18.100A Fall 2018 Problem Set 8 (complete) due **Fri. Nov. 9**

Directions: This is now complete, covering the Fri., Mon., and Wed. classes. Other directions are the same as for the earlier problem sets.

Reading Fri: 20.6; (review 20.3-.4) Growth rate of functions; L'Hospital for ∞/∞.

Problem 1. (1) Work 20.3/2 (Using change of variable; verify hypotheses.)

Problem 2. (2) Work P20-4ab for the case $n = 1$ as described below.

Usually integration by parts is used to make an integral look better. In this somewhat bizarre application, it is used to make it look worse.

The idea is to prove Taylor's theorem (17.2), but just for the case $n = 1$: Theorem 16.1B, the Linearization Error Theorem. (The general case for higher values of n is proved by induction the same way, and no harder, just more time-consuming.)

a) Follow the directions in P20-4a, but just for $n = 1$, getting the remainder term $R_1(x)$ as a definite integral ("the integral remainder").

(The integration by parts is used in a strange way. As advised, watch the signs carefully, and remember that x is just an unspecifed constant, as far as the integral is concerned.)

b) Convert the remainder term to the form it has in Theorem 16.1B ("the Lagrange remainder" or "the derivative remainder") following the method described in P20-4b; use one of the formulas in P20-3.

Problem 3. (4: 1,1,2) Parts (a) and (b) might help with (c).

a) Do Q20.6/2, paying attention to the Note below (23) and (24).

b) If $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to \infty$ and f grows faster than g, prove $f - g \to \infty$.

c) Work 20.6/1a; prove your work, determining all four limits needed.

Problem 4 (2) Work 20.6/3ab .

Problem 5 (2) In an earlier P-set, the Prime Number Theorem (PNT) was about the sequence of primes. It says PNT: $p_n \sim n \ln n$, where $p_n =$ the $n-$th prime.

In 20.6/4, it is implicitly stated by two functions: Euler's and Gauss' respectively:

PNT: $\pi(x) \sim \dfrac{x}{\ln x}$; $\pi(x) \sim \mathrm{Li}(x)$; ($\pi(x) =$ the number of primes $\leq x$).

Show that these two PNT forms are equivalent, by showing that $\mathrm{Li}(x) \sim \dfrac{x}{\ln x}$.

b) The Riemann Hypothesis. Gauss' estimation of $\pi(x)$ is more accurate than Euler's: the graph of $x/\ln x$ is always below that of $\pi(x)$, whereas while the graph of $\mathrm{Li}(x)$ starts out above the graph of $\pi(x)$, the two graphs cross each other infinitely often afterwards.

In other words, if we write the estimation in the form $\pi(x) = \mathrm{Li}(x) + e(x)$, then the error term $e(x) < 0$ at the start, but is known today to first become > 0 at some $x < 1.4 \cdot 10^{317}$.

The Riemann Hypothesis (1859), is an estimate of the error term $e(x)$. Its truth is one of the great problems of number theory (\$ 1,000,000 if you can solve it). It has two forms:

$$i)\ \ |e(x)| < C(\sqrt{x}\ln x) \text{ for some constant } C; \qquad ii)\ \ \lim_{x\to\infty} \frac{|e(x)|}{x^{\frac{1}{2}+\epsilon}} = 0 \text{ for all } \epsilon > 0.$$

Which of these is the stronger statement, i.e., which one implies the other? Prove it.

Reading Mon: 21.1-.2 Improper integrals with nonnegative integrands:; comparison tests for convergence; standard comparison integrals.

Problem 6. (2) Assume $f(t)$ is non-negative and integrable on $[0, \infty]$. For improper integrals of the first kind, the analog of the "nth term test for divergence" for infinite series (7.2A) should be: $\displaystyle\int_0^\infty f(t)\,dt$ converges $\Longrightarrow \lim_{t\to\infty} f(t) = 0$.

