SOLUTIONS

1 a) $\displaystyle \lim_{x\to0}\frac{\sqrt{1+x}-1}{\sin 2x} \underset{\text{L'Hosp.}}{=} \lim_{x\to0}\frac{\frac{1}{2\sqrt{1+x}}}{2\cos 2x} = \frac{1/2}{2} = \frac14$

OR: (via approx.) $\displaystyle = \lim_{x\to0}\frac{1+\frac12 x-1}{2x} = \frac14$

b) $\displaystyle \lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{f'(x)}{g'(x)} = \lim_{x\to\infty}\frac{\frac{\ln x-1}{(\ln x)^2}}{\frac{1}{\ln x}}$

$\displaystyle = \lim_{x\to\infty}\frac{\ln x-1}{\ln x} = \lim_{x\to\infty} 1-\frac{1}{\ln x} = 1$

2 $x^2 = \sin u$
$2x\,dx = \cos u$
$x=0 \to u=0$
$x=1 \to u=\pi/2$

$\displaystyle \int_0^1 x\sqrt{1-x^4}\,dx = \int_0^{\pi/2}\frac12\cos u\cdot\cos u\,du = \frac12\int_0^{\pi/2}\cos^2 u\,du$

$\displaystyle = \frac12\int_0^{\pi/2}\frac{1+\cos 2u}{2}\,du = \frac14\left[u+\frac{\sin 2u}{2}\right]_0^{\pi/2}$

$\displaystyle = \frac14\left[\frac{\pi}{2}+0-0\right] = \frac{\pi}{8}.$

3 $\displaystyle \int x\sin x\,dx = -x\cos x - \int -\cos x\,dx$
$= -x\cos x + \sin x + c$

4 $\displaystyle \frac{3x^2+1}{(x^2+1)(x-1)} = \frac{Ax+B}{x^2+1} + \frac{2}{x-1}$ ← by cover-up method

compare numerators on both sides

$3x^2+1 = (Ax+B)(x-1) + 2(x^2+1)$

coef. of x^2 : $3 = A+2$ $\therefore A=1$
of x : $0 = -A+B$ $\therefore B=1$
of 1 : $1 = -B+2$ $\therefore B=1$ ✓
(as check)

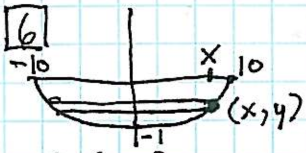
5 $\displaystyle \int_2^6 \sqrt{16+t^4}\,dt = \int_1^3\sqrt{16+16u^4}\cdot 2\,du$

Put $t=2u$
$dt = 2\,du$

$\displaystyle = 8\int_1^3\sqrt{1+u^4}\,du$

$= 8\left[F(3)-F(1)\right]$

6



Vol. disc $= \pi x^2\,dy$
$\delta = 1+y$
mass of disc $= \pi\cdot100(1-y^2)(1+y)\,dy$

$\left(\frac{x}{10}\right)^2 + y^2 = 1$
$x^2 = 100(1-y^2)$

total mass $\displaystyle = \int_{-1}^0 100\pi(1+y-y^2-y^3)\,dy$

$\displaystyle = 100\pi\left[y+\frac{y^2}{2}-\frac{y^3}{3}-\frac{y^4}{4}\right]_{-1}^0 = 100\pi\cdot\frac{5}{12}$
$\displaystyle = \frac{125\pi}{3}$

[volume: $\displaystyle 100\pi\int_{-1}^0(1-y^2)\,dy = 100\pi\left[y-\frac{y^3}{3}\right]_{-1}^0 = \frac{200\pi}{3}$]

7

Area of ring $= 2\pi r\,dr$
#holes in ring $= 2\pi r\cdot dr\cdot\frac{1}{1+r^2}$ density/area

Total # holes $\displaystyle = 2\pi\int_0^{100}\frac{r\,dr}{1+r^2}$

$\displaystyle = 2\pi\cdot\frac12\ln(1+r^2)\Big]_0^{100} = \pi\ln(1+10^4)$ "exact" answer

$\approx \pi\ln(10^4) = 4\pi\ln 10$
$\approx (3.1)\cdot4\cdot(2.3) \approx 30$ (to nearest 10)

8 $x^k\sqrt{x+1} \sim x^k\sqrt{x} = x^{k+1/2}$

$\displaystyle \therefore \frac{1}{x^k\sqrt{x+1}} \sim \frac{1}{x^{k+1/2}}$

$\displaystyle \int_1^\infty\frac{dx}{x^{k+1/2}}$ converges if $k+1/2 > 1$ or $\boxed{k>1/2}$

9 a) $\displaystyle \frac{n^2}{\sqrt{n^5+1}} \sim \frac{n^2}{n^{5/2}} = \frac{1}{n^{1/2}}$ $\therefore$ sum diverges

($\displaystyle \sum_1^\infty\frac{1}{n^p}$ diverges when $p\le1$).

b) $\displaystyle \sum_1^\infty\frac{(-1)^n}{\sqrt n}$ converges since the signs alternate and $\displaystyle \lim_{n\to\infty}\frac{1}{\sqrt n} = 0.$

Convergence is conditional since $\displaystyle \sum_1^\infty\frac{1}{\sqrt n}$ diverges (as in part (a)).

