1.1 Introduction. Real numbers.

Mathematical analysis depends on the properties of the set $\mathbb{R}$ of real numbers, so we should begin by saying something about it.

There are two familiar ways to represent real numbers. Geometrically, they may be pictured as the points on a line, once the two reference points corresponding to 0 and 1 have been picked. For computation, however, we represent a real number as an infinite decimal, consisting of an integer part, followed by infinitely many decimal places:

$$3.14159\ldots, \qquad -.033333\ldots, \qquad 101.2300000\ldots .$$

There are difficulties with decimal representation which we need to think about. The first is that two different infinite decimals can represent the same real number, for according to well-known rules, a decimal having only 9's after some place represents the same real number as a different decimal ending with all 0's (we call such decimals *finite* or *terminating*):

$$26.67999\ldots = 26.68000\ldots = 26.68\ , \qquad -99.999\ldots = -100.$$

This ambiguity is a serious inconvenience in working theoretically with decimals.

Notice that when we write a finite decimal, in mathematics the infinite string of decimal place zeros is dropped, whereas in scientific work, some zeros are retained to indicate how accurately the number has been determined.

Another difficulty with infinite decimals is that it is not immediately obvious how to calculate with them. For finite decimals there is no problem; we just follow the usual rules—add or multiply starting at the right-hand end:

$$\begin{array}{r} 2.389 \\ +\ \underline{2.389} \\ \ldots 78 \end{array} \qquad\qquad \begin{array}{r} 2.849 \\ \times\ \underline{\quad .09} \\ \ldots 41 \end{array}$$

But an infinite decimal has no right-hand end...

To get around this, instead of calculating with the infinite decimal, we use its truncations to finite decimals, viewing these as approximations to the infinite decimal. For instance, the increasing sequence of finite decimals

$$(1) \qquad 3, \quad 3.1, \quad 3.14, \quad 3.141, \quad 3.1415, \quad \ldots$$

gives ever closer approximations to the infinite decimal $\pi = 3.1415926\ldots$; we say that π is the *limit* of this sequence (a definition of “limit” will come soon).

To see how this allows us to calculate with infinite decimals, suppose for instance we want to calculate

$$\pi + \sqrt[3]{2}\ .$$

We write the sequences of finite decimals which approximate these two numbers:

$$\begin{array}{lllllllll} \pi & \text{is the limit of} & 3, & 3.1, & 3.14, & 3.141, & 3.1415, & 3.14159, \ldots; \\ \sqrt[3]{2} & \text{is the limit of} & 1, & 1.2, & 1.25, & 1.259, & 1.2599, & 1.25992, \ldots; \end{array}$$

then we add together the successive decimal approximations:

$$\begin{array}{lllllllll} \pi+\sqrt[3]{2} & \text{is the limit of} & 4, & 4.3, & 4.39, & 4.400, & 4.4014, & 4.40151, \ldots, \end{array}$$

obtaining a sequence of numbers which also increases.

The decimal representation of this increase isn't as simple as it was for the sequence representing π, since as each new decimal digit is added on, the earlier ones may change. For instance, in the fourth step of the last row, the first decimal place changes from 3 to 4. Nonetheless, as we compute to more and more places, the earlier part of the decimals in this sequence ultimately doesn't change any more, and in this way we get the decimal expansion of a new number; we then define the sum $\pi + \sqrt[3]{2}$ to be this number, $4.4015137\ldots$.

We can define multiplication the same way. To get $\pi \times \sqrt[3]{2}$, for example, multiply the two sequences above for these numbers, getting the sequence

$$3, \quad 3.72, \quad 3.9250, \quad 3.954519, \quad \ldots\ . \tag{2}$$

Here too as we use more decimal places in the computation, the earlier part of the numbers in the sequence (2) ultimately stops changing, and we define the number $\pi \times \sqrt[3]{2}$ to be the limit of the sequence (2).

As the above shows, even the simplest operations with real numbers require an understanding of sequences and their limits. These appear in analysis whenever you get an answer not at once, but rather by making closer and closer approximations to it. Since they give a quick insight into some of the most important ideas in analysis, they will be our starting point, beginning with the sequences whose terms keep increasing (as in (1) and (2) above), or keep decreasing. In some ways these are simpler than other types of sequences.

Appendix A.0 contains a brief review of set notation, and also describes the most essential things about the different number systems we will be using: the integers, rational numbers, and real numbers, as well as their relation to each other. Look through it now just to make sure you know these things.

Questions 1.1

(Answers to the Questions for each section of this book can be found at the end of the corresponding chapter.)

1. In the sequence above for $\pi + \sqrt[3]{2}$, the first decimal place of the final answer is not correct until four steps have been performed. Give an example of addition where the first decimal place of the final answer is not correct until k steps have been performed. (Here k is a given positive integer.)

1.4 Example: the number e

We saw in Section 1.1 how the notion of limit lets us define addition and multiplication of positive real numbers. But it also gives us an important and powerful method for constructing particular real numbers. This section and the next give examples. They require some serious analytic thinking and give us our first proofs.

