper) \( k \)-coloring if \( c(u) \neq c(v) \) for any edge \((u, v) \in E\).

**Examples.**

1. \( G = K_3 = \triangle \)
   - no 1-colorings
     
     A graph \( G \) has a 1-coloring iff \( E = \emptyset \).

2. no 2-colorings
   
   A graph has a 2-coloring iff \( G \) is a bipartite graph.

3. \( \exists \) a 3-coloring
   
   \( (1 = \bullet, \ 2 = \circ, \ 3 = \cdot) \)
   
   \( K_3 \) has \( 6 \) 3-colorings,

\[ G = \begin{array}{c}
\end{array} \]

1. no 1-colorings

2. \( G \) has 2 2-colorings:

3. \( G \) has 3 \( 2 \cdot 2 = 12 \) 3-colorings:
Lemma. There exists a unique polynomial $X_G(t)$ such that, for any positive integer $k$, $X_G(k)$ equals the number of $k$-colorings of $G$.

Moreover, $X_G(t)$ has integer coefficients.

Definition. $X_G(t)$ is called the chromatic polynomial of graph $G$.

The minimal number $k \in \mathbb{Z}_{\geq 0}$ such that $X_G(k) \neq 0$ is called the chromatic number.

Examples

1. $G = K_3$  
   \# k-colorings = $k \cdot (k-1) \cdot (k-2)$  
   \[
   X_{K_3}(t) = t \cdot (t-1) \cdot (t-2)
   \]  
   The chromatic number is 3.

2. $G = \quad \quad$  
   \# k-colorings = $k \cdot (k-1) \cdot (k-1)$  
   \[
   X_{\quad \quad}(t) = t \cdot (t-1) \cdot (t-1)
   \]  
   The chromatic number is 2.
Proof of Lemma. The uniqueness claim is clear. If two polynomials coincide at infinitely many points, then the polynomials are equal to each other.

The proof of existence is by induction on $|E|$. For $|E|=0$, $G$ is a graph with no edges.

Deletion - Contraction

Assume that $E \neq \emptyset$, and pick one edge $e \in E$.

$G$ is the graph with edge $e$. $G/e$ is the graph with edge $e$ deleted.

Examples:

1. $G = e \begin{array}{c} \downarrow \end{array}$
   $G/e = e \begin{array}{c} \downarrow \end{array}$

2. $G = e \begin{array}{c} \downarrow \end{array}$. $G/e = e \begin{array}{c} \downarrow \end{array}$

Notice that $G/e$ might contain multiple edges even if $G$ has no multiple edges.

However, for the chromatic polynomial $\chi_G(x)$, multiple edges don’t matter.

For example, $\chi_G(x) = \chi_{G/e}(x)$.

We can remove all multiple edges without affecting $\chi_G(x)$. 

Deletion-contraction recurrence for $k$-colorings:

$$\# \mathcal{C}_k\text{-colorings at } G \setminus e =$$

$$= \# \mathcal{C}_k\text{-colorings at } G \setminus e' - \# \mathcal{C}_k\text{-colorings at } G / e,$$

Indeed, $G = \begin{array}{c}
\includegraphics{graph.png}
\end{array}$

the $k$-colorings of $G \setminus e$ which are not $k$-colorings of $G$, are exactly the $k$-colorings $c : V \rightarrow \{1, 2, \ldots, k\}$ such that $c(u) = c(v)$. These correspond to the $k$-colorings of $G / e$ (contraction).

... back to the proof of lemma.

Induction on $|E|$. 

Base case: $E = \emptyset$. 

$G = \begin{array}{c}
\includegraphics{empty_graph.png}
\end{array}$

(The empty graph on $n$ vertices)

$\# k$-colorings equals $k^n$ (a polynomial in $k$)

So $x_G(t) = t^n$. 
By induction \( \chi_{G/e}(t) \) and 
\( \chi_{G/e'}(t) \) are polynomials, 
so 
\[ \chi_G(t) = \chi_{G/e}(t) - \chi_{G/e'}(t) \]
is a polynomial.
Moreover, we prove by induction that \( \chi_G(t) \) has 
integer coefficients.

**Acyclic orientations**

**Definition.** An acyclic orientation 
of an (undirected) graph \( G \) 
is a way to direct its edges 
so that the resulting directed graph 
has no directed cycles.

**Examples.**
1. \( G = K_3 \) has 6 acyclic orientations:
   - \( \triangle \) \( \triangle \) \( \triangle \)
   - \( \triangle \) \( \triangle \) \( \triangle \)
   - \( \triangle \) \( \triangle \) \( \triangle \)
   - \( \triangle \) \( \triangle \) \( \triangle \)
   - \( \triangle \) \( \triangle \) \( \triangle \)
   - \( \triangle \) \( \triangle \) \( \triangle \)

2. \( G = \) has 4 acyclic orient
   - \( \rightarrow \) \( \rightarrow \) \( \rightarrow \) \( \rightarrow \)
   (All orientations are acyclic)

3. More generally, a forest has \( 2^{\vert E \vert} \) acyclic orientations.

4. An \( n \)-cycle \( G = \) has
   \[ 2^n - 2 \] acyclic orientations.

All orientations are acyclic, 
except the two orientations:

\[ \text{and} \]
Theorem (R. Stanley, 1973)

Let $G$ be a graph on $n$ vertices. Then the number of acyclic orientations of $G$, equals $(−1)^n \chi_G(−1)$.

