Theory of Partitions (cont’d)

last time: we proved Euler’s

Pentagonal Theorem:

\[
\prod_{n=1}^{\infty} (1 - q^n) = \sum_{k=-\infty}^{+\infty} (-1)^{k} q^{k(3k-1)/2}
\]

The proof that we gave is due to

Fabian Franklin

(1853 - 1939)

Today we’ll talk about a

generalization of Euler’s

pentagonal theorem...
Jacobi Triple Product

Theorem [Jacobi, 1829]

\[
\prod_{n=1}^{\infty} \frac{1-x^{2n}}{(1-x^{2n-1})(1-x^{2n-1}/y^2)} = \sum_{k=-\infty}^{+\infty} x^{k^2} y^{2k}.
\]

Carl Gustav Jacob Jacobi
(1804–1851)

Euler's pentagonal number theorem is a special case of Jacobi triple product for \( x = q^{3/2}, y^2 = -q^{1/2} \).

We'll give a combinatorial proof. First, let's slightly rewrite the triple product formula by making the substitution:

\[ x^2 = q, \quad y^2 = z \cdot q^{1/2} \]

We get the following equivalent formula:

\[
\prod_{n=1}^{\infty} (1-q^n)(1+zq^n)(1+z^{-1}q^{n-1}) = \sum_{r=-\infty}^{+\infty} z^r q^{r(2r+1)/2}.
\]

Let's move the first term in the triple product to the R.H.S.
Theorem. \( \prod_{n \geq 1} (1 + zq^n)(1 + z^{-1}q^{n-1}) \)

\[(2) \quad = \left( \sum_{r = -\infty}^{+\infty} z^r q^{r(r+1)/2} \right) \prod_{n \geq 1} \frac{1}{1-q^n}. \]

Proof. The coefficient of \( z^a \)
in \( \prod_{n \geq 1} (1 + zq^n) \)
equals

\[ \sum_{\mu \text{ partition}} q^{\left| \mu \right|} \]
with a distinct parts

Similarly, the coefficient of \( z^{-b} \)
in \( \prod_{n \geq 1} (1 + z^{-1}q^{n-1}) \)
equals

\[ \sum_{\nu \text{ partition}} q^{121 - b} \]
with b distinct parts

Of course, \( \prod_{n \geq 1} \frac{1}{1-q^n} \)
equals

\[ \sum_{\lambda \text{ any partition}} q^{\left| \lambda \right|} \]
In order to prove (4) we need to match all terms in the L.H.S. with all terms in the R.H.S.

We need to construct a bijection

\[
\begin{pmatrix} a, b, \mu, \nu \end{pmatrix} \leftrightarrow \begin{pmatrix} r, \lambda \end{pmatrix}
\]

Where

- **LHS:** \( a, b \in \mathbb{Z}_{\geq 0} \)
  - \( \mu \) partition w/ a dist. parts
  - \( \nu \) partition w/ b dist. parts

- **RHS:** \( r \in \mathbb{Z} \) any integer
  - \( \lambda \) any partition

Such that the corresponding monomials are the same:

\[
a - b = |\mu| + |\nu| - b = r \quad \text{such that } |\mu| - \frac{r(r+1)}{2} \geq 0
\]

- \( a - b = r \)
- \( |\mu| + |\nu| - b = \frac{r(r+1)}{2} + |\lambda| \).
Let's do this by example:

Example. \( \mu = (4, 6, 4, 3, 1), \nu = (6, 5, 3). \)

\[
\begin{align*}
a &= 5, & \mu &= (4, 6, 4, 3, 1), \\
b &= 3, & \nu &= (6, 5, 3).
\end{align*}
\]

Convert \( \mu \) & \( \nu \) into shifted Young diagrams \( \tilde{\mu} \) & \( \tilde{\nu} \):

\[
\begin{align*}
\tilde{\mu} &= \\
\tilde{\nu} &= 
\end{align*}
\]

Shifted Young diagrams

They have the same number of boxes in each row as \( \mu \) & \( \nu \), but the leftmost boxes in rows (marked as \( \bullet \)) form a staircase.

Let's transpose \( \tilde{\nu} \) and remove its \( b \) diagonal boxes:

\[
\tilde{\nu} = 
\quad \implies \quad \tilde{\nu}' = 
\]
Let's now "glue" the shapes $\tilde{\lambda}$ and $\tilde{\lambda}'$ into a single shape by identifying the last min $(a,b)$ diagonal boxes (marked with $\cdot$):

$\tilde{\lambda}$ $\mapsto$ $\tilde{\lambda}'$

Finally, let's chop off the "little triangle" in the upper left corner:

$\tilde{\lambda}$ $\&$ $\tilde{\lambda}'$ combined $\lambda$

We obtain the pair

$r = \# \text{ boxes on the side of the "little triangle"}$

$\lambda = \tilde{\lambda} \cup \tilde{\lambda}'$ with chopped off "little triangle"
This construction

\((a, b, \lambda, \nu) \rightarrow (r, \lambda)\)

is the needed bijection.

In the above example we had \(a \geq b\). Let's give another example with \(a < b\).

**Example #2.** \(M = (5, 9, 2), \nu = (9, 8, \ast, 9, 3, 1)\)

\(a = 3\) \hspace{1cm} \(\tilde{\nu} = \)

\(b = 6\) \hspace{1cm} \(\tilde{\nu} = \)

\(\tilde{\lambda} = \)

\(r = a - b = -3\)

\(\lambda = (8, 8, 7, 5, 3, 3)\)
This operation is clearly invertible.