The aim in each proof is to present an uncluttered, clear, and convincing argument based upon what most readers already know or should be willing to accept as clearly true. The first proof for example refers explicitly to the binomial theorem

$$(8)\qquad (1+x)^k = 1+kx+\ldots+\binom{k}{i}x^i+\ldots+x^n, \qquad \binom{k}{i} = \frac{k(k-1)\cdots(k-i+1)}{i!}\ ,$$

which you should know. But it also uses without comment the result

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^n}\ <\ 2,$$

which can be seen by picturing the successive points $1,\ 1\frac{1}{2},\ 1\frac{3}{4},\ldots$ on the line. It also follows from the formula for the geometric sum (taking $r = 1/2$):

$$1+r+r^2+\ldots+r^n = \frac{1-r^{n+1}}{1-r}\ .$$

If you didn't think of the picture and didn't remember or think of using the formula, you will feel a step has been skipped. One person's meat is another person's gristle; just keep chewing and it will ultimately go down.

As motivation for this first example, we recall the compound interest formula: invest P dollars at the annual interest rate r, with the interest compounded at equal time intervals n times a year; by the end of the year it grows to the amount

$$A_n = P\Big(1+\frac{r}{n}\Big)^n\ .$$

Thus if we invest one dollar at the rate $r = 1$ (i.e., 100% annual interest), and we keep recalculating the amount at the end of the year, each time doubling the frequency of compounding, we get a sequence beginning with

$$\begin{array}{llll} A_1 = 1+1 & = & 2 & \textit{simple interest;} \\ A_2 = (1+1/2)^2 & = & 2.25 & \textit{compounded semiannually;} \\ A_4 = (1+1/4)^4 & \approx & 2.44 & \textit{compounded quarterly.} \end{array}$$

Folk wisdom suggests that successive doubling of the frequency should steadily increase the amount at year's end, but within bounds, since banks do manage to stay in business even when offering daily compounding. This should make the following proposition plausible. (The limit is e.)

Proposition 1.4 *The sequence* $\quad a_n = \Big(1+\dfrac{1}{2^n}\Big)^{2^n}\quad$ *has a limit.*

Proof. By Theorem 1.3, it suffices to prove $\{a_n\}$ is increasing and bounded above.

To show it is increasing, if $b \neq 0$ we have $b^2 > 0$, and therefore,

$$(1+b)^2\ >\ 1+2b\ ;$$

raising both sides to the 2^n power, we get

$$(1+b)^{2\cdot 2^n}\ >\ (1+2b)^{2^n}\ .$$

If we now put $b = 1/2^{n+1}$, this last inequality becomes $\ a_{n+1} > a_n\ .$ $\square$

To show that a_n is bounded above, we will prove a stronger statement ("stronger" because it implies that a_n is bounded above: cf. Appendix A):

$$(9) \qquad \left(1+\frac{1}{k}\right)^k \le 3 \qquad \text{for any integer } k \ge 1 .$$

To see this, we have by the binomial theorem (8),

$$(10) \quad \left(1+\frac{1}{k}\right)^k = 1 + k\left(\frac{1}{k}\right) + \ldots + \frac{k(k-1)\cdots(k-i+1)}{i!}\left(\frac{1}{k}\right)^i + \ldots + \frac{k!}{k!}\left(\frac{1}{k}\right)^k .$$

To estimate the terms in the sum on the right, we note that

$$k(k-1)\cdots(k-i+1) \le k^i , \qquad i = 1, \ldots, k ,$$

since there are i factors on the left, each at most k; and by similar reasoning,

$$(11) \qquad \frac{1}{i!} = \frac{1}{i}\cdot\frac{1}{i-1}\cdot \;\cdots\; \cdot\frac{1}{2} \le \left(\frac{1}{2}\right)^{i-1}, \quad i = 2, \ldots, k .$$

Therefore, for $i = 2, \ldots, k$ (and $i = 1$ also, as you can check),

$$(12) \qquad \frac{k(k-1)\cdots(k-i+1)}{i!}\cdot\left(\frac{1}{k}\right)^i \le \frac{1}{2^{i-1}} .$$

Using (12) to estimate the terms on the right in (10), we get, for $k \ge 2$,

$$(13) \qquad \begin{aligned} \left(1+\frac{1}{k}\right)^k &\le 1 + 1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^{k-1}} ; \\ &\le 1 + 2 ; \end{aligned}$$

and this is true for $k = 1$ as well. □□

Remarks.

1. Euler was the first to encounter the number $\lim a_n$; he named it e because of its significance for the exponential function (or maybe after himself).

2. In the proof that a_n is increasing, the b could have been dispensed with, and replaced from the start with $1/2^{n+1}$. But this makes the proof harder to read, and obscures the simple algebra. Also, for greater clarity the proof is presented (as are many proofs) backwards from the natural procedure by which it would have been discovered; cf. Question 1.4/1.

3. In the proof that a_n is bounded by 3, it is easy enough to guess from the form of a_n that one should try the binomial theorem. Subsequent success then depends on a good estimation like (12), which shows the terms of the sum (10) are small. In general, this estimating lies at the very heart of analysis; it's an art which you learn by studying examples and working problems.

4. Notice how the three inequalities after line (10) as well as the two in line (13) are lined up one under the other. This makes the proof much easier to read and understand. When you write up your arguments, do the same thing: use separate lines and line up the $=$ and $\le$ symbols, so the proof can be read as successive transformations of the two sides of the equation or inequality.

