A polynomial $A(x)$ is a sequence of constant numbers $a_i$ such that $A(1) = A$. (The classical list)

$q$-numbers:

\[ [n]_q = \frac{q^n - 1}{q - 1} \]

$q$-factorials:

\[ [n]!_q = [1]_q [2]_q \cdots [n]_q \]

Example:

\[ [3]!_q = [1]_q [2]_q [3]_q = (1 + q)(1 + q + q^2) \]

Here we have already seen $[n]_q$ in the context of permutations:

\[ \text{Theorem 21: } \sigma \in S_n \quad \Rightarrow \quad [\pi]_q = \sum_{\sigma \in S_n} \sigma \]

Also $\sum_{\sigma \in S_n} \sigma = [n]!_q$ if $q = 1$.

$q$-binomial coefficients take the Gaussian coefficients

\[ \binom{n}{k}^q = \frac{[n]!_q}{[k]!_q [n-k]!_q} \]

Example:

\[ \binom{3}{2}^q = \frac{[3]!_q}{[2]!_q [1]!_q} = \frac{(1 + q)(1 + q + q^2)}{(1 + q)(1 + q^2)} = 1 + q^2 + q^3 + q^4 \]

This means that $A(x)$ can be a polynomial with positive integer coefficients. But in general, $A(x)$ need not be. Thus, for instance, $A(x)$ divisible by $x^{ab}$. This is discussed by the Macmillans.
Theorem.
\[ \sum_{k=0}^{n} \binom{n}{k} q^k = \prod_{\lambda \in \mathcal{P}(n-k)} \left( 1 + q \right)^{\lambda} \]

The sum over Young diagrams \( \lambda \) that fit inside the \( \mathcal{P}(n-k) \) rectangle. The coefficients of this expression are exactly the rank numbers \( r \) of the Young's lattice \( L (K, n-k) \).

Example \[ \sum_{k=0}^{2} \binom{2}{k} q^k = \]

\[ 1 + q + q^2 + q^2 + q^3 + q^4 \]

Proof #1. It is not hard to prove it by induction on \( n \). One can easily show that both L.H.S. & R.H.S. satisfy the \( q \)-Pascal's recurrence relation:
\[ \binom{n}{k} = \binom{n-1}{k-1} + q \binom{n-1}{k} \]

Exercise. Check this.

Remark. This is an easy but not very conceptual proof. We'll give another more interesting proof.
q - Pascal's triangle

Voici le q-analogue de mon triangle

Blaise Pascal
1623 - 1662
The Sperner property of $L(m,n)$ can be reformulated like this:

Let $P_1, P_2, \ldots, P_N$ be any collection of lattice paths from the lower left to the upper right corners in the $m \times n$ rectangle $S$.

Any two paths $P_i$ & $P_j$ intersect.

For example,

```
3
```

Then $N \leq \binom{m+n}{\frac{m+n}{2}}$:  

\[
\begin{align*}
\text{Young diagram:} & \\
\text{such that} & \\
|x| & = \lfloor \frac{n}{2} \rfloor \\
\end{align*}
\]
A more conceptual proof that
\[ \binom{n}{k}_q = \sum_{\lambda \in \text{Sym}(n-k)} q^{|\lambda|} \]

**Proof #2.** Both sides are rational expressions in \( q \). It is enough to check that they are equal to each other for infinitely many values of \( q \).

We'll prove this when \( q = p^r \) (a power of a prime number \( p \)).

There are infinitely many prime numbers!

Euclid 300 BC

Let \( \mathbb{F}_q \) be the finite field with \( q \) elements.

Finite fields are also known as Galois fields.

If not that estate duel I would have discovered a lot of other genius things

Evariste Galois
1811 - 1832

\( \text{GL}_n(\mathbb{F}_q) \) - all invertible \( n \times n \) matrices with elements in \( \mathbb{F}_q \).

What is \( \# \text{GL}_n(\mathbb{F}_q) \)?