10 Ratio test:

$$\left|\frac{x^{n+1}}{2^{2(n+1)}\sqrt{n+1}}\right|\cdot\left|\frac{2^{2n}\sqrt{n}}{x^n}\right| = \frac{|x|}{2^2}\sqrt{\frac{n}{n+1}}$$

$$\lim_{n\to\infty}\frac{|x|}{4}\sqrt{\frac{n}{n+1}} = \frac{|x|}{4} < 1 \text{ if } |x|<4$$

$\therefore\ R = 4.$

11

$$\frac{\sin t}{1+t} \approx \underbrace{\left(t-\frac{t^3}{6}\right)}_{\sin t}\underbrace{\left(1-t+\frac{t^2}{2}-\frac{t^3}{6}\right)}_{\frac{1}{1+t}}$$

$$\approx t - t^2 + t^3\left(-\frac{1}{6}+\frac{1}{2}+\frac{1}{2}\right)$$

$$= t - t^2 + \frac{5}{6}t^3$$

12

$$f(x) = \sum_0^\infty \frac{x^{2n+1}}{2n+1} = x + \frac{x^3}{3} + \dots$$

$$f'(x) = \sum_0^\infty x^{2n} = 1 + x^2 + \dots = \frac{1}{1-x^2}$$

$$\therefore f(x) = \int\frac{dx}{1-x^2} = \int\left(\frac{1/2}{1+x}+\frac{1/2}{1-x}\right)dx$$

(by partial fractions)

$$= \frac{1}{2}\left[\ln(1+x) - \ln(1-x)\right]$$

$$= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

[No additive constant since $f(0) = 0$]

13 a) $A\int_0^1 x^2(1-x)\,dx = A\left[\frac{x^3}{3}-\frac{x^4}{4}\right]_0^1 = A\cdot\frac{1}{12}$

To be a prob. density,

$A\cdot\frac{1}{12} = 1, \quad \therefore A = 12.$

b) $\underset{\text{variance}}{v} = \int_0^1 12x^2\cdot x^2(1-x)\,dx - m^2$

$$= 12\left[\frac{x^5}{5}-\frac{x^6}{6}\right]_0^1 = 12\cdot\frac{1}{30} - (.6)^2$$

$$= .40 - .36$$

$\therefore\ \sigma = \sqrt{.04} = .2$

[mean: $m = \int_0^1 12x\cdot x^2(1-x)\,dx$

$= 12\left[\frac{x^4}{4}-\frac{x^5}{5}\right] = \frac{12}{20} = .6$]

14 Use Poisson R.V. X, with $m=2$

$P(X=0 \text{ or } 1) = e^{-2}\left(\frac{2^0}{0!}+\frac{2^1}{1!}\right) = \frac{3}{e^2}$

(forgets less often)

$P(X>2) = 1 - P(X=0,1,2) =$

(forgets more often) $\quad 1 - e^{-2}\left(1+2+\frac{2^2}{2!}\right)$

$= 1 - \frac{5}{e^2}$

and $\underset{\text{(less)}}{\frac{3}{e^2}} > \underset{\text{(more)}}{1-\frac{5}{e^2}}$ since $\frac{8}{e^2} \approx \frac{8}{7.4} > 1$

15 Using minutes as the units:

Prob. of a call (at least one) in a 5-min. period is (using exponential R.V. with a mean = 20)

$$\underset{Pr(0<T<5)}{\int_0^5 \frac{1}{20}e^{-t/20}\,dt} = -e^{-t/20}\Big]_0^5 = -e^{-1/4}+1$$

$$= 1 - \left[1 - \tfrac{1}{4} + \frac{(1/4)^2}{2!}\right]$$

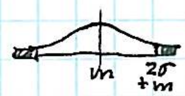
$$= \frac{1}{4} - \frac{1}{32} = .22$$

16 Use normal random variable X with $m=200$, $\sigma = 20$

a) Since $240 = 200 + 2\sigma$:

$Pr(|X-m| < 2\sigma) \approx 95\% = .95$

$\therefore\ Pr(X - 200 > 2\sigma) \approx \frac{1}{2}(.05) \approx .025$



(area of 2 tails ≈ .05 ∴ area of right tail ≈ .025)

More accurate:

$P(X>240) = 1 - P(X<240)$

$= 1 - P\left(\frac{X-200}{20} < \frac{40}{20}\right) =$

$= 1 - P(Z<2) = 1 - \Phi(2)$

$= 1 - .9772 \approx .023$

b) Choose $\underset{\text{(no. of aphids)}}{A}$ so that $P(X<A) = .99$

$P(X<A) = P\left(\frac{X-200}{20} < \frac{A-200}{20}\right) = .99$

want $\therefore\ \Phi\left(\frac{A-200}{20}\right) \approx .99$

From table (interpolating mentally),

$\frac{A-200}{20} \approx 2.34 \quad \therefore A \approx 247$