last time:

**Flajolet's Fundamental Lemma**

\[ a_1, a_2, a_3, \ldots ; b_1, b_2, b_3, \ldots \text{ some weights} \]

(or just formal variables)

For a Dyck path \( P \), define

\[
\text{wt}(P) := \prod_{S \text{ is an up step in } P} a_{h(S)}
\]

More generally, for a Motzkin path \( P \), define

\[
\text{wt}(P) := \prod_{S \text{ is an up step in } P} a_{h(S)} \prod_{S' \text{ is a horizontal step}} b_{h(S')}
\]

**Examples**

- \( P = \)

  \[
  \text{wt}(P) = a_1 a_1 a_2 a_3 a_2
  \]

- \( P = \)

  \[
  \text{wt}(P) = a_1 b_1 a_1 a_2 b_3 a_2 a_3 b_2 b_3
  \]
Theorem (P. Flajolet, 1980)

We have the following identities for formal power series:

\[ \sum_{n \geq 0} \left( \sum_{P \text{ Pyck}} \text{wt}(P) \right) x^n = \]
\[ = \frac{1}{1 - a_1 x} \]
\[ \quad \frac{1 - a_2 x}{1 - a_3 x} \]
\[ \quad \frac{1 - \cdots}{1 - \cdots} \]

\[ \sum_{n \geq 0} \left( \sum_{P \text{ Motzkin}} \text{wt}(P) \right) x^n = \]
\[ = \frac{1}{1 - b_1 x - \frac{a_1 x^2}{1 - b_2 x - \frac{a_2 x^2}{1 - b_3 x - a_3 x^2}} \cdots} \]

(The first formula is a special case of the second for \( b_1 = b_2 = \ldots = 0 \) and \( x^2 \to x \).)
Examples. 1. $a_1 = a_2 = \ldots = 1, b_1 = b_2 = \ldots = 0$

$$\sum_{n \geq 0} C_n x^n = \frac{1}{1-x} \frac{1}{1-x} \frac{1}{1-x} \ldots$$

$$1 + \frac{1}{1 + \frac{1}{1 + \ldots}} = \varphi = \frac{1 + \sqrt{5}}{2}$$

(The golden ratio)

It satisfies the eqn. $\varphi = 1 + \frac{1}{\varphi}$.

So the "alternating sum of the Catalan numbers" is $\varphi^+$. 

$$C_0 - C_1 + C_2 - C_3 + \ldots = \left( \sum_{n \geq 0} C_n x^n \right) |_{x = -1}$$

These series diverge.

The value of the analytic continuation of

$$\sum_{n \geq 0} C_n x^n = \frac{1-\sqrt{1-4x}}{2x}$$

at $x = -1$

$$= \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \ldots}}} = \varphi^+ = \frac{\sqrt{5} - 1}{2}$$

Remarks.

Recall, that the golden ratio $\varphi$ is also related to the Fibonacci numbers $F_n$

$$(F_n = F_{n-1} + F_{n-2}, n \geq 2, F_0 = 0, F_1 = 1)$$

$$\lim_{n \to \infty} \frac{F_n}{F_{n-1}} = \varphi$$
\[ a_i = q^i, \quad b_i = b_2 = \cdots = 0 \]

\[ \sum_{n \geq 0} C_n(q) x^n = \frac{1}{1 - x} \frac{1}{1 - q x} \frac{1}{1 - q^2 x} \]

where \[ C_n(q) := \sum_{P \text{ Dyck path}} \text{Area below } P \]

\[ = 1 - q^3 x \]

\[ = \sum_{P \text{ Dyck path}} q \]

\[ e.g., \text{ for } n = 3, \]

\[ C_3(q) = 1 + 2q + q^2 + q^3 \]
3. \( a_i = i (i+1) \quad b_1 = b_2 = ... = 0 \)

\[
\sum_{n \geq 0} A_{2n+1} x^n =
\]
\[
= \frac{1}{1 - 1 \cdot 2 \cdot x}
\]
\[
= \frac{1}{1 - 2 \cdot 3 \cdot x}
\]
\[
= \frac{1}{1 - 3 \cdot 4 \cdot x}
\]
\[
= \frac{1}{1 - ...}
\]

