A lattice is a poset for which two operations \( \vee \) (join) and \( \wedge \) (meet) are well defined.

A poset \( P \) with two binary operations \( \vee \) and \( \wedge \) satisfying several axioms (e.g., commutativity, associativity, absorption, and absorption law)

\[ \begin{align*}
A lattice \text{ is a poset with two operations } \vee \text{ and } \wedge \\
\text{such that:} \\
1. \text{ Commutativity: } a \vee b = b \vee a \\
2. \text{ Associativity: } (a \vee b) \vee c = a \vee (b \vee c) \\
3. \text{ Absorption: } a \vee (a \wedge b) = a \text{ and } a \wedge (a \vee b) = a \\
4. \text{ Idempotency: } a \vee a = a \text{ and } a \wedge a = a
\end{align*} \]

The lattice of order ideals

Let \( P \) be any poset (not necessarily a lattice).

An order ideal in \( P \) is a subset \( I \subseteq P \) of elements of \( P \) such that

\[ \begin{align*}
x \leq y & \implies x \in I \\
x \in I & \text{ and } y \in I \implies x \vee y \in I
\end{align*} \]

Example:

\[ \begin{align*}
\text{Let } P(\{a, b, c, d, e, f\}) \text{ be a lattice.
}\end{align*} \]

Definition:

Let \( J(P) \) be the poset of all order ideals in \( P \) ordered by containment.

\[ \begin{align*}
I \subseteq J & \iff I \subseteq J \\
J(P) & \text{ is called the lattice of order ideals of } P.
\end{align*} \]

Lemma: \( J(P) \) is a lattice.

Proof: For order ideals \( I, J \),

\[ \begin{align*}
I \cup J & = \{ x \in P \mid x \leq I \text{ or } x \leq J \} \\
I \cap J & = \{ x \in P \mid x \leq I \text{ and } x \leq J \}
\end{align*} \]

It is easy to see that the set theoretic union/intersection of two order ideals is an order ideal.

Examples:

\[ \begin{align*}
P = & \{ a, b, c, d, e, f \} \\
J(P) = & \{ \emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, e\}, \{a, f\}, \{b, c\}, \{b, d\}, \{b, e\}, \{b, f\}, \{c, d\}, \{c, e\}, \{c, f\}, \{d, e\}, \{d, f\}, \{e, f\}, \{a, b, c\}, \{a, b, d\}, \{a, b, e\}, \{a, b, f\}, \{a, c, d\}, \{a, c, e\}, \{a, c, f\}, \{a, d, e\}, \{a, d, f\}, \{a, e, f\}, \{b, c, d\}, \{b, c, e\}, \{b, c, f\}, \{b, d, e\}, \{b, d, f\}, \{b, e, f\}, \{c, d, e\}, \{c, d, f\}, \{c, e, f\}, \{d, e, f\}, \{a, b, c, d\}, \{a, b, c, e\}, \{a, b, c, f\}, \{a, b, d, e\}, \{a, b, d, f\}, \{a, b, e, f\}, \{a, c, d, e\}, \{a, c, d, f\}, \{a, c, e, f\}, \{a, d, e, f\}, \{b, c, d, e\}, \{b, c, d, f\}, \{b, c, e, f\}, \{b, d, e, f\}, \{c, d, e, f\}, \{a, b, c, d, e\}, \{a, b, c, d, f\}, \{a, b, c, e, f\}, \{a, b, d, e, f\}, \{a, c, d, e, f\}, \{b, c, d, e, f\}, \{a, b, c, d, e, f\} \}
\end{align*} \]

In general, \( J(P(\{a, b, c\})) = \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c\} \cdot \{a, b, c}
Example. Let \( \mathbb{Z}_0 = \{1, 2, 3, \ldots\} \) be the poset of positive integers with the usual order \( 1 < 2 < 3 < \ldots \) (an infinite chain).

Then \( J(\mathbb{Z}_0) \cong \mathbb{Z}_0 \)

\( J(\mathbb{Z}_0 \times \mathbb{Z}_0) \cong \mathbb{Y} \)  

Young's lattice.

\( \mathbb{Z}_0 \times \mathbb{Z}_0 = \)

Identify an element \((i, j) \in \mathbb{Z}_0 \times \mathbb{Z}_0\) with a box in position \((i, j)\).

Then an order ideal in \( \mathbb{Z}_0 \times \mathbb{Z}_0 \) corresponds to a collection of boxes in a Young diagram.

Example. Let \( Q_n \) be the "empty poset" on \( n \) elements, i.e., any two elements of \( Q_n \) are incomparable. That is, \( Q_n \) consists of a single antichain.

