18.01A Practice Problems for Exam 1 (70 minutes, 140 pts, 2 pts./min.)

Directions: No books, notes, or calculators. (The actual exam will be 50 minutes.)

1. (30) Evaluate: a) $\lim_{x\to 0} \frac{\sin x}{1-\sqrt{1-x}}$ b) $\int_0^{\pi/2} \cos^3 u \sin u \, du$ c) $\int \frac{\ln^2 x \, dx}{2x}$

2. (10) Find the best linear and the best quadratic approximations to $f(x) = e^{-2x}\cos x$, for values of x near 0. (Sugestion: use algebraic methods.)

3. (20; 10, 10) The bottom of a small shallow circular pond has the shape of the parabola $y = (x/10)^2$, rotated around the y-axis (units: meters). Its radius is 10 meters; its depth in the center is one meter. Algae are in the pond, but not spread uniformly.

Set up (but **do not evaluate**) two definite integrals which give the total mass of algae in the pond if the concentration C of algae (in g/m^3) at each point P in the pond is as given below (two cases):

a) C is numerically equal to the depth (in meters) of the point P.

b) C is numerically equal to $\frac{1}{1+r^2}$, where r is the horizontal distance of P from the center of the pond.

4. (10) The decay law for mitium is $A = A_0e^{-kt}$ (A gms., t yrs.; $k > 0$ constant)

At time $t = 0$, A_0 grams of mitium are are left to decay for two years. What's the average amount (over time) of undecayed mitium present during those two years?

5. (10) If $\int_0^x f(t)\,dt = e^{2x}\cos x + c$, determine the constant c and the function $f(t)$.

6. (10) Differentiate $\tan^{-1}(x^2)$ and use the result to evaluate $\int_0^1 \frac{x\,dx}{1+16x^4}$.

7. (15) If $\int \frac{x^2+2}{(x+1)^2(x+4)}dx = f(x) + \ln(g(x))$, find the rational functions f and g.

8. (15) Evaluate $\int_0^3 \frac{dx}{(x^2+9)^2}$ by substituting $x = 3\tan u$.

9. (20) The bowl of a martini glass has the shape obtained by rotating the graph of $y = \sin x$, $0 \le x \le \pi/2$, about the y-axis. Find its volume.

Other possible types of problems not represented above:

Applications of integration to calculating Work, Arclength, Mass;

Other applications of integration (as in Exercises 4J);

Finding properties of functions defined by a definite integral (like 3D-3).

18.01A Practice Probs, Exam 1: Sol'ns

1 a) $\lim_{x\to 0} \frac{\sin x}{1-\sqrt{1-x}} \overset{\text{L'Hosp}}{=} \frac{\cos x}{\frac{1}{2}(1-x)^{-1/2}}\Big|_0 = \frac{1}{1/2} = 2$

or: via lin approx: $\approx \frac{x}{1-(1-\frac{1}{2}x)} = \frac{x}{\frac{1}{2}x} = 2$

b) $\int_0^{\pi/2} \cos^3 u \sin u\, du = -\frac{1}{4}\cos^4 u\Big]_0^{\pi/2} = \frac{1}{4}$

c) $\int \frac{\ln^2 x}{2x}dx = \frac{1}{2}\cdot\frac{\ln^3 x}{3} = \frac{\ln^3 x}{6}$

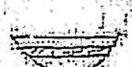
2 Algebraically: $e^{-2x} \approx 1-2x+\frac{4x^2}{2!}$

$\cos x \approx 1 - x^2/2$

$e^{-2x}\cos x \approx (1-2x+2x^2)(1-x^2/2)$

$\approx 1-2x+\frac{3}{2}x^2$

[Calculating derivatives $f'(0)$, $f''(0)$ is tedious, but also works.]