Questions 1.4

1. Write down the proof that the sequence a_n is increasing as you think you would have discovered it. (In the Answers is one possibility, with a discussion of the problems of writing it up. Read it.)

2. Define $b_n = 1 + 1/1! + 1/2! + 1/3! + \ldots + 1/n!$; prove $\{b_n\}$ has a limit (it is e). (Hint: study the second half of the proof of Prop. 1.4.)

3. In the proof that $(1+1/k)^k$ is bounded above, the upper estimate 3 could be improved (i.e., lowered) by using more accurate estimates for the beginning terms of the sum on the right side of (10). If one only uses the estimate (11) when $i \geq 5$, what new upper bound does this give for $(1 + 1/k)^k$?

Exercises

(The exercises go with the indicated section of the chapter.)

1.2

1. For each of the a_n below, tell if the sequence $\{a_n\}$, $n \geq 1$, is increasing (strictly?), decreasing (strictly?), or neither; show reasoning.

(If simple inspection fails, try considering the difference $a_{n+1} - a_n$, or the ratio a_{n+1}/a_n, or relate the sequence to the values of a function $f(x)$ known to be increasing or decreasing.)

(a) $1 - \frac{1}{2} + \frac{1}{3} - \ldots + (-1)^{n-1}\frac{1}{n}$ (b) $n/(n+1)$

(c) $\sum_1^n \sin^2 k$ (d) $\sum_1^n \sin k$

(e) $\tan(1/n)$ (f) $\sqrt{1 + 1/n^2}$

1.4

1. Consider the sequence $\{a_n\}$, where

$$a_n = 1 + \frac{1}{1\cdot 3} + \frac{1}{1\cdot 3\cdot 5} + \frac{1}{1\cdot 3\cdot 5\cdot 7} + \ldots + \frac{1}{1\cdot 3\cdot \ldots \cdot (2n-1)} .$$

Decide whether $\{a_n\}$ is bounded above or not, and prove your answer is correct. (Hint: cf. Question 1.4/2 .)

2. Prove the sequence $a_n = n^n/n!$, $n \geq 1$, is

(a) increasing; (b) not bounded above (show $a_n \geq n$).

2.3 Proving boundedness.

Rather than an upper and lower estimate, often you want an estimate just in one direction. This was the case with our work with sequences in Sections 1.4 and 1.5. Using the language of estimations, we summarize the principles used there; for simplicity, we assume the sequences are non-negative: $a_n \geq 0$.

To show $\{a_n\}$ is bounded above, get one upper estimate: $a_n \leq B$, for all n;

to show $\{a_n\}$ is not bounded above, get a lower estimate for each term: $a_n \geq B_n$, such that B_n tends to ∞ as $n \to \infty$.

The method for showing unboundedness is rather restricted, but it's good enough most of the time. We will give a precise definition of "tend to ∞" in the next chapter; for now we will use it intuitively. The examples we gave in Sections 1.4 and 1.5 illustrate these principles.

Example 1.4 (9) $\quad b_k = \left(1 + \dfrac{1}{k}\right)^k$.

We showed $\{b_k\}$ bounded above by the upper estimate: $b_k < 3$ for all k.

Example 1.5A $\quad a_n = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \ldots + \dfrac{1}{n}$.

Here we showed $\{a_n\}$ was not bounded above, by proving the lower estimate $a_n > \ln(n+1)$, and relying on the known fact that $\ln n$ tends to ∞ as $n \to \infty$.

These examples show that to estimate, you may have to first decide what kind of estimate you are looking for: upper or lower. If you aren't told this in advance, deciding may not be so easy. For example, by using the rectangles in Example 1.5A a bit differently, we could easily have obtained the *upper* estimate

$$a_n < 1 + \ln(n+1)$$

for the terms of the sequence. But this would have been useless for showing the sequence was unbounded, since the estimate goes in the wrong direction. Nor does it show the sequence is bounded, since the estimate $1 + \ln(n+1)$ depends on n and tends to ∞, instead of being the same for all the terms.

As another example of the difficulty in deciding which type of estimate to look for, consider the sequence formed like the one above in Example 1.5, but using only the prime numbers in the denominators (here p_n denotes the n-th prime):

$$a_n = \ 1/2 + 1/3 + 1/5 + 1/7 + 1/11 + \ldots + 1/p_n \ .$$

Should we look for an upper estimate that will show it is bounded, or a lower estimate that clearly tends to infinity? Anyone who can answer this without having studied number theory is a mathematical star of zeroth magnitude.

Questions 2.3

1. Let $a_n = 1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{n^2}$; is it bounded or unbounded; i.e., should one look for an upper or a lower estimate for the a_n? (cf. Example 1.5A)

2.6 The terminology "for n large".

In estimating or approximating the terms of a sequence $\{a_n\}$, sometimes the estimate is not valid for all terms of the sequence; for example, it might fail for the first few terms, but be valid for the later terms. In such a case, one has to specify the values of n for which the estimate holds.

Example 2.6A Let $a_n = \dfrac{5n}{n^2-2}$, $n \geq 2$; for what n is $|a_n| < 1$?