\( Q_n = \{1, 2, 3, \ldots, n\} \)

Then \( B_n = J(Q_n) \)

The Boolean lattice.

Indeed, any subset \( I \subseteq \{1, 2, \ldots, n\} \) is an order ideal of \( Q_n \).
Distributive Lattices

Actually, \( J(B) \) is not only just a lattice, but also it belongs to an especially nice class of lattices.

**Definition.** A lattice \( L \) is called a **distributive** lattice if it satisfies the two distributive laws:

- \( x \lor (y \land z) = (x \lor y) \land (x \lor z) \)
- \( x \land (y \lor z) = (x \land y) \lor (x \land z) \)

**Remark.** For numbers (say, in \( \mathbb{R} \)) and the usual operations of addition \( + \) and multiplication \( \cdot \), we do have the distributive law:

\[
x \cdot (y + z) = (x \cdot y) + (x \cdot z),
\]

but we don't have the "second distributive law"

\[
x + (y \cdot z) \neq (x + y) \cdot (x + z)
\]

This is usually not correct.

So \( (\mathbb{R}, +, \cdot) \) is not a distributive lattice, but it shares some similarities with distributive lattices.
Lemma. \( J(P) \) is a distributive lattice.

Proof: It is easy to check that the basic distributive laws hold for the union \( \cup \) and intersection \( \cap \) of sets. Recall, that for \( I, J, \in S \)

\[
I \cup J \ni I \cap J \ni I \cup J \ni I \cap J
\]

Clearly, the following posets are distributive lattices:

- n-chain \( I \)
- the Boolean lattice \( B_n \)
- Young's lattice \( Y \).

**Fundamental Theorem on Finite Distributive Lattices (a.k.a. Birkhoff's Representation Theorem), 1934.**

For a finite poset \( P \),

\( J(P) \) is a finite distributive lattice, and any finite distributive lattice \( L \) is isomorphic to \( J(P) \) for some finite poset \( P \).

Remark. Basically, this theorem says:

\[ P \leftrightarrow J(P) \]

is a one-to-one correspondence between finite posets and finite distributive lattices.

We need to be a little careful here. In order to talk about \( 1:1 \) correspondence we need to have some sets. But there is no such thing as the "set of all finite posets".

Strictly speaking, the above statement does not make sense mathematically.

In order to rigorously formulate it, we need to talk about the "categories" of finite posets, finite distributive lattices, but I want to avoid talking about category theory in this class.
Idea of proof. For any finite distributive lattice \( L \), we need to find a finite poset \( P \) such that \( L \cong J(P) \).

An element \( z \) of \( L \) is called join-irreducible if \( z \) is not a minimal element of \( L \) and we cannot write it as \( z = x \lor y \) for some \( x, y \leq z \).

Let \( P \) be the poset of all join-irreducible elements in \( L \).

Then one can deduce, using the axioms of distributive lattices, that \( L \cong J(P) \).

Exercise. Prove this.

Example

\[
\begin{array}{c}
\begin{tikzpicture}
  \node (a) at (0,0) {a};
  \node (b) at (1,0) {b};
  \node (c) at (2,0) {c};
  \node (d) at (0,-1) {d};
  \node (e) at (1,-1) {e};
  \node (f) at (2,-1) {f};
  \draw (a) -- (b) -- (c) -- (a);
  \draw (d) -- (e) -- (f) -- (d);
  \draw (a) -- (d);
  \draw (b) -- (e);
  \draw (c) -- (f);
  \node[draw, circle, fill=white] (x) at (1.5,-2) {x};
  \node[draw, circle, fill=white] (y) at (1.5,-3) {y};
  \node[draw, circle, fill=white] (z) at (1.5,-4) {z};
  \draw (x) -- (y) -- (z) -- (x);
  \node[draw, circle, fill=white] (i) at (0,-5) {i};
  \node[draw, circle, fill=white] (j) at (1,-5) {j};
  \node[draw, circle, fill=white] (k) at (2,-5) {k};
  \node[draw, circle, fill=white] (l) at (3,-5) {l};
  \draw (i) -- (j) -- (k) -- (l) -- (i);
  \node[draw, circle, fill=white] (m) at (1.5,-6) {m};
  \node[draw, circle, fill=white] (n) at (1.5,-7) {n};
  \draw (m) -- (n);
  \end{tikzpicture}
\end{array}
\]

This is the lattice of all Young diagrams that fit inside the 2x2 square.
Young's lattice $\mathcal{Y}$ is not a finite lattice. But for any Young diagram $\lambda$, $\mathcal{Y}$ has a finite sub-lattice:

$$\mathcal{Y}_\lambda := \text{the poset of all Young diagrams } \mu \text{ that fit inside } \lambda, \text{ ordered by inclusion}$$

$$= \text{the interval } [\emptyset, \lambda]_\mathcal{Y} \text{ in Young's lattice } \mathcal{Y}.$$ 

Also denote by $P_\lambda$ the poset on the set of boxes of $\lambda$, ordered such that $x \leq y$ if the box $y$ is to the South, East, or South-East of the box $x$.