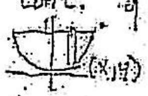
3 a) Divide into horizontal discs 
since algae conc. is constant on a disc

$\text{algae in disc} = \underset{\text{conc.}}{(1-y)} \cdot \underset{\text{area disc}}{\pi x^2} \cdot \underset{\text{thickness}}{dy}$

$\text{total} = \int_0^1 (1-y)\pi \cdot 100y\, dy$

b) Divide into cylindrical shells since algae conc. is constant on shell

$\text{algae in shell} = \underset{\text{conc.}}{\frac{1}{1+x^2}} \cdot \underset{\text{area of ring}}{2\pi x\, dx} \cdot \underset{\text{ht of shell}}{(1-y)}$



$\text{total} = \int_0^{10} \frac{1}{1+x^2}\cdot 2\pi x\left(1-\frac{x^2}{100}\right)dx$

4 average over $[0,2]$ $= \frac{1}{2}\int_0^2 A_0 e^{-kt}dt$

$= \frac{A_0}{2}\frac{e^{-kt}}{-k}\Big]_0^2 = \frac{A_0}{2k}\left[1-e^{-2k}\right]$

5 By 2nd F.T. $\frac{d}{dx}\int_0^x f(t)\,dt = f(x)$, so

$f(x) = \frac{d}{dx}e^{2x}\cos x = e^{2x}(2\cos x - \sin x)$

To find c: put $x=0$. $f(t) = ?$ but use t, not x

$\int_0^0 f(t)\,dt = 0 = e^{2\cdot 0}\cos 0 + c$

$= 1 + c$ $\therefore$ $c = -1$.

6 $D\tan^{-1}(x^2) = \frac{1}{1+(x^2)^2}\cdot 2x$

$= \frac{2x}{1+x^4}$

$\int_0^1 \frac{x\,dx}{1+16x^4} = \frac{1}{4}\int_0^2 \frac{u\,du}{1+u^4}$

(put $u = 2x$, $du = 2dx$, $u\,du = 4x\,dx$) $= \frac{1}{8}\tan^{-1}(u^2)\Big]_0^2$

$= \frac{1}{8}\tan^{-1}4$

7 $\frac{x^2+2}{(x+1)^2(x+4)} = \frac{1}{(x+1)^2} - \frac{1}{x+1} + \frac{2}{x+4}$

(Find 1st + 3rd coeffs. by cover-up; middle one by putting $x=0$ or by undetermined coeffs. method.)

$\therefore \int \frac{x^2+2}{(x+1)^2(x+4)}dx = \frac{-1}{x+1} - \ln(x+1) + 2\ln(x+4)$

$= \frac{-1}{x+1} + \ln\frac{(x+4)^2}{x+1} + c$

8 $\int_0^3 \frac{dx}{(x^2+9)^2} = \int_0^{\pi/4} \frac{3\sec^2 u\, du}{9^2(\sec^2 u)^2}$

$x = 3\tan u$ $\quad x=0: u=0$

$dx = 3\sec^2 u$ $\quad x=3: u=\pi/4$

$= \frac{1}{27}\int_0^{\pi/4}\cos^2 u\,du = \frac{1}{27}\int_0^{\pi/4}\frac{1+\cos 2u}{2}du$

$= \frac{1}{27}\left[\frac{u}{2}+\frac{\sin 2u}{4}\right]_0^{\pi/4} = \frac{1}{27}\left[\frac{\pi}{8}+\frac{1}{4}\right]$

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By shells,

$V = \int_0^{\pi/2} 2\pi x(1-\sin x)\,dx$

$= \int_0^{\pi/2} 2\pi x - 2\pi\int_0^{\pi/2} x\sin x\, dx$

By parts:

$\int_0^{\pi/2} x \sin x\, dx = -x\cos x\Big]_0^{\pi/2} - \int_0^{\pi/2} -\cos x\, dx$

$= -x\cos x\Big]_0^{\pi/2} + \sin x\Big]_0^{\pi/2}$

$= 0 - 0 + 1 - 0$

$V = 2\pi\left[\frac{1}{2}\left(\frac{\pi}{2}\right)^2 - 1\right] = 2\pi\left[\frac{\pi^2}{8}-1\right]$