Solution. For $n = 1$, the estimate is not valid; if $n > 1$, then a_n is positive, so we can drop the absolute value. Then we have

$$\frac{5n}{n^2-2} < 1 \quad \Leftrightarrow \quad 5n < n^2 - 2 \quad \Leftrightarrow \quad 5 < n - \frac{2}{n} ,$$

and by inspection, one sees this last inequality holds for all $n \geq 6$. □

Example 2.6B In the sequence $a_n = \dfrac{n^2+2n}{n^2-2}$, for what n is $a_n \underset{.1}{\approx} 1$?

Solution. $$\left|\frac{n^2+2n}{n^2-2} - 1\right| = \frac{2n+2}{n^2-2} ,$$

$$\text{which is } < \frac{1}{10} \quad \Leftrightarrow \quad n(n-20) > 22 ,$$

and by inspection, this last inequality holds for all $n \geq 22$. □

The above is a good illustration of Warning 2.4. To show a_n and 1 are close, we get a small upper estimate of the difference by first transforming it algebraically; trying instead to use the triangle inequality, in either the sum or difference form, would produce nothing useful.

As the above examples illustrate, sometimes a property of a sequence a_n is not true for the first few terms, but only starts to hold after a certain place in the sequence. In this case, a special terminology is used.

Definition 2.6 The sequence $\{a_n\}$ has property $\mathcal{P}$ **for n large** if

(10) there is a number N such that a_n has property $\mathcal{P}$ for all $n \geq N$.

One can say instead *for large n, for n sufficiently large,* etc. Most of the time we will use the symbolic notation *for* $n \gg 1$, which can be read in any of the above ways.

Note that in the definition, N need not be an integer.

To illustrate the definition, Examples 2.6A and 2.6B show, respectively:

$$\text{if } a_n = \frac{5n}{n^2-2}, \quad \text{then } |a_n| < 1 \quad \text{for } n \gg 1 ;$$

$$\text{if } a_n = \frac{n^2+2n}{n^2-2}, \quad \text{then } |a_n| \underset{.1}{\approx} 1 \quad \text{for } n \gg 1 .$$

In both examples, we gave the smallest integer value of N (it was 6 and 22, respectively), such that the stated property of a_n held for all $n \geq N$. In general, this is overkill: to show something is true for n large, one only has to give *some* number N which works, not the "best possible" N. And as we said, it need not be an integer.

The symbols $a \gg b$, with $a, b > 0$, have the meaning "a is relatively large compared with b", that is, a/b is large. Thus we do not write "for $n \gg 0$"; intuitively, every positive integer is relatively large compared with 0.

Example 2.6C Show the sequence $\{\sin 10/n\}$ is decreasing for large n.

Proof. The function $\sin x$ is increasing on the interval $0 < x < \pi/2$, i.e.,

$$a < b \;\Rightarrow\; \sin a < \sin b, \qquad \text{for } 0 < a < b < \pi/2 \;.$$

Thus

$$\sin \frac{10}{n+1} \;<\; \sin \frac{10}{n}\;, \quad \text{if} \;\; \frac{10}{n} \;<\; \frac{\pi}{2}\;, \;\; \text{i.e., if} \;\; n > \frac{20}{\pi}\;. \qquad \square$$

Remarks. This completes the argument, since it shows we can use $N = 20/\pi$. If you prefer an integer value, take $N = 7$, the first integer after $20/\pi$.

In this example, it was no trouble to find the exact integer value $N = 7$ at which the sequence starts to be decreasing. However this is in general not necessary. Any value for N greater than 7 would do just as well in showing the sequence is decreasing for large n.

Thus, for example, if it turned out for some sequence a_n that

$$a_{n+1} < \; a_n \qquad \text{if } n^2 + n > 100 \;,$$

to show the sequence is decreasing for large n, it is a waste of time to solve the quadratic equation $n^2 + n = 100$; by inspection one sees that if $n \geq 10$, then $n^2 + n > 100$, so that one can take $N = 10$.

3.1 Definition of limit.

In Chapter 1 we discussed the limit of sequences that were monotone; this restriction allowed some short-cuts and gave a quick introduction to the concept. But many important sequences are not monotone—numerical methods, for instance, often lead to sequences which approach the desired answer alternately from above and below. For such sequences, the methods we used in Chapter 1 won't work. For instance, the sequence

$$1.1, \;.9, \;1.01, \;.99, \;1.001, \;.999, \;\ldots$$

has 1 as its limit, yet neither the integer part nor any of the decimal places of the numbers in the sequence eventually becomes constant. We need a more generally applicable definition of the limit.

We abandon therefore the decimal expansions, and replace them by the approximation viewpoint, in which "the limit of $\{a_n\}$ is L" means roughly

a_n is a good approximation to L , when n is large.

The following definition makes this precise. After the definition, most of the rest of the chapter will consist of examples in which the limit of a sequence is calculated directly from this definition. There are "limit theorems" which help in determining a limit; we will present some in Chapter 5. Even if you know them, don't use them yet, since the purpose here is to get familiar with the definition.

Definition 3.1 The number L is the **limit** of the sequence $\{a_n\}$ if

(1) $$\text{given } \epsilon > 0, \qquad a_n \underset{\epsilon}{\approx} L \quad \text{for } n \gg 1.$$

If such an L exists, we say $\{a_n\}$ **converges**, or *is convergent*; if not, $\{a_n\}$ **diverges**, or *is divergent*. The two notations for the limit of a sequence are:

$$\lim_{n\to\infty} \{a_n\} = L\ ; \qquad\qquad a_n \to L \text{ as } n \to \infty\ .$$

These are often abbreviated to: $\lim a_n = L$ or $a_n \to L$.

