18.100A Fall 2018: Problem Set 9 (complete) due Mon. Nov. 19

Directions: If you collaborate, list collaborators but write up independently. It is illegal to consult solutions from previous semesters.

Reading Mon.: 22.1, 22.2 (through Example 22.2C only)

Pointwise vs. uniform convergence of sequences $\{f_n(x)\}$ and series $\sum_0^\infty u_k(x)$ on an x-interval I. Weierstrass M-test for uniform convergence of series.

1. (4: 1,2,1) a) 22.1/1a, b) like 22.1/1c, but use $\dfrac{n^2x}{1+n^2x^2}$, $[a,\infty)$, $a>0$.

c) 22.1/3 (omit the proof of non-uniform convergence on $(-\infty,\infty)$)

2. (2) 22.2/2b (estimate $|\cos(x/n)-\cos(0)|$)

3. (2) Read the statement of the M-test, then use it to work: a) 22.2/2d b) 22.2 /4

4. (1) Read just the statement of Th'm 22.2C; then prove it – treat it like a Question (if stuck take a quick look at the book's proof for a hint).

5. (2)) a) First do 7.3/1a (switching the roles of n and k), to prove the infinite triangle inequality.

– Start from the extended triangle inequality, $|\sum_0^n a_k| \le \sum_0^n |a_k|$, and assume the infinite series is absolutely convergent. Prove (in order) that as $n\to\infty$,
the right side has as its limit the right side of the infinite inequality (this is easy), the left side has as its limit the left side of the infinite inequality (this needs 3 citations), and then show the two limits are connected by $\le$, which needs one final citation.

(This proof revisits convergence theorems for numerical series, and theorems about continuity and limits.)

b) Treating it as a Question, prove the M-test, given below in skeleton form, by supplying reasons for each step – definitions or theorems. Assume the hypotheses and notation used in the book (Theorem 22.2B).

(i) On I, let $f(x)=\sum_0^\infty u_k(x)$. (Why can you say this?)

(ii) Let $s_n(x)=\sum_0^n u_k(x)$, $S_n=\sum_0^n M_k$, $S=\sum_0^\infty M_k$. Then given $\epsilon>0$,

$$|f(x)-s_n(x)| \;=\; |\sum_{n+1}^\infty u_k(x)| \;\le\; \sum_{n+1}^\infty |u_k(x)| \;\le\; \sum_{n+1}^\infty M_k \;=\; |S-S_n| \;<\; \epsilon, \text{ for } n>N_\epsilon.$$

(Why does this prove uniform convergence?)

Reading Wed.: 22.3 (read statement only of Cor. 22.3)

6. (1) Treating it as a Question: read the statement of Corollary 22.3 and prove it.

7. (4: 1,1.5,1.5) a) 22.3/1 b) 22.3/2 c) 22.3/3

8. (2) Work 22.3/4a,b (use $I=[0,\,1]$ as the interval). (See directions on reverse side.)

4a) You don't have to copy the proof of Theorem 22.3 onto your paper; just indicate what modifications are necessary to the "chain of approximations" in the third displayed line, and prove the modifications lead to the desired conclusion.

4b) Include a graph of a typical approximating function $f_n(x)$.

There are two hypotheses and one conclusion.

Review section 13.5. Cite relevant definitions, theorems, and examples as needed.

Reading: Fri.: 22.4 (skip Thm. 22.4A until you've tried Prob. 9 below)
Integrating infinite series of functions term-by-term

9. (1) Work 22.4/1; Treat it like a Question, without reading the similar Thm. 22.4A.

10. (2) (This is about the basic definitions; follow the directions.)

a) Prove that on any interval $[-a, a]$, where $0 < a < 1$, the geometric series $\sum_0^\infty t^n$ converges uniformly to its sum $1/(1-t)$.

(Use the definitions; do not use the M-test. See (4) p. 52 for the formulas needed.)

b) Using partial sums and Problem 9 (not Theorem 22.4B and/or the M-test) prove

$$\sum_1^\infty \frac{x^n}{n} = -\ln(1-x) \text{ converges uniformly on any } [-a, a], \text{ where } 0 < a < 1.$$

11. (2) Work 22.4/3. For the last step, Chapter 17 gives some standard Taylor series around the point 0 (in other words, taking $a = 0$).

12. (3) Work Problem 22-3b (Uses and reviews 22.4 mostly.)

Background: The function $J_0(x)$ is the function whose graph over an interval $[-a, a]$ gives at a single moment in time the cross-sectional shape along a diameter of the vibrating membrane of a bass drum in a marching band, shortly after the drummer hits it exactly in the middle. (The number a can be thought of as the radius of the drumhead.)

From this, one can derive a differential equation that this cross-sectional shape must satisfy – Bessel's ODE, given in Problem 22-2; it gives also the power series solution to the ODE (including the initial and boundary conditions), which is the therefore also the power series for $J_0(x)$, since there can be only one solution – the drumhead knows how it vibrates.

Actually, Bessel was an astronomer, and he discovered the function in a different context, and in the strange-looking form given in P22-3b. So the purpose of this problem is to show that the two representations of $J_0(x)$ – as a series, and as a definite integral – are equivalent.

Procedure: You will need the power series form of $J_0(x)$, as given in P22-2, and also the standard definite integrals in 22-3a.

Start with the power series for $\cos u$ (cf. 17.1). After $x \sin\theta$ is substituted for u, it is no longer a power series, but the theorems in 22.3,.4 are still usable, since they are phrased in terms of general functions $u_k(x)$.

The main confusion is likely to come from the notation. This problem uses the standard notation in the literature for the integral form of the Bessel function, but that makes it an exception to the general rule followed in the textbook that "uniform whatever" is always with respect to the variable x. If you take this warning to heart, mind your θ's, x's, and u's and don't make any errors in calculation, it should all work out in the end, despite some possible dark moments in the middle.

Show your understanding of Chapter 22 by citing at the right points in the proof the key theorems you are using; verify that their hypotheses are satisfied here – this is an important step (i.e, it's at least -1 if you fail to do it.)