Examples. \[ G = \begin{array}{c}
\end{array} \]

\[ \chi_{K_3}(t) = t \cdot (t-1)(t-2). \]

\[ (−1)^3 \chi_{K_3}(-1) = -(-1)(-2)(-3) = 6. \]

$K_3$ has 6 acyclic orientations.

(2) \[ G = \begin{array}{c}
\end{array} \]

\[ \chi_G(t) = t \cdot (t-1)^2 \]

\[ (−1)^3 \chi_G(-1) = -(-1)(-2)^2 = 4. \]

$G$ has 4 acyclic orientations.
What $k$-colorings & acyclic orientations have in common?

Deletion - Contraction

Let $AO(G) := \# \text{ acyclic orientations of } G$.

**Lemma.** For an edge $e$ of $G$

$$AO(G) = AO(G \setminus e) + AO(G / e)$$

**Proof.** Let $O$ be an acyclic orientation of $G \setminus e$. Then can be 2 or 1 way to extend $O$ to an acyclic orientation of $G$.

![Diagram of an acyclic orientation with edge $e$.]

1. If $O$ does not have a directed path from $u$ to $s$, or from $s$ to $u$, then we can orient the edge $e$ in either of the 2 ways. ($2$ ways to extend an acyclic orientation)

2. If $G$ has a directed path between $u$ and $s$, say, a path from $u$ to $s$, then there is only one way to extend the acyclic orientation. Namely, the edge $e$ should be oriented from $u$ to $s$.

Now observe that, if we contract the edge $e$, in case (I) we get an acyclic orientation of $G / e$, but in case (II) we get an orientation of $G / e$ with a directed cycle.

So we deduce that

$$AO(G) = AO(G \setminus e) + AO(G / e)$$
Notice that we have slightly different deletion-contraction recurrences:

\[ X_G(t) = X_{G\setminus e}(t) - X_{G\setminus e'}(t) \]

\[ AO(G) = AO(G\setminus e) + AO(G\setminus e) \]

The factor \((-1)^n\) takes care of this.

**Proof of Steuken's theorem**

\[ AO(G) = (-1)^n \cdot X_G(-1) \]

**Induction on |E|**.

**Base** \( E = \emptyset \)

\[ AO(G) = 1, \quad X_G(t) = t^n \]

\[ (-1)^n \cdot (-1)^n = 1 \quad \checkmark \]

**Induction Step** \( e \in E \)

\[ AO(G) = AO(G\setminus e) + AO(G\setminus e') \]

\[ = (-1)^n \cdot X_{G\setminus e}(-1) + (-1)^{n-1} \cdot X_{G\setminus e}(-1) \]

\[ = (-1)^n \left( X_{G\setminus e}(-1) - X_{G\setminus e}(-1) \right) \]

\[ = (-1)^n \cdot X_G(-1), \quad \text{as needed.} \]
How to calculate \( \chi_G^+(+) \)?

There is a nice class of graphs, for which \( \chi_G^+(+) \) is given by a simple product formula.

**Chordal Graphs**

**Definition.** A simple graph \( G \) is called **chordal** if any cycle \( u_1, u_2, \ldots, u_r \) in \( G \) of length \( r \geq 4 \) has a chord, i.e. pair of vertices \( u_i, u_j \) (\( j = i \pm 1 \mod r \)) connected by an edge.

![A chord in a 5-cycle](image)

**Examples.**

- is chordal

- is not chordal

  (the 4-cycle does not have a chord)
Definition A perfect elimination ordering of a graph $G$ is an ordering $S_1, \ldots, S_n$ of all its vertices such that

$$\forall i = 2, 3, \ldots, n$$

the subset of vertices

$$\bigcup_{j < i} S_j$$

is an edge in $G$.

forms a clique, i.e., a complete subgraph in $G$.

Example

\[ G = \]

a perfect elimination ordering of vertex in $G$
Theorem (Fulkerson–Gross 1965)

A simple graph $G$ is chordal if and only if it has a perfect elimination ordering of vertices.

Remark One direction ($\Leftarrow$) is easy.

For a perfect elimination ordering $s_1, \ldots, s_n$ of $G$ define the numbers $a_1, a_2, \ldots, a_n \in \mathbb{Z}_{\geq 0}$

\[ a_i := \# \{ j < i \mid (s_j, s_i) \text{ is an edge} \} \]  

(we always have $a_1 = 0$)

Example

\[ G = \left( \begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\end{array} \right) \]

Another perfect elimination ordering of the same graph

\[ G = \left( \begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\end{array} \right) \]

We obtain two different sequences:

\[(a_1, \ldots, a_8) = (0, 1, 1, 2, 1, 3, 2)\]

\[(a'_1, \ldots, a'_8) = (0, 1, 2, 1, 2, 3)\]

Notice that these two sequences are permutations of each other.

