Directions: This is the first third of P-set 8, covering Friday’s class, except we didn’t get to the proof of L’Hospital’s rule for determining limits of \( f(x)/g(x) \) when naive evaluation gives you the indeterminate form \( \infty/\infty \).

Other directions are the same as for the earlier problem sets.

Reading Fri: 20.6; (review 20.3-.4) Growth rate of functions; L’Hospital for \( \infty/\infty \).

Problem 1. (1) Work 20.3/2 (Using change of variable; verify hypotheses.)

Problem 2. (2) Work P20-4ab for the case \( n = 1 \) as described below.

Usually integration by parts is used to make an integral look better. In this somewhat bizarre application, it is used to make it look worse.

The idea is to prove Taylor’s theorem (17.2), but just for the case \( n = 1 \): Theorem 16.1B, the Linearization Error Theorem. (The general case for higher values of \( n \) is proved by induction the same way, and no harder, just more time-consuming.)

a) Follow the directions in P20-4a, but just for \( n = 1 \), getting the remainder term \( R_1(x) \) as a definite integral (“the integral remainder”).

(The integration by parts is used in a strange way. As advised, watch the signs carefully, and remember that \( x \) is just an unspecified constant, as far as the integral is concerned.)

b) Convert the remainder term to the form it has in Theorem 16.1B (“the Lagrange remainder” or “the derivative remainder”) following the method described in P20-4b; use one of the formulas in P20-3.

Problem 3. (4: 1,1,2) Parts (a) and (b) might help with (c).

a) Do Q20.6/2, paying attention to the Note below (23) and (24).

b) If \( f(x) \to \infty \) and \( g(x) \to \infty \) as \( x \to \infty \) and \( f \) grows faster than \( g \), prove \( f - g \to \infty \).

c) Work 20.6/1a; prove your work, determining all four limits needed.

Problem 4 (2) Work 20.6/3ab.

Problem 5 (2) In an earlier P-set, the Prime Number Theorem (PNT) was about the sequence of primes: \( p_n \sim n \ln n \), where \( p_n \) is the \( n \)-th prime.

In 20.6/4, it is implicitly stated by two functions: Euler’s and Gauss’ respectively:

PNT: \( \pi(x) \sim \frac{x}{\ln x} \); \( \pi(x) \sim \text{Li}(x) \); \( (\pi(x) = \text{the number of primes} \leq x) \).

Show that the two PNT forms are equivalent, by showing that \( \text{Li}(x) \sim \frac{x}{\ln x} \).

b) The Riemann Hypothesis. Gauss’ estimation of \( \pi(x) \) is more accurate than Euler’s: the graph of \( x/\ln x \) is always below that of \( \pi(x) \), whereas while the graph of \( \text{Li}(x) \) starts out above the graph of \( \pi(x) \), the two graphs cross each other infinitely often afterwards.

In other words, if we write the estimation in the form \( \pi(x) = \text{Li}(x) + e(x) \), then the error term \( e(x) < 0 \) at the start, but is known today to first become \( > 0 \) at some \( x < 1.4 \cdot 10^{317} \).

The Riemann Hypothesis (1859), is an estimate of the error term \( e(x) \). Its truth is one of the great problems of number theory ($ 1,000,000 if you can solve it). It has two forms:

i) \( |e(x)| < C(\sqrt{x} \ln x) \) for some constant \( C \);

\( \lim_{x \to \infty} \frac{|e(x)|}{x^{\delta}} = 0 \) for all \( \epsilon > 0 \).

Which of these is the stronger statement, i.e., which one implies the other? Prove it.