Statement (1) looks short, but it is actually fairly complicated, and a few remarks about it may be helpful. We repeat the definition, then build it in three stages, listed in order of increasing complexity; and with each, an English version.

Definition 3.1 $\lim a_n = L$ if: given $\epsilon > 0$, $a_n \underset{\epsilon}{\approx} L$ for $n \gg 1$.

Building this up in three succesive stages:

(i) $a_n \underset{\epsilon}{\approx} L$ (a_n approximates L to within ϵ);

(ii) $a_n \underset{\epsilon}{\approx} L$ for $n \gg 1$ (the approximation holds for all a_n far enough out in the sequence);

(iii) given $\epsilon > 0$, $a_n \underset{\epsilon}{\approx} L$ for $n \gg 1$

(the approximation can be made as close as desired, provided we go far enough out in the sequence—the smaller ϵ is, the farther out we must go, in general).

The heart of the limit definition is the approximation (i); the rest consists of the if's, and's, and but's. First we give an example.

Example 3.1A Show $\displaystyle\lim_{n\to\infty} \frac{n-1}{n+1} = 1$, directly from definition 3.1.

Solution. According to definition 3.1, we must show:

(2) $$\text{given } \epsilon > 0, \qquad \frac{n-1}{n+1} \underset{\epsilon}{\approx} 1 \quad \text{for } n \gg 1\ .$$

We begin by examining the size of the difference, and simplifying it:

$$\left|\frac{n-1}{n+1} - 1\right| = \left|\frac{-2}{n+1}\right| = \frac{2}{n+1}\ .$$

We want to show this difference is small if $n \gg 1$. Use the inequality laws:

$$\frac{2}{n+1} < \epsilon \quad \text{if} \quad n+1 > \frac{2}{\epsilon}\ , \quad \text{i.e., if} \quad n > N, \text{ where } N = \frac{2}{\epsilon} - 1\ ;$$

this proves (2), in view of the definition (2.6) of "for $n \gg 1$". □

Writing it on one line (ungrammatical, but easier to write and read this way):

Solution. Given $\epsilon > 0$, $\displaystyle\left|\frac{n-1}{n+1} - 1\right| = \frac{2}{n+1} < \epsilon$, if $\displaystyle n > \frac{2}{\epsilon} - 1$. □

Remarks on limit proofs.

1. The heart of a limit proof is in the approximation statement, i.e., in getting a small upper estimate for $|a_n - L|$. Often most of the work will consist in showing how to rewrite this difference so that a good upper estimate can be made. (The triangle inequality may or may not be helpful here.)

Note that in doing this, you *must* use $|\quad|$; you can drop the absolute value signs only if it is clear that the quantity you are estimating is non-negative.

2. In giving the proof, you must exhibit a value for the N which is lurking in the phrase "for $n \gg 1$". You need not give the smallest possible N; in example 3.1A, it was $2/\epsilon - 1$, but any bigger number would do, for example $N = 2/\epsilon$.

Note that N depends on ϵ: in general, the smaller ϵ is, the bigger N is, i.e., the further out you must go for the approximation to be valid within ϵ .

3. In Definition 3.1 of limit, the phrase "given $\epsilon > 0$" has at least five equivalent forms; by convention, all have the same meaning, and any of them can be used. They are:

for all $\epsilon > 0$,   for every $\epsilon > 0$,   for any $\epsilon > 0$;
given $\epsilon > 0$,   given any $\epsilon > 0$.

The most standard of these phrases is "for all $\epsilon > 0$", but we feel that if you are meeting (1) for the first time, the phrases in the second line more nearly capture the psychological meaning. Think of a **limit demon** whose only purpose in life is to make it hard for you to show that limits exist; it always picks unpleasantly small values for ϵ. Your task is, given any ϵ the limit demon hands you, to find a corresponding N (depending on ϵ) such that $a_n \underset{\epsilon}{\approx} L$ for $n > N$.

Remember: the limit demon supplies the ϵ; you cannot choose it yourself.

4. In writing up the proof, good mathematical grammar requires that you write "given $\epsilon > 0$" (or one of its equivalents) *at the beginning*; get in the habit now of doing it. We will discuss this later in more detail; briefly, the reason is that the N depends on ϵ, which means ϵ must be named first.

Example 3.1B Show $\lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n}) = 0$. (This one's tricky.)

Solution. We use the identity $A - B = \dfrac{A^2 - B^2}{A + B}$, which tells us that

$$\left|(\sqrt{n+1} - \sqrt{n})\right| = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{2\sqrt{n}} \; ; \tag{3}$$

given $\epsilon > 0$, $\quad \dfrac{1}{2\sqrt{n}} < \epsilon \quad$ if $\quad \dfrac{1}{4n} < \epsilon^2, \quad$ i.e., if $\quad n > \dfrac{1}{4\epsilon^2}$. $\square$

Note that here we need not use absolute values since all the quantities are positive.

It is not at all clear how to estimate the size of $\sqrt{n+1} - \sqrt{n}$; the triangle inequality is useless. Line (3) is thus the key step in the argument: the expression must first be transformed by using the identity. Even after doing this, line (3) gives a further simplifying inequality to make finding an N easier; just try getting an N without this step! The simplification means we don't get the smallest possible N; who cares?