**Inverse Construction**

\[(r, \lambda) \mapsto (a, b, p, \gamma)\]

- If \( r \geq 0 \), then attach the "little triangle" \((r, r-1, r-2, \ldots)\) to \( \lambda \) as follows:

- If \( r < 0 \), then attach the "little triangle" \((-r+1, -r+2, -r+3, \ldots)\) to \( \lambda \) as follows:

Notice that in both cases

\[\text{# boxes in the little triangle} = \frac{r(r+1)}{2}.\]
Then cut the resulting shape along the main diagonal to get shifted shapes $\chi$ & $\chi'$.

Then get $\mu$ & $\lambda$.

So we've got the bijection $\{(a, b, \mu, \lambda)\} \longleftrightarrow \{(r, \lambda)\}$.

The needed properties are clear from the construction:

- $a - b = r$
- $|\mu| + |\lambda| - b = \frac{r(r+1)}{2} + |\lambda|$
Theorem [Gauss]

(1) \( \left( \prod_{k \geq 1} (1 - q^k) \right)^3 = \sum_{k = 0}^{+\infty} (-1)^k q^k \)

(2) \( \prod_{n = 1}^{+\infty} \frac{1 - q^n}{1 + q^n} = \sum_{k = -\infty}^{+\infty} (-1)^k q^{k^2} \)

Proof. (1) Specialize Jacobi triple product (x) for \( z = -t \), \( q = t^{1/2} \)
(2) Specialize Jacobi triple product (x) for \( z = -1 \)

Remark. All this stuff has deep links with:

- Number theory: modular forms, \( \Theta \)-functions, etc.
- Representation Theory of infinite Lie algebras, Kac-Moody algebras.

This stuff is out of scope of this course. But if later some of you will study modular forms and/or Kac-Moody theory, you might revisit Euler's pentagonal number formula, Gauss identities & Jacobi triple product.

There are generalizations of Jacobi triple product to affine Kac-Moody algebras, called Macdonald Identities.
Remark: We obtained:

Euler: \[
\prod_{n \geq 1} (1 - q^n) = \sum_{k = -\infty}^{+\infty} \frac{k}{(3k-1)/2} (-1)^k q^k
\]

Gauss: \[
\left( \prod_{n \geq 1} (1 - q^n) \right)^3 = \sum_{k = -\infty}^{+\infty} \frac{k}{(k+1)/2} (-1)^k q^k
\]

How about \[
\left( \prod_{n \geq 1} (1 - q^n) \right)^2
\]

Actually, if we expand it, we get total mess.

Unlike \[
\prod_{n \geq 1} (1 - q^n)
\]

whose expansions contain very sparse sets of terms,

the expansion of \[
\left( \prod_{n \geq 1} (1 - q^n) \right)^2
\]

contain many terms & there is no simple formula for its coefficients.

An explanation of this phenomenon is given in theory of Kac-Moody algebras.
Gaussian $q$-binomial coefficients again...

\[
\binom{n}{k}_q := \frac{[n]_q!}{[k]_q! [n-k]_q!}
\]

\[
= \sum_{\lambda \subseteq k \times (n-k)} q^{|\lambda|}
\]

Young diagram
that fit into
rectangle

Theorem (q-binomial formula)

\[
(1+x)(1+xq)(1+xq^2) \ldots (1+xq^{n-1})
\]

\[
= \sum_{k=0}^{n} \binom{n}{k}_q \cdot q^{\frac{k(k-1)}{2}}
\]
Take the limit of this as \( n \to \infty \)

\[
\lim_{n \to \infty} \left[ \sum_{k=1}^{n} \frac{1}{k^2} \right] = \frac{1}{(1-q)(1-q^2) \cdots (1-q^k)}
\]

\[
= \sum_{k=1}^{\infty} q^{k-1}
\]

\( \text{with at most } k \text{ parts} \)

\( \text{encode such } x \text{ by } (n_1, n_2, \ldots, n_k) \in \mathbb{Z}_{\geq 0}^k \)

where \( n_i = \# \text{ columns with } i \text{ boxes in } x \)

So \( \sum_{k=1}^{\infty} q^{k-1} = \sum_{(n_1, n_2, \ldots, n_k) \in \mathbb{Z}_{\geq 0}^k} q^{n_1 + n_2 + \cdots + n_k} \)

\[
= \frac{1}{1-q} \frac{1}{1-q^2} \cdots \frac{1}{1-q^k}
\]

**Corollary**

\[
\prod_{n=0}^{\infty} (1+xq^n)
\]

\[
= \sum_{k \geq 0} \frac{q^k}{(1-q)(1-q^2) \cdots (1-q^k)} x^k
\]
Proof. Let's give a combinatorial proof of this formula.

\[ \text{L.H.S. } \sum_{n=0}^{\infty} x^n \cdot \left( \sum_{\lambda} q^{\lambda} \right) \]

where \( \lambda \) runs over all partitions with \( k \) distinct parts one of which can be 0.

\( \lambda = (14, 13, 11, 8, 6, 5, 3, 0) \)

Let's convert it into a shifted Young diagram \( \lambda' \) and then cut off its "staircase part".

So we get L.H.S.

\[ \sum_{k=0}^{\infty} x^k \cdot \frac{K(k+1)/k}{(1-q) \cdots (1-q^k)} \]

comes from the staircase part \( \sum_{\lambda' \text{ with at most } k \text{ rows}} q^{\lambda'} \)

as needed. \( \square \)
Here is the $q$-analog of the identity
\[
\sum_{k=0}^{n} (q^k)^2 = \binom{2n}{n}.
\]

**Theorem**

\[
\sum_{n \geq 0} q^n \left\lfloor \frac{n}{k} \right\rfloor = \sum_{k=0}^{n} q^{k^2} \left( \left\lfloor \frac{n}{k} \right\rfloor \right)^2.
\]

It is not hard to prove the last 2 theorems, if you express them in terms of partitions.

**Exercise** Prove them.