Example. $P_\lambda = \begin{array}{c c c c}
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\end{array}$

Clearly, we have

Corollary. $\mathcal{Y}_\lambda = J(P_\lambda)$ is a finite distributive lattice.

Examples:

$$\mathcal{Y}_\lambda = J(\mathcal{Y}) = \begin{array}{c c c c}
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\end{array}$$

$$\mathcal{Y}_\lambda = J(\square) = \begin{array}{c c c c}
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\end{array}$$

Replace the boxes of $\lambda$ by dots & rotate the picture by 180°.
Linear extensions of posets

For a finite poset \( P \) with \( n \) elements, a linear extension of \( P \) is a bijective map \( f : P \rightarrow \{1, 2, \ldots, n\} \) such that if \( x \leq_P y \), then \( f(x) \leq f(y) \).

Let \( \text{ext}(P) \) denote the set of linear extensions of \( P \).

Example. \( P = \begin{array}{c|c|c}
\emptyset & \emptyset & \emptyset \\
\emptyset & \emptyset & \emptyset \\
\emptyset & \emptyset & \emptyset \\
\end{array} \)

\( \text{ext}(P) = \{(1, 2, 3)\} \).

Proposition. For a finite poset \( P \) with \( n \) elements, \( \text{ext}(P) \) equals the number of saturated chains from the minimal element \( \emptyset \) to the maximal element \( \emptyset \) in \( \mathcal{J}(P) \).

In particular,

\[
\text{P}_{\emptyset} = \text{ext}(\mathcal{J}(\emptyset)) = \#	ext{ saturated chains from } \emptyset \text{ to } \emptyset = \emptyset
\]

So we can think of \( \text{SYT} \) as an alternative description of \( \text{ext}(\emptyset) \).

Example. \( P = \begin{array}{c|c|c}
\emptyset & \emptyset & \emptyset \\
\emptyset & \emptyset & \emptyset \\
\emptyset & \emptyset & \emptyset \\
\end{array} \)

\( \mathcal{J}(P) = \begin{array}{c|c|c|c|c|c|c|c|c|c}
\emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset \\
\end{array} \)
Let $L(m,n) := J(C_m \times C_n) \cong \prod_{m \times n}$ the product of chain.

The product of Young diagrams inside the $m \times n$ rectangle.

Examples,

$L(2,2) = \begin{array}{c}
\begin{array}{c}
\hline
\hline
\hline
\end{array}
\end{array}$

$L(3,2) = \begin{array}{c}
\begin{array}{c}
\hline
\hline
\hline
\end{array}
\end{array}$

More generally,

$L(n,2) = \begin{array}{c}
\begin{array}{c}
\hline
\hline
\hline
\end{array}
\end{array}$

Corollary. # saturated chains from $\delta \rightarrow \uparrow$ in $L(n,2)$

$= \# SYTs of shape $n \times 2$.

$= \text{Catalan number } \frac{1}{n+1} \binom{2n}{n}$. 

Clearly, saturated chains from $\delta \rightarrow \uparrow$ in $L(n,2)$ are literally Dyck paths.

Clearly, $L(n,2)$ has symmetric chain decomposition (SCD):

$L(n,2) = \begin{array}{c}
\begin{array}{c}
\hline
\hline
\hline
\end{array}
\end{array}$

How about an arbitrary $L(m,n)$?
The Gaussian coefficients

Clearly, Young's lattice \( L(m) \) is a graded poset with the rank function \( P(x) = N \).

Let \( \tau^n_r (L(m)) \in [0, 1, \ldots, m^n] \) denote the rank numbers of Young's lattice \( L(m) \).

Explicitly, \( \tau^n_r (L(m)) = \lambda \)

\[
\begin{align*}
\text{Young diagram } \lambda & \text{ in } \mathbb{N} \\
\text{in } [N, \ldots, 1] \\
\end{align*}
\]

The generating function
\[
\tau^n_r (L(m)) = \frac{1}{(1 - q)^{N+1}} 
\]

is known as the Gaussian generating coefficient.