Why?
Theorem: Let \( G \) be a chordal graph. Let \((a_1, a_2, \ldots, a_n)\) be the sequence obtained from any perfect elimination ordering of \( G \).

Then
\[
\chi(G) = (t-a_1)(t-a_2)\cdots(t-a_n).
\]

Corollary: The acyclic orientation of a chordal graph equals \((a_1)(a_2+1)\cdots(a_n+1)\).

Example: For the above graph \( G \), we have
\[
\chi(G) = t(t-1)^2(t-2)^2(t-3),
\]
and \( AO(G) = 1 \cdot 2 \cdot 3 \cdot 3 \cdot 4 \).

Proof of Theorem: Let color the vertices \( v_1, v_2, \ldots, v_n \) of \( G \) one by one starting from:

1. \( k \)-colorings of \( G \):
   - \( k = k-a_1 \) ways to color \( v_1 \),
   - \( k-a_2 \) ways to color \( v_2 \),
   - \( k-a_3 \) ways to color \( v_3 \), etc.

Notice that, at each step, there are exactly \( a_i \) colors which we cannot use to color \( v_i \). These are the colors of the proceeding vertices \( v_{i-1}, v_{i-2}, \ldots, v_1 \) connected to \( v_i \).

Since these \( a_i \) vertices form a clique in \( G \), they all have different colors.

Thus, \( S \) is a \( k \)-coloring of \( G \) is
\[
(k-a_1)(k-a_2)\cdots(k-a_n),
\]
and
\[
\chi(G) = \frac{1}{(a_1)}(t-a_i).
\]
There are several invariants of graphs that satisfy (a version of deletion-contraction recurrence:

- \# acyclic orientations of G
- chromatic polynomial \(X_G(t)\)
- \# spanning trees \(ST(G)\):

\[ ST(G) = ST(G \cup e) + ST(G \setminus e) \]

* Here the word “invariant” means that all these numbers and polynomials don’t depend on a choice of ordering of the vertices of G.

Is there the most general graphical invariant that satisfies the deletion-contraction recurrence?
The Tutte polynomial $T_G(x,y)$

$G$ - an undirected graph (we allow multiple edges and loops $\circ$)

An edge $e$ in $G$ is called a **bridge** (a.k.a. **isthmus**) if $G - e$ has more connected components than $G$.

Example

![Graph with loops and bridges]

**Theorem (Tutte)** There exists a unique polynomial $T_G(x,y)$, with positive integer coefficients, such that:

- For any edge $e$ in $G$, which is not a loop nor a bridge, we have $T_G = T_G \cdot e + T_{G/e}$.
- If $G$ has a bridge $b$ and $b$ loops (and no other edges), then $T_G = x^n y^b$.

**Example** $G = K_3 = \square$

$$T_{K_3} = T_{\square} + T_0$$

$$= x^2 + (T_{\square} + T_0)$$

$$= x^2 + x + y.$$

All other graphical invariants we discussed today ($\chi_G$, $\lambda(G)$, $ST(G)$) are special values of the Tutte polynomial.
Non-recursive formula for $T_G(x, y)$

Internal and external activities

Fix a total ordering of the set $E$ of edges of $G$.

Let $T \subseteq G$ be a Spanning tree of $G$.

Definition (1) An edge $e \in T$ is called **internally active** if

$A$ smaller edge $e' \subset e$ s.t.

$$(T \setminus e') \cup e'$$

is a Spanning tree.

(2) An edge $e \notin G \setminus T$ is called **externally active** if

$A$ smaller edge $e' \supset e$ s.t.

$$(T \cup e') \setminus e'$$

is a Spanning tree.

Let

$\text{int}(T) = \{ \text{internally active edge } e \text{ s.t. } T \setminus e \subseteq G \}$

$\text{ext}(T) = \{ \text{externally active edge } e \text{ s.t. } T \cup e \subseteq G \}$

**Theorem**

$T_G(x, y) = \sum_{T \text{ is a Spanning tree of } G} x^{\text{int}(T)} y^{\text{ext}(T)}$

**Remark** Clearly $\text{int}(T)$ and $\text{ext}(T)$ depend on a choice of total order of edges in $G$.

But the polynomial $T_G(x, y)$ does not depend on this choice.

**Example**

$G = \begin{array}{c}
\text{\includegraphics[width=2cm]{example.png}}
\end{array}$

$\text{int} = 3 \quad \text{int} = 1 \quad \text{int} = 0$

$\text{ext} = 3 \quad \text{ext} = 0 \quad \text{ext} = 1$

$T_3 = x^2 + x + y$. 
Special Values of $T_G(x, y)$

- $T_G(1, 1) = \#$ spanning trees

- $T_G(t) = (-1)^{y-k} + \sum_{k=1}^{\# \text{ connected components of } G}$

- $T_G(1, 0) = \#$ acyclic orientations of $G$

- $T_G(2, 1) = \#$ forests in $G$

- $T_G(2, 2) = 2^{1_E}$