Matrix Tree Theorem (cont'd)

$G$ connected graph on $n$ vertices labelled $1, 2, ..., n$.

**Adjacency matrix** $A = (a_{ij})_{i, j \in \{1, 2, ..., n\}}$

$a_{ij} = \# \text{ edges between vert. } i \text{ & } j$

($(a_{ii} = 0 \ \forall \ i)$

**Laplacian matrix**

$L = \text{diag} (d_1, ..., d_n) - A$

$d_i = \deg_G(i)$

$\widetilde{L}(i) = L \text{ with } i^{th} \text{ row & } i^{th} \text{ col. removed } (n-1 \times n-1 \text{ matrix})$

MTT: # spanning trees of $G$

$= \det (\widetilde{L}(i))$

for any $i = 1, ..., n$. 

reduced Laplacian matrix
Q: Why is the R.H.S. independent of $i$?

Let $C$ be any symmetric $n \times n$ matrix with zero row & column sums.

$\tilde{C}(i) = C$ with $i^{th}$ row & $i^{th}$ col. removed.

($\det(\tilde{C}(i))$ are called principal cofactors of $C$).

Let $\mu_1, \mu_2, \ldots, \mu_n$ be the eigenvalues of $C$.

(One eigenvalue should be 0, because $\det(C) = 0$)

**Lemma** For any $i = 1, \ldots, n$

$\det(\tilde{C}(i)) = \frac{\mu_1 \mu_2 \cdots \mu_i}{\mu_i}$

all principal cofactors of $C$ are equal to each other

**Exercise** Prove this lemma.

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**Alternative Formulation of MTT.**

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of the Laplacian matrix $L$. Then

$$\# \text{ Spanning Trees of } G = \frac{\lambda_2 \lambda_3 \cdots \lambda_n}{n}$$
Def. The spectrum of graph $G$ is the collection $\lambda_1, \lambda_2, \ldots, \lambda_n$ of its adjacency matrix $A$.

"Spectral Graph Theory" is the area of math that studies property of a graph in terms of its spectrum.

But for MTT we need the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the Laplacian matrix, not the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the adjacency matrix.

Q: Are $\lambda_1, \ldots, \lambda_n$ related to $\lambda_1, \ldots, \lambda_n$ the spectrum of $G$?

A: In general, no.
For regular graphs they are important.

A graph \( G \) is called \( d \)-regular if the degree of any vertex in \( G \) equals \( d \).

Example:

a 3-regular graph

(Actually, this graph is \( K_{3,3} \)).

For a \( d \)-regular graph, the Laplacian matrix is

\[
L = dI - A.
\]

Clearly, we have

Lemma. For a \( d \)-regular graph \( G \), the eigenvalues \( \lambda_i \) of its Laplacian are related to the eigenvalues \( \lambda_j \) of its adjacency matrix (the spectrum of \( G \)) by

\[
\lambda_i = d - \lambda_j.
\]

Proof:

Recall that the eigenvalues of a matrix \( A \) are the roots of its characteristic equation \( \det(A - \lambda I) = 0 \).

The characteristic equation of \( L \) is

\[
\det(L - \lambda I) = \det(dI - A - \lambda I) = \det((-\lambda - d)I - A) = 0.
\]

So its roots are \( d \) minus the roots of the linear equation \( \lambda + A = 0 \).

Remark. It is usually easier to calculate the eigenvalues \( \lambda_j \) of the adjacency matrix \( A \) than the eigenvalues \( \lambda_i \) of the Laplacian matrix.

For \( d \)-regular graphs we can relate \( \lambda_i \) and \( \lambda_j \) expressing the spanning trees in terms of \( \lambda_i \).

Theorem. Let \( G \) be a \( d \)-regular graph and \( \lambda \) be an eigenvalue of its spectrum.

Then \( \lambda = d \) (the largest eigenvalue equals \( d \)).

\# spanning trees of \( G \) is

\[
\frac{1}{d(d-\lambda_1)(d-\lambda_2)\ldots(d-\lambda_k)}
\]
Hypercubes

Let’s now calculate the number of spanning trees of the d-hypercube graph $H_d$.

$H_d$ is the 1-skeleton of the d-dimensional hypercube.