**Example** \( q = 2, \ n = 2 \)

\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

\( \# \text{GL}_2(\mathbb{F}_2) = 6. \)
Let's construct an invertible \( n \times n \) matrix by picking its rows one by one.

\[
\begin{array}{c}
\text{row 1} \\
\text{row 2} \\
\vdots \\
\text{row n}
\end{array}
\]

Row 1 can be any \( n \)-vector over \( \mathbb{F}_q \), except \((0, 0, \ldots, 0)\).

So we have \( q^n - 1 \) choices.

Row 2 can be any \( n \)-vector over \( \mathbb{F}_q \), except \( \text{Row 1} \) rescaled by an element of \( \mathbb{F}_q \).

We have \( q^n - q \) choices.

Row 3 can be any \( n \)-vector, except a linear combination of Row 1 & Row 2.

We have \( q^n - q \) choices.

etc.

**Theorem:** There are exactly

\[
(q^n-1)(q^n-q)(q^n-q^2)\ldots(q^n-q^{k-1})
\]

\( k \times n \) matrices over \( \mathbb{F}_q \)

that have maximal possible rank \( r \).

In particular,

\[
\# \text{GL}_n(\mathbb{F}_q) = (q^n-1)(q^n-q)(q^n-q^2)\ldots(q^n-q^{n-1}) = q^{\binom{n}{2}}(q^n-1)^n \cdot L_n(q).
\]
The Grassmannian $\text{Gr}(k, n; F)$ is the space of $k$-dim linear subspaces in $F^n$.

Example. $k=1, n=2$. The elements of $\text{Gr}(1, 2)$ (aka the projective line) are lines that pass through the origin.

$\text{Gr}(1, 2; \mathbb{R}) \sim \text{a circle where any two opposite points are identified} \sim \text{just a circle.}$

More concretely, $\text{Gr}(k, n; F_q)$ is the space of $k \times n$ matrices over $F_q$ of rank $k$ modulo row operations.

So $\# \text{Gr}(k, n; F_q) =$

$$\frac{(q^n-1)(q^n-q)(q^n-q^2) \cdots (q^n-q^{k-1})}{(q^k-1)(q^{k-1})(q^{k-2}) \cdots (q^{k-k})}$$

Theorem $\# \text{Gr}(k, n; F_q) =$

$$= \prod_{i=1}^k (q-1).$$
Let's see a different way:

**Gaussian Elimination**

Is there a way to avoid using the method of Gaussian elimination by row operations into reduced row echelon form?

**Example.** $R_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

- Any element of $E_i$ - it is enough to match
- $x_i$ - variable
- $x_i$ - variable
- $E_i$ - matrix

So there should be answers:

- $x_1 = x_2 = x_3 = x_4 = 0$

The reduced form looks better if we replace the pivot columns.

- $E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Note that the reduced form of $E_1$ is

(a) The set of pivot columns
(b) A basis vector $\lambda = (x_1, x_2, x_3, x_4)$

We obtain:

$E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Comparing this with the previous expression:

$E_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

we get the result.
We’ve got

\[ \sum_{\omega \in S_n} i_{\omega}(x) q^{\omega} = \sum_{\mathbf{n} \in \mathbb{N}^n} \frac{\mathbf{n}!}{n_1! n_2! \cdots n_r!} \]

How about the multinomial coefficients?

Let \( n_1 + \cdots + n_r = n \) (a composition of \( n \))

The multinomial coefficients are defined as

\[ \binom{n}{n_1, n_2, \ldots, n_r} := \frac{n!}{n_1! n_2! \cdots n_r!} \]

Proposition:

\[ \binom{n}{n_1, n_2, \ldots, n_r} = \#	ext{ permutations of the multi set } \{1^{n_1}, 2^{n_2}, \ldots, r^{n_r}\} \]

i.e. words with exactly \( n_1 \) 1’s, \( n_2 \) 2’s, etc.
For a permutation of multiset (or word) \( w = w_1, \ldots, w_n \)
define the number of inversions

\[ \text{inv}(w) := \# \{ 1 \leq i < j \leq n \mid w_i > w_j \} \]

Clearly, if \( w \in S_n \) is a usual permutation of \( 1, 2, \ldots, n \), then \( \text{inv}(w) \) is the inversion number of \( w \) that we discussed before.