4. \( a_i = i^2 \quad b_1 = b_2 = ... = 0 \)

\[
\sum_{n \geq 0} A_{2n} x^n =
\]
\[
= \frac{1}{1 - 1^2 \cdot x}
\]
\[
= \frac{1}{1 - 2^2 \cdot x}
\]
\[
= \frac{1}{1 - 3^2 \cdot x}
\]
\[
= \frac{1}{1 - ...}
\]
5. \( a_i = \frac{i(i+1)}{2} \) [Some weights as in Fréconou-Viennot bijection]

\[
\sum_{n \geq 0} \frac{(n+1)!}{n!} x^n = \frac{1}{1-2x-1.2x^2 - 4x - 2.3x^2 - 6x - 3.4x^2 - \ldots}
\]

Remark: Examples 3, 4, 5 are exceptions to the rule that we should use exponential generating series for labelled objects. (Permutations and alternating permutations are labelled objects.) Earlier we discussed exponential generating functions

\[
\sum_{n \geq 0} A_{2n} \frac{x^{2n}}{(2n)!} = \sec(x),
\]

\[
\sum_{n \geq 0} A_{2n+1} \frac{x^{2n+1}}{(2n+1)!} = \tan(x).
\]

Notice that the series in Examples 3, 4, 5 \underline{diverge} for all non-zero values of \( x \).

These identities are for \underline{formal} power series. They are \underline{not} analytic functions.
Let's move to the next topic...

**Lindström's Lemma**  
a.k.a. **Gessel–Viennot method**

Counting non-crossing paths.

A typical problem:

![Diagram of non-crossing lattice paths](image)

Calculate # of pairs of non-crossing lattice paths connecting points $A_1$ and $A_2$ with $B_1$ and $B_2$. 
Let $G$ be a directed graph such that:

- $G$ has no directed cycles.
- $G$ is drawn on the plane.
- All edges $e$ of $G$ have positive weights $w_e$.
- There are $n$ selected vertices $A_1, \ldots, A_n$ on the "left side" of $G$ and $n$ selected vertices $B_1, \ldots, B_n$ on the "right side" of $G$. Both $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ are arranged from top to bottom.

Example:

![Directed Graph Example](image)

- A graph drawn inside a rectangle.
- Vertices $A_i, B_i$ on the left side of the boundary.
- Vertices $A_i, B_i$ on the right side of the boundary.

We want to count all ways (or weighted sum over) all collections of non-crossing paths $P_1, \ldots, P_n$ that connect vertices $A_1, A_n$ with $B_1, B_n$.

**Definition.** A collection of directed paths $P_1, P_2, \ldots, P_n$ is non-crossing if $P_i$ and $P_j$ have no common vertices for any $i \neq j$.

Here is one possible way to connect $A_i$'s with $B_i$'s by non-crossing paths.
For a directed path $P$ in $G$, let

$$\text{weight}(P) := \prod_{e \text{ edge in } P} e$$

Let $C = (c_{ij})$ be the $n \times n$ matrix such that

$$c_{ij} = \sum_{P \text{ directed path from } A_i \text{ to } B_j} \text{weight}(P)$$

Lindström–Gessel–Viennot Lemma

$$\sum_{(P_1, \ldots, P_n) \text{ collection of non-crossing directed paths in } G, \ s.t. \ P_i : A_i \to B_i} \prod_{i=1}^{n} \text{weight}(P_i) = \det(C)$$
Example

G square grid graph with edges directed right & up.
All edge weights $x_e$ are 1.

# directed paths

- From $A_1$ to $B_1$ is $\binom{7}{3}$
- From $A_1$ to $B_2$ is $\binom{8}{2}$
- From $A_2$ to $B_1$ is $\binom{8}{5}$
- From $A_2$ to $B_2$ is $\binom{9}{4}$

So # non-crossing pairs of paths connecting $A_1, A_2$ with $B_1, B_2$ equals

$$\text{det} \begin{bmatrix} \binom{7}{3} & \binom{8}{2} \\ \binom{8}{5} & \binom{9}{4} \end{bmatrix} = \binom{7}{3} \binom{9}{4} - \binom{8}{2} \binom{8}{5}.$$
Another Example (with weights)

\[
\det(C) = \begin{vmatrix} x + yz & yu \\ tz & t + u + s \end{vmatrix} = (x + yz)(t + u + s) - tz \cdot yu
\]

Notice that all terms in the expansion correspond to all possible ways to connect \(A_1 \& A_2\) with \(B_1 \& B_2\) by paths (including crossing paths). But the term corresponding to crossing pairs of paths cancel each other, and only non-crossing pairs of paths remain.