**Examples**

$H_1 = \bullet$ 1 spanning tree

$H_2 = \begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet
\end{array}
\end{array}$ 4 spanning trees

$H_3 = \begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet
\end{array}
\end{array}$

$H_4 = \begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet
\end{array}
\end{array}$

**Observation** $H_d$ is a d-regular graph.

So here “d” stands both for “dimension” and “degree.”
Product of graphs

\[ G = (V_1, E_1), \ H = (V_2, E_2) \]

\[ G \times H := (V, E) \]

where \( V = V_1 \times V_2 \)

\[ = \{ (u, v) \mid u \in V_1, v \in V_2 \} \]

\[ E = \{ (u, v), (u', v') \mid u, u' \in V_1; \ v, v' \in V_2 \]

\[ u = u', \ \text{and} \ (v, v') \in E_2, \ \text{or} \]

\[ v = v', \ \text{and} \ (u, u') \in E_1 \]
In general, the hypercube graph is the product:

\[ H_d = \underbrace{\times \cdots \times}_d \]

Let \( A(G) \) be the adjacency matrix of \( G \).

**Lemma.** Suppose that

- \( A(G) \) has eigenvalues \( \lambda_1, \ldots, \lambda_m \); and
- \( A(H) \) has eigenvalues \( \beta_1, \beta_2, \ldots, \beta_n \).

Then \( A(G \times H) \) has \( mn \) eigenvalues \( \lambda_i + \beta_j \),

\( i \in [m], \ j \in [n] \).

**Idea of proof.** (Use Linear Algebra)

We can explicitly express the eigenvectors of \( A(G \times H) \) in terms of the eigenvectors of \( A(G) \) and \( A(H) \), and see that the eigenvalues of \( A(G \times H) \) as expressed as above. \( \square \)

**Remark**

This is related to tensor products of matrices

\[ A(G \times H) = A(G) \otimes I_n + I_m \otimes A(H). \]

**Exercise.** Prove this lemma.
Example

\[
A(0) = \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\]

eigenvectors are \([1, 1]^T\) and \([1, -1]^T\)

and the eigenvalues are \(\frac{1}{2}, -\frac{1}{2}\)

\[
A(G \times H) = A(G) \otimes A(H)
\]

eigenvectors of this matrix
are \(\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}\)

The eigenvalues are \(1+1, -1-1, -1+1, 1-1\)

\[
= 2, 0, 0, -2
\]

Example

\[
H_3 = (\quad \times \quad) \times \quad
\]

The adjacency matrix \(A(H_3)\)
has eigenvalues \(\pm 1, \pm 1, \pm 1\)

(3 choices for the signs):

\[
\begin{bmatrix}
1+1 & -1+1 & -1+1 & -1+1 \\
-1+1 & -1+1 & -1+1 & 1+1 \\
-1+1 & -1+1 & 1+1 & -1+1 \\
1+1 & -1+1 & -1+1 & -1+1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2
\end{bmatrix}
\]

In general, \(A(H_d)\) has

eigenvalues

\[
\pm 1, \pm 1, \pm 1, \ldots, \pm 1
\]

\(d\) terms

2nd choice for

Signs
Lemma. \( A(H_d) \) has

Eigenvalues \(-d + 2k\) with

multiplicity \((d)\) for \(k = 0, \ldots, d\).

Since \( H_d \) is a \( d \)-regular graph,

Corollary. The eigenvalues of the Laplacian matrix \( L \) of \( H_d \) are

\( 2k \) with multiplicity \((d)\)

for \( k = 0, 1, 2, \ldots, d\).

Now we deduce.

Theorem (R. Stanley, Enumerative Combinatorics Vol. 2, p. 82)

The number of spanning trees of the hypercube graph \( H_d \) is

\[
\frac{d}{2^d} \prod_{k=1}^{d} (2k)^{(d)}
\]

Example. \( H_3 \)

has \( 2^3 - 3 - 1 = 7 \) spanning trees.

Remark. Stanley asked for a bijective proof of this formula. This was solved by Olivier Bernardi in 2012.
Grid graphs

The product of $m$-chain and $n$-chain is the $mn$ grid graph.

\[ \times \quad = \]

6x5 grid

Can we use the previous approach and get a closed formula for the number of spanning trees of the $mn$ grid graph?

One problem is that the grid graph is not a regular graph.

We can calculate the spectrum (eigenvalues of the adjacency matrix). But we cannot express the eigenvalue of the Laplacian matrix in terms of them.

However, if the take the product of $m$-cycles & $n$-cycle, we do get a regular graph.

Example

\[ \times \]

6x5 "toric grid graph"

Exercise. Find a closed formula for the number of spanning trees of the $mn$ toric grid graph.
Reciprocity for Spanning trees

Let \( G = (V, E) \) be a simple graph, and \( \overline{G} = (V, \overline{E}) \) be its complementary graph.

For \( u \neq v \in V \), if \((u, v) \in E\) then \((u, v) \notin \overline{E}\) and vice versa.