Example:
\[
L(5, 2) = \begin{array}{c}
\text{Diagram}
\end{array}
\]

The rank numbers \( \tau^n_r (L(5, 2)) \) are:
1, 1, 2, 2, 3, 3, 3, 2, 2, 1.

The chromatic polynomial of \( L(5, 2) \) is:
\[
\chi_G(z) = z^1 + z^2 + 2z^3 + 3z^4 + 2z^5 + z^6 + z^7 + z^8 + z^9 + z^{10}
\]

Theorem. Young's lattice \( L(m) \) is:

- Plane symmetric, i.e., \( P(x) \) is:
- Rank invariant:
- \( P(x) = \cdots = P(x, \ldots, x) \)
- Speiser.

Properties of the plane symmetry is trivial: A \( \sim B \) if:

- \( \text{The complemented } \lambda \text{ in the main magma } \)

- \( \text{The unimodularity of the Gaussian coefficients is a surprisingly hard result.} \)

The first proof was given by Speiser in 1958. The first constructive proof was given more than 20 years later by K. R. M. \( 1980 \)

- The Speiser property of \( (m) \) was proved by R. Stanley, 1980.

- \( \text{Gauss in (m) has a symmetric chain decomposition?} \)

Unknown, for general \( m \).
The Sperner property of $L(m,n)$ can be reformulated like this:

Let $P_1, P_2, \ldots, P_n$ be any collection of lattice paths from the lower left to the upper right corners in the $m \times n$ rectangle such that any two paths $P_i$ and $P_j$ intersect.

For example,

\[
\begin{array}{|c|c|c|c|c|}
\hline
& & & & \text{4} \\
& & & 3 & \\
& & 2 & 1 & \\
\hline
\end{array}
\]

Then $N \leq \text{\# } L_{\frac{m+n}{2}}$ with

\[
\begin{align*}
\text{Young diagram} \\
x \leq m \times n \\
|x| = \frac{m \times n}{2}
\end{align*}
\]
$q$-analog

A $q$-analog of some classical combinatorial number $A$ is a polynomial $A(q) = a_0 + a_1 q + \ldots + a_n q^n$ (typically, with non-negative integer coefficients $a_i$) such that $A(1) = A$ (the "classical limit").

$q$-numbers

$[n] q := 1 + q + q^2 + \ldots + q^n = 1 - q^n$.

$q$-factorials

$[n]_q ! := [1]_q [2]_q \ldots [n]_q$.

Example:

$[3]_q ! = (1+q)(1+q+q^2) = 1 + 2q + 2q^2 + q^3$.

$q$-binomial coefficients

$\left[ \begin{array}{c} n \\ k \end{array} \right]_q = \frac{[n]_q !}{[k]_q ![n-k]_q}.

Example: $\left[ \begin{array}{c} 3 \\ 2 \end{array} \right]_q = \frac{[3]_q !}{[2]_q ![1]_q !} = \frac{(1+q)(1+q+q^2)(1+q+q^2+q^3)}{(1+q)} = (1+q^2)(1+q+q^2) = 1 + q + 2q^2 + q^3 + q^4$.

This turned out to be a polynomial with positive integer coefficients. But in general, this is not obvious. Why should the numerator be divisible by the denominator?
Theorem.
\[ \sum_{\lambda \subseteq K \times (n-k)} q^\lambda = \prod_{k \geq 1} \left( 1 + q + q^2 + \ldots + q^k \right) \]

The sum is over Young diagrams \( \lambda \) that fit inside the \( K \times (n-k) \) rectangle.

The coefficients of this expression are exactly the rank numbers \( r_k \) of the Young’s lattice \( \mathcal{L}(K, n-k) \).

Example
\[ \sum_{\lambda \subseteq 2 \times 3} q^\lambda = \]
\[
\begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

\[ 1 + q + q^2 + q^2 + q^3 + q^4 \]

Proof. It is not hard to prove it by induction on \( n \). One can easily show that both L.H.S. & R.H.S. satisfy the \( q \)-Pascal’s recurrence relation:
\[ \sum_{\lambda \subseteq K \times (n-k)} q^\lambda = \sum_{\lambda \subseteq (n-1-k) \times (n-k)} q^\lambda + q^{\lambda_{(n-1)}} \]

Exercise. Check this.

Remark. This is an easy but not very conceptual proof. We’ll give another more interesting proof.
$q$ - Pascal's triangle:

\[
\begin{array}{cccccc}
 & & & 0 & & \\
 & & 1 & & 1 & \\
 & 1 & & 2 & & 1 \\
1 & & 3 & & 3 & & 1 \\
\end{array}
\]