Questions 3.1

1. Directly from the definition of limit (i.e., without using theorems about limits you learned in calculus), prove that

(a) $\dfrac{n}{n+1} \to 1$ (b) $\dfrac{\cos na}{n} \to 0$ (a is a fixed number)

(c) $\dfrac{n^2+1}{n^2-1} \to 1$ (d) $\dfrac{n^2}{n^3+1} \to 0$ $\left(\begin{array}{l}\text{cf. Example 3.1B: make}\\ \text{a simplifying inequality}\end{array}\right)$

2. Prove that, for any sequence $\{a_n\}$, $\lim a_n = 0 \quad \Leftrightarrow \quad \lim |a_n| = 0$.
(This is a simple but important fact you can use from now on.)

3. Why does the definition of limit say $\epsilon > 0$, rather than $\epsilon \geq 0$?

3.2 The uniqueness of limits. The K-ϵ principle.

Can a sequence have more than one limit? Common sense says no: if there were two different limits L and L', the a_n could not be arbitrarily close to both, since L and L' themselves are at a fixed distance from each other. This is the idea behind the proof of our first theorem about limits. The theorem shows that if $\{a_n\}$ is convergent, the notation $\lim a_n$ makes sense; there's no ambiguity about the value of the limit. The proof is a good exercise in using the definition of limit in a theoretical argument. Try proving it yourself first.

Theorem 3.2A Uniqueness theorem for limits.

A sequence a_n has at most one limit: $a_n \to L$ *and* $a_n \to L' \;\Rightarrow\; L = L'$.

Proof. By hypothesis, given $\epsilon > 0$,

$$a_n \underset{\epsilon}{\approx} L \text{ for } n \gg 1, \quad \text{and} \quad a_n \underset{\epsilon}{\approx} L' \text{ for } n \gg 1.$$

Therefore, given $\epsilon > 0$, we can choose some large number k such that

$$L \underset{\epsilon}{\approx} a_k \underset{\epsilon}{\approx} L' .$$

By the transitive law of approximation (2.5 (8)), it follows that

(4) $$\text{given } \epsilon > 0, \quad L \underset{2\epsilon}{\approx} L' .$$

To conclude that $L = L'$, we reason indirectly (cf. Appendix A.2).

Suppose $L \neq L'$; choose $\epsilon = \frac{1}{2}|L - L'|$. We then have

$$|L - L'| < 2\epsilon, \qquad \text{by (4); i.e.,}$$
$$|L - L'| < |L - L'|, \quad \text{a contradiction.} \qquad \square$$

Remarks.

1. The line (4) says that the two numbers L and L' are arbitrarily close. The rest of the argument says that this is nonsense if $L \neq L'$, since they cannot be closer than $|L - L'|$.

2. Before, we emphasized that the limit demon chooses the ϵ; you cannot choose it yourself. Yet in the proof we chose $\epsilon = \frac{1}{2}|L - L'|$. Are we blowing hot and cold?

The difference is this. Earlier, we were trying to prove a limit existed, i.e., were trying to prove a statement of the form:

$$\text{given } \epsilon > 0, \quad \text{some statement involving } \epsilon \text{ is true.}$$

To do this, you must be able to prove the truth no matter what ϵ you are given.

Here on the other hand, we don't have to prove (4)—we already deduced it from the hypothesis. It's a true statement. That means we're allowed to *use* it, and since it says something is true for every $\epsilon > 0$, we can choose a particular value of ϵ and make use of its truth for that particular value.

To reinforce these ideas and give more practice, here is a second theorem which makes use of the same principle, also in an indirect proof. The theorem is "obvious" using the definition of limit we started with in Chapter 1, but we are committed now and for the rest of the book to using the newer Definition 3.1 of limit, and therefore the theorem requires proof.

Theorem 3.2B $\{a_n\}$ *increasing*, $L = \lim a_n \;\Rightarrow\; a_n \le L$ *for all* n;
$\{a_n\}$ *decreasing*, $L = \lim a_n \;\Rightarrow\; a_n \ge L$ *for all* n.

Proof. Both cases are handled similarly; we do the first.

Reasoning indirectly, suppose there were a term a_N of the sequence such that $a_N > L$. Choose $\epsilon = \frac{1}{2}(a_N - L)$. Then since $\{a_n\}$ is increasing,

$$a_n - L \;\ge\; a_N - L \;>\; \epsilon, \quad \text{for all } n \ge N,$$

contradicting the Definition 3.1 of $L = \lim a_n$. □

The K-ϵ principle.

In the proof of Theorem 3.2A, note the appearance of 2ϵ in line (4). It often happens in analysis that arguments turn out to involve not just ϵ but a constant multiple of it. This may occur for instance when the limit involves a sum or several arithmetic processes. Here is a typical example.

Example 3.2 Let $a_n \;=\; \dfrac{1}{n} + \dfrac{\sin n}{n+1}$. Show $a_n \to 0$, from the definition.