We'll use the involution principle.

By the definition of det:

\[
\det(C) = \sum_{w = w_1 \ldots w_n} (-1)^{l(w)} C_{1w_1} C_{2w_2} \ldots C_{nw_n},
\]

permutation \( w \in S_n \)

\[
= \sum_{w \in S_n} (-1)^{l(w)} \sum_{i=1}^{n} \frac{1}{\text{weight}(P_i)}
\]

\( P_1 : A_1 \to B_{w_1} \)

\( P_2 : A_2 \to B_{w_2} \)

\[ \vdots \]

\( P_n : A_n \to B_{w_n} \)

Here we have an signed sum over arbitrary collections of directed paths \( P_1, \ldots, P_n \) connecting \( A_1, \ldots, A_n \) with \( B_1, B_2, \ldots, B_n \).
We want to cancel all terms corresponding to collections of paths $P_{i_1}, \ldots, P_{i_n}$ with at least one crossing.

(i.e. $(P_{i_1}, P_{i_2})$ st. $\exists i < j$ st. $P_i \& P_j$ have a common vertex.)

- Find the "first intersection point" $c$ of some $P_i \& P_j$

- "Swap the tails" of $P_i \& P_j$ at $c$, i.e. replace $P_i \& P_j$ with $\tilde{P}_i \& \tilde{P}_j$.

- Don't change the other paths $P_k$, $k \neq i, j$.

$$\sigma \cdot (P_{i_1}, P_{i_2}, \ldots, P_{i_j}, P_{i_n}) \rightarrow (P_{i_1}, P_{i_2}, \tilde{P}_{i_j}, \ldots, P_{i_n})$$

Notice that the weight of the weight $\sigma$ does not change, but the sign $(-1)^{\sigma(w)}$ reverses.

So the terms corresponding to the two collections of paths $(P_{i_1}, \ldots, P_{i_n})$ and $(P_{i_1}, \tilde{P}_{i_2}, \tilde{P}_{i_j}, \ldots)$ cancel each other.
The only subtle part of the construction of the "swap of tails" operation $\varphi$ is how to define the "first intersection point" $C$.

We want to consistently define the point $C$ for any collection of paths $(P_1, \ldots, P_n)$ with at least one crossing such that

$\varphi$ is an involution i.e. if

$\varphi : (P_1, \ldots, P_n) \mapsto (P_1 \ldots \tilde{P_i} \ldots \tilde{P_j} \ldots P_n)$

then $\varphi : (P_1 \ldots \tilde{P_i} \ldots \tilde{P_j} \ldots P_n) \mapsto (P_1 \ldots P_n)$

This collection of paths should have the same point $C$ as for the original collection $(P_1, \ldots, P_n)$. 
Example

What intersection point at $p_i$ & $p_j$ should we take?

Let’s try:

- Find the lexicographically minimal pair $(i,j)$ s.t. $p_i$ & $p_j$ have a common vertex
- Find the first common vertex $c$ of $p_i$ & $p_j$

This does not work...

A different "first intersection point" $c$. 
Here is the correct construction of map $\phi$:

- Find the minimal $i$ such that $P_i$ has a common vertex with another path.
- Find the first vertex $c$ of $P_i$ that belongs to another path.
- Find the minimal $j \neq c$ such that $P_j$ passes through vertex $c$.

Then swap the tails of $P_i$ & $P_j$ at vertex $c$.

$$\phi: (P_b, P_i, \ldots, P_j, \ldots, P_n) \mapsto (P_b, P_i, \ldots, P_j, \ldots, P_n)$$
It is easy to see that \( \sigma \) defined like this is an involution.

So it matches all the "bad guys" in pairs that cancel each other.

Only terms corresponding to non-crossing collections of paths \( P_1, \ldots, P_n \) remain.

All non-crossing collections correspond to the identity permutation \( \omega \in S_n \), i.e., \( P_i : A_i \rightarrow B_i \). If \( i \) the sign of this permutation is +1. So we get the needed identity:

\[
\det(C) = \sum_{P_1 \ldots P_n} \prod_{i=1}^n \text{weight}(P_i)
\]

\[\text{non-crossing}] \]
Plane Partitions

Fix $m, n, k \geq 1$

A plane partition (of shape $m \times n$) is a filling of the $m \times n$ rectangle by non-negative integers $\in \{0, 1, \ldots, k\}$ (equiv. a $m \times n$ matrix) such that the entries weakly decrease in rows & columns.