But this test for divergence is false. Give a counterexample to it by a carefully drawn graph of a non-negative continuous function $f(t)$ with the first few necessary t-values and $f(t)$ values clearly indicated.

Prove that its integral $F(R)$ on $[0, R]$ converges, by giving the relevant properties of $F(R)$ on $[0, R]$ and citing relevant theorems needed. (You can omit continuity if the graph makes it clear.)

But show it doesn't satisfy the conclusion of this "test", since $\lim_{t\to\infty} f(t) \neq 0$.

(For half-credit, give a discontinuous $f(t)$ counterexample.)

(There is also no useful analog – for improper integrals of the first kind – of the ratio test and n-th root test for convergence of infinite series.

Problem 7. (2) Work 21.1/2; do it two different ways:

a) Change the variable to convert it to an integral of the first kind, then use comparison.

b) Do the integration, then evaluate the limit you get, either by changing variable, or converting it to a standard indeterminate form.

Problem 8. (3: 1.5, 1.5) In both of these, k is any real number. Show work or reasoning.

a) Work 21.2/1c, using asymptotic comparison at both ends.

b) Work $\displaystyle\int_0^\infty t^k e^{-t}\,dt$, telling at the end for what values of k it converges. Use comparison and/or asymptotic comparison. Example 21.2B should help with comparison.

Reading Wed.: 20.5, 21.3 Graphical and Analytic representations of $n!$:
Stirling's asymptotic approximation and Euler's Gamma function.

Problem 9. (2)) Two problems using just Stirling's approximation formula, not its proof.

a) Work 20.5/1a, which asks for the Stirling approximation to $\displaystyle\binom{2n}{n} = \frac{(2n)!}{(n!)(n!)}$

This expression can be interpreted both as the the middle binomial coefficient in $(1+x)^{2n}$ and the number of ways of choosing n things from a set of $2n$ different things.

Simplify your answer as much as possible.

b) In P-set 3, Problem 10b, you showed that for the power series $\displaystyle\sum_0^\infty \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{n!}\,x^n$,

the radius of convergence was $R = 1/2$.

But testing the endpoints $x = \pm 1/2$ was incomplete, since for $x = 1/2$, though the terms were strictly decreasing, and had limit 0, none of the standard tests showed whether it converged or diverged.

Fill in this gap by using Stirling's formula and the asymptotic comparison test.

Problem 10. (2)

a) Work 21.3/1a You will need G3, G6, and the definition of $\lim_{x \to 0^+} f(x) = \infty$.

b) Section 21.3 (8) gives an improper integral $\displaystyle\int_0^\infty f_n(t)\,dt$ whose exact value is $n!$. There are advanced techniques that derive Stirling's formula from this integral, and even improve upon the formula. This exercise aims at just the start of these, showing how the mysterious quotient $(n/e)^n$ in the formula is related to the integral.

Study the integrand $f_n(t)$ – show it has a unique critical point P on $[0, \infty]$, show from the behavior of $f_n'(t)$ near the critical point that the graph is concave (no second-derivatives needed, please), so it's a maximum point, which thus gives the corresponding maximum value $\bar{m}$ of $f_n(t)$ on the interval.

Then get a lower bound for the total area under the graph of $f_n(t)$ by making the point Q one unit to the right of P and finding the area of the trapezoid lying over PQ and just unde the concave graph.

Simplify the trapezoidal area so it looks like a familiar expression in n and constants, multiplied by a simple estimated rational fraction. More advanced methods get more information from the integral, use it to complete Stirling's formula and add improvements to it.

Problem 11. (3) (Read first the proof of Stirling's formula, except for the exact value of the constant $K = \sqrt{2\pi}$).

Work 20.5/2, except change the cube roots to square roots, and call the sum S_n.

Follow the same procedure and use the same notation as in the proof of Stirling's formula. Denote by B_n the sum of the little "bows" (or "slivers") on top of the trapezoids.

Get a little more accuracy by estimating L visually: about how much of the square over $[0, 1]$ do the n bows take up?