18.211 PROBLEM SET 3 (due Friday, October 18, 2019)

Problem 1. Use the Inclusion-Exclusion Principle to calculate the number positive integers between 1 and 1000 which are not divisible by 2, 3, 5, or 7.

Problem 2. Find an explicit expression for the sequence of numbers A_n, $n \geq 0$, given by the recurrence relation

$$A_n = A_{n-1} + 6A_{n-2}, \text{ for } n \geq 2,$$

with the initial values $A_0 = 2$, $A_1 = 1$.

Problem 3. Find an explicit expression for the sequence of numbers B_n, $n \geq 0$, given by the recurrence relation

$$B_n = n\, B_{n-1} + 3\, n!, \text{ for } n \geq 1,$$

with the initial value $B_0 = 0$.

Problem 4. Consider the sequence of numbers $C_n = 3^n + 4^n$, for $n \geq 0$.
(1) Calculate the ordinary generating function for C_n.
(2) Find a recurrence relation for these numbers of the form

$$C_n = a\, C_{n-1} + b\, C_{n-2}, \text{ for } n \geq 2,$$

where a and b are some constants.

Problem 5. Prove two recurrence relations for the number D_n of derangements in the symmetric group S_n:
(1) $D_n = (n-1)(D_{n-1} + D_{n-2})$, for $n \geq 2$.
(2) $D_n = n\, D_{n-1} + (-1)^n$, for $n \geq 1$.

Problem 6. For any n, decide which of the two numbers is bigger: the number of permutations in S_n without fixed points or the number of permutations in S_n with exactly one fixed point.

Problem 7. Calculate the exponential generating functions $\sum_{n\geq 0} D_n \frac{x^n}{n!}$ for the derangement numbers D_n.

Problem 8. Let P_n be the number of set partitions of $[n]$ such that each block contains $1, 2$ or 3 elements. Calculate the exponential generating function $\sum_{n\geq 0} P_n \frac{x^n}{n!}$.

Problem 9. You probably know that $1 + 2 + \cdots + n = \binom{n+1}{2}$ and that $1^2 + 2^2 + \cdots + n^2 = n(n+1)(2n+1)/6 = \binom{n+1}{2} + 2\binom{n+1}{3}$. Prove the general formula, for any positive integers n and k,

$$1^k + 2^k + \cdots + n^k = \sum_{r=1}^{k} r!\, S(k,r) \binom{n+1}{r+1},$$

where $S(k, r)$ is the Stirling number of the second kind.

Problem 10. A *super-rook* is a new kind of chess piece that can attack vertically and horizontally (as a normal rook) and in addition to this it can attack one step diagonally (as a king) but only in the South-West or North-East direction. Let SR_n be the number of ways to place n pairwise non-attacking super-rooks on the $n \times n$ chessboard. (For example, $\mathrm{SR}_1 = 1$, $\mathrm{SR}_2 = 1$, $\mathrm{SR}_3 = 3$.)

(1) Give an explicit expression for the number SR_n using the Inclusion-Exclusion principle. (This expression may involve a summation.)

(2) Show that $SR_n = D_n + D_{n-1}$, where D_n is the derangement number.

Problem 11. (Bonus) A *royal rook* is a chess piece that can attack as a normal rook and as a king. (It is a stronger chess piece than a super-rook.) Find an expression for the number of ways to place n pairwise non-attacking royal rooks on the $n \times n$ chessboard.