Solution To show a_n is small in size, use the triangle inequality:

$$\left|\frac{1}{n} + \frac{\sin n}{n+1}\right| \;\le\; \left|\frac{1}{n}\right| \;+\; \left|\frac{\sin n}{n+1}\right| \;.$$

At this point, the natural thing to do is to make the separate estimations

$$\left|\frac{1}{n}\right| \;<\; \epsilon, \quad \text{for } n > \frac{1}{\epsilon}\;; \qquad\qquad \left|\frac{\sin n}{n+1}\right| \;<\; \epsilon, \quad \text{for } n > \frac{1}{\epsilon} - 1\;;$$

so that, given $\epsilon > 0$,

$$\left|\frac{1}{n} + \frac{\sin n}{n+1}\right| \;<\; 2\epsilon\;, \quad \text{for } n > \frac{1}{\epsilon}\;.$$

This is close, but we were supposed to show $|a_n| < \epsilon$. Is 2ϵ just as good?

The usual way of handling this would be to start with the given ϵ, then put $\epsilon' = \epsilon/2$, and give the same proof, but working always with ϵ' instead of ϵ. At the end, the proof shows

$$\left|\frac{1}{n} + \frac{\sin n}{n+1}\right| \;<\; 2\epsilon', \quad \text{for } n > \frac{1}{\epsilon'} \;;$$

and since $2\epsilon' = \epsilon$, the limit definition is satisfied.

Instead of doing this, let's once and for all agree that if you come out in the end with 2ϵ, or 22ϵ, that's just as good as coming out with ϵ. If ϵ is an arbitrary small number, so is 22ϵ. Therefore, if you can prove something is less than 22ϵ, you have shown that it can be made as small as desired.

We formulate this as a general principle, the "K-ϵ principle". This isn't a standard term in analysis, so don't use it when you go to your next mathematics congress, but it is useful to name an idea that will recur often.

Principle 3.2 The K-ϵ principle.

Suppose that $\{a_n\}$ is a given sequence, and you can prove that

$$\text{(5)} \qquad \textit{given any } \epsilon > 0, \quad a_n \underset{K\epsilon}{\approx} L \quad \textit{for } n \gg 1 \;,$$

where $K > 0$ is a fixed constant, i.e., a number not depending on n or ϵ.

Then $\lim\limits_{n\to\infty} a_n = L$.

The K-ϵ principle is here formulated for sequences, but we will use it for a variety of other limits as well. In all of these uses, the essential point is that K must truly be a constant, and not depend on any of the variables or parameters.

Questions 3.2

1. In the last (indirect) part of the proof of the Uniqueness Theorem, where did we use the hypothesis $L \neq L'$?

2. Show from the definition of limit that if $a_n \to L$, then $ca_n \to cL$, where c is a fixed non-zero constant. Do it both with and without the K-ϵ principle.

3. Show from the definition of limit that $\lim\left(\dfrac{1}{n+1} - \dfrac{2}{n-1}\right) = 0$.

3.5 Writing limit proofs.

Get in the habit of writing your limit proofs using correct mathematical grammar. The proofs in the body of the text and some of the answers to questions (those answers which aren't just brief indications) are meant to serve as models. But it may also help to point out some common errors.

One frequently sees the following usages involving "for n large" on student papers. Your teacher may know what you mean, but the mathematical grammar is wrong, and technically, they make no sense; avoid them.

Wrong	*Right*
$a_n \to \ 0 \ \text{ for } n \gg 1;$	$a_n \to \ 0 \ \text{ as } n \to \infty;$
$\lim 2^n \ = \ \infty \ \text{ for } n \gg 1;$	$\lim 2^n = \infty;$
$\lim(1/n) \ = \ 0 \ \text{ if } n > 1/\epsilon;$	$\begin{cases} \lim(1/n) \ = \ 0; \\ (1/n) \underset{\epsilon}{\approx} 0 \quad \text{if } n > 1/\epsilon. \end{cases}$

In the first two, the limit statement applies to the sequence as a whole, whereas "for $n >$ some N" can only apply to individual terms of the sequence. The third is just a general mess; two alternatives are offered, depending on what was originally meant.

As we said earlier, "given $\epsilon > 0$" or "given $M > 0$" must come first:

Poor: $1/n < \epsilon$, for $n \gg 1$ (what is ϵ, and who picked it?)

Wrong: For $n \gg 1$, given $\epsilon > 0$, $1/n \ < \ \epsilon$.

This latter statement is wrong, because according to mathematical conventions, it would mean that the N concealed in "for $n \gg 1$" should not depend on ϵ. This point is more fully explained in Appendix B; rather than try to study it there at this point, you will be better off for now just remembering to first present ϵ or M, and then write the rest of the statement.

Another point: write up your arguments using plenty of space on your paper (sorry, clarity is worth a tree). Often in the book's examples and proofs, the inequality and equality signs are lined up below each other, rather than strung out on one line; it is like properly-written computer code. See how it makes the argument clearer, and imitate it in your own work. If equalities and inequalities both occur, the convention we will follow is:

$$\begin{aligned} A \ &< \ n(n+1) \\ &= \ n^2 + n \ , \end{aligned} \qquad \text{rather than} \qquad \begin{aligned} A \ &< \ n(n+1) \\ &< \ n^2 + n \ . \end{aligned}$$

The form on the right doesn't tell you explicitly where the second line came from; in the form on the left, the desired conclusion $A < n^2 + n$ isn't explicitly stated, but it is easily inferred.

3.6 Some limits involving integrals.

To broaden the range of applications and get you thinking in some new directions, we look at a different type of limit which involves definite integrals.