**Examples:**

1. \( G = \begin{array}{c}
\begin{array}{c}
\text{1} \\
\text{2} \\
\text{3} \\
\text{4} \\
\end{array}
\end{array} \quad \overline{G} = \begin{array}{c}
\begin{array}{c}
\text{1} \\
\text{2} \\
\text{3} \\
\text{4} \\
\end{array}
\end{array} \)

2. \( G = \begin{array}{c}
\begin{array}{c}
\text{1} \\
\text{2} \\
\text{3} \\
\text{4} \\
\text{5} \\
\end{array}
\end{array} \quad \overline{G} = \begin{array}{c}
\begin{array}{c}
\text{1} \\
\text{2} \\
\text{3} \\
\text{4} \\
\text{5} \\
\end{array}
\end{array} \)

**Q:** Are spanning trees of \( G \) & \( \overline{G} \) somehow related to each other?
Assume that $G$ is a simple graph on $n$ vertices
1, 2, ..., $n$.

Let $G^+$ be the graph on $n+1$ vertices $0, 1, 2, ..., n$
obtained from $G$ by adding the vertex $0$
connected with all other
vertices 1, 2, ..., $n$ by edges.

Similarly, we have the graph $\overline{G^+}$.

Example, $G = \begin{array}{cc}
1 & 2 \\
3 & 4 \\
\end{array}$

$G^+ = \begin{array}{cc}
0 & 1 \\
2 & 3 \\
4 & 0 \\
\end{array}$

$\overline{G^+} = \begin{array}{cc}
1 & 2 \\
3 & 4 \\
\end{array}$
Let \( F_G(x) = \sum x^{\deg_1(x) - 1} \)

\[ \text{T spanning tree of } G^- \]

Example \( G = \)

\[ F_G(x) = x^3 + 4.2x^2 + (4.3 + 2.2.2)x + 4.4 \]

\[ F_G(x) = x^3 + 4x^2 + 4x + 0. \]

Glearly, \( F_G(0) = \)

\[ = n \cdot \# \text{ spanning trees of } G^- \]

The vertex 0 is connected to one of the vertices 1, 2, ..., n, a spanning tree of \( G^- \).
Reciprocity Formula
[S.D. Bedrosian, 1964]

\[ F_G(x) = (-1)^{n-1} F_{\bar{G}}(-x - n) \]

Example: \( G = \square \)

\[ F_G(x) = x^3 + 8x^2 + 20x + 16 \]
\[ = (-1)^3 F_{\bar{G}}(-x - 4) \]
\[ = - \left( (-x-4)^3 + 4(-x-4)^2 + 4(-x-4) \right). \]

Exercise: Prove the reciprocity formula. (For example, you can deduce it from the MTT, or give a direct combinatorial proof.)
Suppose we know nothing about spanning trees, except the reciprocity formula.

We can use it to deduce many other formulas.

Some examples

1. Let $O_n = ([n], \emptyset)$ be the empty graph on $n$ vertices.

\[ O_n^+ = \begin{array}{ccccccc}
1 & 2 & 3 & 4 & \ldots & n
\end{array} \]

\[ F_{O_n}(x) = x^{n-1} \]

\[ O_n = K_n \]

\[ F_{K_n}(x) = (-1)^{n-1}(x-n)^{n-1} = (x+n)^{n-1} \]

# Spanning trees in $K_n$

\[ = \frac{1}{n} F_{K_n}(x) = \frac{1}{n} \cdot n^{n-1} \]

\[ = \left\lfloor \frac{n^{n-2}}{n} \right\rfloor \]

So we’ve got Cayley formula.
2. If $G$ and $H$ is the disjoint union of two graphs, then

$$F_{G\cup H}(x) = F_G(x) \cdot F_H(x)$$

So

$$F_{K_n \cup K_n}(x) = x \cdot F_{K_n}(x) \cdot F_{K_n}(x)$$

by the previous example.

$$= x \cdot (X + m)^{n-1} \cdot (X + n)^{n-1}$$

Now $K_n \cup K_n = K_{2n}$, the complete bipartite graph.

So we obtain

$$F_{K_{2n}} = (-1)^{n+1} \left( -X - m - n \right)^{n-1} \cdot (-X - m + n)^{n-1}$$

$$= (X + m + n) \cdot (X + n) \cdot (X + m)^{n-1}$$

In particular, the number of spanning trees of $K_{2n}$ is

$$\frac{1}{m+n} F_{K_{2n}}(0)$$

$$= \frac{1}{m+n} \cdot (m+n)^{n-1} \cdot n^{n-1}$$

$$= \prod_{i=1}^{n-1} i$$

Etc.