**Example.** \( r = 2 \)

Let us identify a Young diagram \( \lambda \) that fits inside the \( n_1 \times n_2 \) rectangle via the following permutation \( w \) of the multiset \( \{1, 2, \ldots, n_1 + n_2 - 3\} \):

\[ n_1 = 5, \quad n_2 = 7 \]

\[ \lambda = (6, 4, 4, 3, 1) \]

\[ w = 2, 1, 2, 2, 1, 1, 2, 2, 2, 1, 2 \]

Then

\[ \text{inv}(w) = |\lambda| \]

Walk along the border of Shape \( \lambda \) from the upper right corner to the lower left corner — inversions of \( w \) are in bijection with boxes of \( \lambda \).
\[ q - \text{multinomial coefficients} \]

\[
\left[ \begin{array}{c}
 n \\
 n_1, n_2, \ldots, n_r \end{array} \right]_q = \frac{[n]_q!}{[n_1]_q! \cdots [n_r]_q!}
\]

**Theorem.** \[
\left[ \begin{array}{c}
 n \\
 n_1, n_2, \ldots, n_r \end{array} \right]_q
\]

is a polynomial in \( q \) with non-negative integer coefficients.

\[
\left[ \begin{array}{c}
 n \\
 n_1, \ldots, n_r \end{array} \right]_q = \sum_{w \text{ is a perm of the multiset}} q^{\text{inv}(w)}
\]

The 2 proofs that for the \( q \)-binomial coefficients (based on recurrence relation and on finite field) can be easily extended to all \( q \)-multinomial coeffts...

It is also easy to deduce the polynomiality of the \( q \)-multinomial coeffts. From the polynomiality of \( q \)-binomial coeffts.

Indeed,

\[
\left[ \begin{array}{c}
 n \\
 n_1, n_2, \ldots, n_r \end{array} \right]_q =
\]

\[
= \left[ \begin{array}{c}
 n \\
 n_1 \end{array} \right]_q \left[ \begin{array}{c}
 n-1 \\
 n_1-1, n_2, \ldots, n_r \end{array} \right]_q \cdots \left[ \begin{array}{c}
 n_r-1 \\
 n_r-1, \ldots, n_1 \end{array} \right]_q
\]

\[
\square
\]
Example \( u, v, w = 1, 1, 2 \)

Terms of the matrix \( \sum_{i,j} 1, 2, 3, 3 \)

\[
\begin{array}{cccc}
1 & 2 & 3 & 0 \\
1 & 3 & 2 & 3 \\
1 & 3 & 3 & 3 \\
2 & 1 & 3 & 3 \\
3 & 3 & 2 & 3 \\
3 & 2 & 1 & 3 \\
3 & 2 & 3 & 3 \\
3 & 3 & 3 & 3 \\
\end{array}
\]

\[
\text{inv}(w) = 0, 1, 2, 3, 4, 5, 6, 7
\]

\[
\sum_{i,j} 1, 2, 3 \quad q = 1 + eq + 3q^2 + 9q^3 + 2q^4 + 5q^5
\]

\[
\begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5 \\
6 \\
7 \\
\end{array}
\]

Notice that the coefficients 1, 2, 3, 3, 2, 1 are symmetric & unimodal.

The symmetry is easy, the unimodality is hard.

Next week we'll discuss how to prove the unimodality.

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Question. Is there a generalization of the major index \( \text{maj}(w) \) to permutations of multisets, which is equidistributed with the inversion number \( \text{inv}(w) \)?

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Percy Alexander MacMahon
1854 - 1929
Don't forget to upload your Problem Set!