**Example** $m = 3, n = 4, k = 2$

\[
\begin{array}{ccc}
2 & 2 & 1 \\
2 & 1 & 0 \\
2 & 1 & 0 \\
\end{array}
\]

... a plane partition of shape $3 \times 4$

with entries $\in \{0, 1, 2\}$

Find the number of plane partitions (for given $m, n, k$).

**Case** $k = 1$.

\[
\begin{array}{c}
\# \text{ plane partitions} = \\
\# \text{ lattice paths from A to B} \\
= \binom{m+n}{m}
\end{array}
\]
Case $k = 2$

\[
\begin{array}{cccccc}
2 & 2 & 2 & 2 & 2 & 1 \\
2 & 2 & 1 & 1 & 1 & 0 \\
2 & 1 & 0 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 & 0 & 0
\end{array}
\]

$\#$ plane partitions $= \#$ pairs of lattice paths from $A$ to $B$ that cannot cross each other (in the strict sense) but can "touch" each other.

Let's shift the second path by 1 step in the direction "\downarrow".

$B_1 = (n, m)$

$B_2 = (n+1, m-1)$

Now we get a pair of non-crossing paths from $A_1 \^A_2$ to $B_1 \& B_2$, which we can count using Lindström–Gessel–Viennot Lemma:

\[
\begin{vmatrix}
\binom{m+n}{m} & \binom{m+n}{m-1} \\
\binom{m+n}{m+1} & \binom{m+n}{m}
\end{vmatrix}
\]
We can do a similar construction for any $k$.
Shift the second path by $(1, -1)$.
Shift the third path by $(2, -2), \ldots, (k, -k)$.

For general $k$, the plane partitions with entries
$G(x_1, \ldots, x_k)$ equals
the $k$-tuples of non-crossing paths connecting $A_i$'s with $B_i$'s.

$A_i = (i, i-k, i-1, i-2), B_i = (m+1, i, m, i-1)$
for $i = 1, 2, \ldots, k$.

Example $(m, q, k) = (5, 4, 3)$

$$
\begin{array}{c}
\text{A} \\
\text{B}_1 \\
\text{B}_2 \\
\text{B}_3 \\
\text{B}_4 \\
\text{B}_5 \\
\text{A}_1 \\
\text{A}_2 \\
\text{A}_3 \\
\text{A}_4 \\
\text{A}_5
\end{array}
$$

Shift the paths.

Liinnström- Gessel- Viennot lemma implies.

Theorem. The plane partition $E_{m,n}$ at shape $m \times n$ with entries
$E(x_1, \ldots, x_k)$ equals
the determinant of $k 	imes k$ matrix

$$
\begin{vmatrix}
(m, m+1) & (m+1, m+2) & \cdots & (m+k-1, m+k) \\
(m, m+2) & (m+2, m+3) & \cdots & (m+k-1, m+k+1) \\
\vdots & \vdots & \ddots & \vdots \\
(m, m+k) & (m+k, m+k+1) & \cdots & (m+k, m+k)
\end{vmatrix}
$$
Actually there is a more explicit formula for 
the plane partitions.

Theorem (A. MacMahon 1916)

\[
\text{# plane partitions of shape } m \times n \times k \\
\text{with entries } e \in \{0, 1, \ldots, k^2\} \text{ equals}
\]

\[
\begin{array}{c|c|c|c}
\hline
m & n & k & a+b+c=1 \\
\hline
a & b & c & a+b+c=2 \\
\hline
\end{array}
\]

\[a=1 \quad b=1 \quad c=1\]

Remark. Notice that the resulting formula is symmetric in \(m, n, k\). This symmetry becomes clear if we represent plane partitions as "3-dimensional Young diagrams."

Example. \((m, n, k) = (3, 4, 3)\)

\[
\begin{array}{ccc}
3 & 3 & 2 \\
3 & 1 & 1 \\
2 & 1 & 0 \\
\end{array}
\]

A plane partition

If a box of the plane partition is filled with \(e\), then put \(e\) \(1 \times 1 \times 1\) cubes above it.

We get a "3-dim Young diagram" that should fit inside the \(m \times n \times k\) box.

It is now clear that such diagrams should be symmetric in \(m, n, k\).