# PRACTICE PROBS. FOR 18.01A EXAM 1 — SOLUTIONS

$\boxed{1}$ a) $\displaystyle \lim_{x\to 0} \frac{\sin x}{1-\sqrt{1-x}} \overset{\text{L'Hosp}}{=} \frac{\cos x}{\frac{1}{2}(1-x)^{-1/2}}\Bigg|_0 = \frac{1}{1/2} = 2$

OR: using lin'l approx: $\displaystyle \approx \frac{x}{1-(1-\frac{1}{2}x)} = \frac{x}{\frac{1}{2}x} = 2$

b) $\displaystyle \int_0^{\pi/2} \cos^3 u \sin u \, du = -\frac{1}{4}\cos^4 u \Big]_0^{\pi/2} = \frac{1}{4}$

c) $\displaystyle \int \frac{\ln^2 x}{2x}\,dx = \frac{1}{2}\cdot\frac{\ln^3 x}{3} = \frac{\ln^3 x}{6}$

$\boxed{2}$ Algebraically: $e^{-2x} \approx 1 - 2x + \frac{4x^2}{2!}$

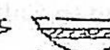
$\cos x \approx 1 - x^2/2$

$e^{-2x}\cos x \approx (1-2x+2x^2)(1-x^2/2)$

$\approx 1 - 2x + \frac{3}{2}x^2$

[calculating derivatives $f'(0)$, $f''(0)$ is tedious, but also works]

$\boxed{3}$ a) Divide into horizontal discs since algae conc. is constant on a disc

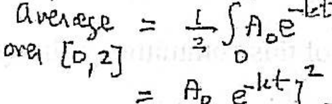


$\displaystyle \text{algae in disc} = \underbrace{(1-y)}_{\text{conc.}} \cdot \underbrace{\pi x^2}_{\text{area disc}} \cdot \underbrace{dy}_{\text{thickness}}$

$\displaystyle \text{total} = \int_0^1 (1-y)\pi\cdot 100y\,dy.$

b) Divide into cylindrical shells since algae conc. is constant on shell

$\displaystyle \text{algae in shell} = \underbrace{\frac{1}{1+x^2}}_{\text{conc.}} \cdot \underbrace{2\pi x\,dx}_{\text{area of ring}} \cdot \underbrace{(1-y)}_{\text{ht. of shell}}$




$\displaystyle \text{total} = \int_0^{10} \frac{1}{1+x^2}\cdot 2\pi x\left(1-\frac{x^2}{100}\right)dx$

$\boxed{4}$ Average over $[0,2]$ $\displaystyle = \frac{1}{2}\int_0^2 A_0 e^{-kt}\,dt$

$\displaystyle = \frac{A_0}{2}\,\frac{e^{-kt}}{-k}\Big]_0^2 = \frac{A_0}{2k}\left[1-e^{-2k}\right]$

$\boxed{5}$ By 2nd F.T. $\displaystyle \frac{d}{dx}\int_0^x f(t)\,dt = f(x)$, so

$\displaystyle f(x) = \frac{d}{dx}e^{2x}\cos x = e^{2x}(2\cos x - \sin x)$

$f(t) = ?$ but use $t$, not $x$

To find c: put $x=0$

$\displaystyle \int_0^0 f(t)\,dt = 0 = e^{2\cdot 0}\cos 0 + c$

$= 1 + c \quad \therefore\ c = -1$

$\boxed{6}$ $\displaystyle D\tan^{-1}(x^2) = \frac{1}{1+(x^2)^2}\cdot 2x = \frac{2x}{1+x^4}$

$\displaystyle \int_0^1 \frac{x\,dx}{1+16x^4} = \frac{1}{4}\int_0^2 \frac{u\,du}{1+u^4}$

[put $u = 2x$, $du = 2dx$, $u\,du = 4x\,dx$] $\displaystyle = \frac{1}{8}\tan^{-1}(u^2)\Big]_0^2$

$\displaystyle = \frac{1}{8}\tan^{-1} 4$

$\boxed{7}$ $\displaystyle \frac{x^2+2}{(x+1)^2(x+4)} = \frac{1}{(x+1)^2} - \frac{1}{x+1} + \frac{2}{x+4}$

(Find 1st & 3rd coeffs. by cover-up's, middle one by putting $x=0$ or by undetermined coeffs. method)

$\displaystyle \therefore \int \frac{x^2+2}{(x+1)^2(x+4)}\,dx = \frac{-1}{x+1} - \ln(x+1) + 2\ln(x+4)$

$\displaystyle = \frac{-1}{x+1} + \ln\frac{(x+4)^2}{x+1} + c$

$\boxed{8}$ $\displaystyle \int_0^3 \frac{dx}{(x^2+9)^2} = \int_0^{\pi/4} \frac{3\sec^2 u\,du}{9^2(\sec^2 u)^2}$

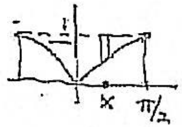
$x = 3\tan u \quad x=0:\ u=0$

$dx = 3\sec^2 u \quad x=3:\ u=\pi/4$

$\displaystyle = \frac{1}{27}\int_0^{\pi/4}\cos^2 u\,du = \frac{1}{27}\int_0^{\pi/4}\frac{1+\cos 2u}{2}\,du$

$\displaystyle = \frac{1}{27}\left[\frac{u}{2}+\frac{\sin 2u}{4}\right]_0^{\pi/4} = \frac{1}{27}\left[\frac{\pi}{8}+\frac{1}{4}\right]$

$\boxed{9}$




By shells,

$$V = \int_0^{\pi/2} 2\pi x(1-\sin x)\,dx$$

$$= \int_0^{\pi/2} 2\pi x - 2\pi\int_0^{\pi/2} x\sin x\,dx$$

By parts:

$$\int_0^{\pi/2} \underset{D}{x}\,\underset{I}{\sin x}\,dx = -x\cos x\Big]_0^{\pi/2} - \int_0^{\pi/2} -\cos x\,dx$$

$$= -x\cos x\Big]_0^{\pi/2} + \sin x\Big]_0^{\pi/2}$$

$$= 0-0+1-0$$

$$\therefore V = 2\pi\left[\frac{1}{2}\left(\frac{\pi}{2}\right)^2 - 1\right] = 2\pi\left[\frac{\pi^2}{8} - 1\right]$$