Example 3.6A Let $a_n = \displaystyle\int_0^1 (x^2+2)^n dx$. Show that $\displaystyle\lim_{n\to\infty} a_n = \infty$.

The way *not* to do this is to try to evaluate the integral, which would just produce an unwieldy expression in n that would be hard to interpret and estimate. To show that the integral tends to infinity, all we have to do is get a lower estimate for it that tends to infinity.

Solution. We estimate the integral by estimating the integrand.

$$x^2+2 \ \geq\ 2 \qquad \text{for all } x;$$

therefore,
$$(x^2+2)^n \ \geq\ 2^n \qquad \text{for all } x \text{ and all } n \geq 0.$$

Thus
$$\int_0^1 (x^2+2)^n dx \ \geq\ \int_0^1 2^n dx \ =\ 2^n \ .$$

Since $\lim 2^n = \infty$ by Theorem 3.4, the definite integral must tend to ∞ also:

$$\text{given } M>0, \quad \int_0^1 (x^2+2)^n dx \ \geq\ 2^n \ \geq\ M, \ \text{ for } n > \log_2 M. \qquad \square$$

Example 3.6B Show $\displaystyle\lim_{n\to\infty} \int_0^1 (x^2+1)^n dx \ =\ \infty.$

Solution. Once again, we need a lower estimate for the integral that is large. The previous argument gives the estimate $(x^2+1)^n \geq 1^n = 1$, which is useless. However, it may be modified as follows.

Since x^2+1 is an increasing function which has the value $A = 5/4$ at the point $x = .5$ (any other point on $(0,1)$ would do just as well), we can say

$$x^2+1 \ \geq\ A \ >\ 1 \qquad \text{for } .5 \leq x \leq 1;$$

therefore,
$$(x^2+1)^n \ \geq\ A^n \qquad \text{for } .5 \leq x \leq 1;$$

since $\lim A^n = \infty$ by Theorem 3.4, the definite integral must tend to ∞ also:

$$\text{given } M>0, \quad \int_0^1 (x^2+1)^n dx \ \geq\ \int_{.5}^1 A^n dx \ =\ \frac{A^n}{2} \ \geq\ M, \ \text{ for } n \text{ large.} \qquad \square$$

Questions 3.6

1. By estimating the integrand, show that: $\displaystyle\frac{1}{3} \leq \int_0^1 \frac{x^2+1}{x^4+2} dx \leq 1$.

2. Show without integrating that $\displaystyle\lim_{n\to\infty} \int_0^1 x^n(1-x)^n dx = 0$.

Exercises

3.6 1. Modeling your arguments on the two examples given in this section, prove the following without attempting to evaluate the integrals explicitly.

$$(a)\ \lim_{n\to\infty}\int_1^2 \ln^n x\,dx = 0 \qquad (b)\ \lim_{n\to\infty}\int_2^3 \ln^n x\,dx = \infty\ .$$

3.7 Show $\displaystyle\lim_{n\to\infty}\int_0^1 (1-x^2)^n = 0$ by estimation, as in Example 3.7.

Problems

3-1 Let $\{a_n\}$ be a sequence and $\{b_n\}$ be its sequence of averages:

$$b_n = (a_1 + \ldots + a_n)/n \qquad \text{(cf. Problem 2-1).}$$

(a) Prove that if $a_n \to 0$, then $b_n \to 0$.

(Hint: this uses the same ideas as example 3.7. Given $\epsilon > 0$, show how to break up the expression for b_n into two pieces, both of which are small, but for different reasons.)

(b) Deduce from part (a) in a few lines without repeating the reasoning that if $a_n \to L$, then also $b_n \to L$.

3-2 To prove a^n was large if $a > 1$, we used "Bernoulli's inequality":

$$(1+h)^n \geq\ 1 + nh, \quad \text{if } h \geq 0\ .$$

We deduced it from the binomial theorem. This inequality is actually valid for other values of h however. A sketch of the proof starts:

$$\begin{aligned}
(1+h)^2 &= 1 + 2h + h^2 \ \geq\ 1 + 2h, \qquad \text{since } h^2 \geq 0 \text{ for all } h;\\
(1+h)^3 &= (1+h)^2(1+h) \ \geq\ (1+2h)(1+h), \quad \text{by the previous case,}\\
&= 1 + 3h + 2h^2,\\
&\geq\ 1 + 3h\ .
\end{aligned}$$

(a) Show in the same way that the truth of the inequality for the case n implies its truth for the case $n+1$. (This proves the inequality for all n by mathematical induction, since it is trivially true for $n = 1$.)

(b) For what h is the inequality valid? (Try it when $h = -3,\ n = 5$.) Reconcile this with part (a).

3-3 Prove that if a_n is a bounded increasing sequence and $\lim a_n = L$ in the sense of Definition 1.3A, then $\lim a_n = L$ in the sense of Definition 3.1.

3-4 Prove that a convergent sequence $\{a_n\}$ is bounded.

3-5 Given any $c \in \mathbb{R}$, prove there is a strictly increasing sequence $\{a_n\}$ and a strictly decreasing sequence $\{b_n\}$, both of which converge to c, and such that all the a_n and b_n are: (i) rational numbers; (ii) irrational numbers.

(Theorem 2.5 is helpful).