18.100A Fall 2018 Problem Set 7 due **Fri. Nov. 2**

Directions: This is now complete.

If you collaborate, list collaborators, and write up solutions independently. Do not consult solutions from previous semesters or on the internet.

Reading Wed.: 13.5 Uniform Continuity of $f(x)$ on an interval I: definition, equivalence with ordinary continuity if $I = [a, b]$, i.e., a sequentially compact interval.

The problems below deal with continuous functions $f(x)$ on non-compact intervals I: what sort of further conditions on $f(x)$ or I will guarantee uniform continuity on I?

Problem 1. (2.5: 1.5, .5, .5))

Let $f(x)$ be a function defined on a given interval I of some unspecified type.

Assume that $f(x)$ has *bounded secant slope* on I. By definition, this means there is a fixed constant K such that, given any two points $x' < x$" in I, the slope λ of the secant line segment joining the points $(x', f(x'))$ and $(x'', f(x''))$ on the graph of $f(x)$ satisfies $|\lambda| < K$.

(Try sketching a few graphs to get a feel for what this means.)

a) Prove that $f(x)$ is uniformly continuous on I.

b) Deduce that if a function $f(x)$ is differentiable on an interval I and $f'(x)$ is bounded on I, then $f(x)$ is uniformly continuous on I.

c) Prove $\ln x$ is uniformly continuous on the non-compact interval $[1, \infty)$.

Problem 2. (3: 1,1,1)

a) Assume the interval I is the union $I_1 \cup I_2$ of two intervals I_1 and I_2 which overlap in an interval I' of positive length $k > 0$.

Prove: $f(x)$ uniformly continuous on I_1 and $I_2 \Rightarrow f(x)$ uniformly continuous on I.

(What condition must a subinterval $[a, b]$ of I satisfy if it isn't a subinterval of I_1 or of I_2? Justify your answer in a few words and symbols, and use it in your proof.)

b) Consider $f(x) = \sqrt{x}$ on the non-compact interval $I = [0, \infty)$. Give separate proofs that it doesn't satisfy either condition in Problems 1b and 1a respectively:
(i) $f'(x)$ bounded on $(0, \infty)$ (ii) $f(x)$ has bounded secant slope on I. (cf. Example 3.1)

c) Prove using part (a) that nonetheless, $f(x)$ is uniformly continuous on I.

Problem 3. (2.5: 1.5, 1) Work P13-6ab. This ties together uniform continuity with the earlier notion of Cauchy sequence in Section 6.4 .

For part (b), you can use an earlier P-set problem: if a sequence $\{x_n\}$ converges, then it is a Cauchy sequence.)

Reading Fri.:

18.1-.2 Partitions, Riemann-integrability definition.

18.3: read Theorem 18.3B only (skip Theorem 18.3A, which is Problem 4, here being treated as a Question.)

18.4 Properties of integrability: read the statements, skip the proofs.

Problem 4. (2)

a) Prove: (Theorem 18.3A): If $f(x)$ is decreasing on $[a, b]$, it is integrable on $[a, b]$.

(Try to do this without consulting the proof in the book. Using the graph of $f(x)$, represent the difference $U(P) - L(P)$ as a sum of rectangles, and show they can all be fitted without overlapping into a rectangle of width ϵ and fixed height, if the mesh $|P| < \epsilon$ and the function $f(x)$ is decreasing on $[a, b]$.

If you need to consult the book's proof for a further hint, let some time elapse before you write up your own version of it, in your own words.

Problem 5. (2) Work: 18.3/1, which is an example of a function with k discontinuities that is integrable.

(Some older printings of the book have n where the current printing has k; but you need n for the n-partitions in your proof. Be careful with the argument – note that a point can lie in more than one subinterval of a partition.)

Problem 6. (1) Using the result of Problem 4, give an example of an integrable function having an infinity of jump discontinuities on $[0, 1]$.

Reading: (Mon.) 19.1-.4 (skip proofs in 19.3-.4 –will be given in Wed. class, but the def'ns and statements are used in Problems 7,8,9.)

The Riemann integral: def'n, exists for integrable functions; Riemann sums: def'n; use in calculating integrals.

Problem 7. (2) Work 19.2/1, using lower sums (not upper sums, as in older printings).

You may assume e^x is continuous for all x.

To make the calculation of the limit needed at the end more transparent, you can substitute the continuous variable h for $1/n$; what theorems guarantee that

$$\lim_{h \to 0} f(h) = \lim_{n \to \infty} f(1/n) \ ?$$

(Note that the left side is a function, the right side a sequence.)

Problem 8. (2) Work 19.3/3ab.

The trapezoidal rule is a standard numerical method used for evaluating integrals, in all calculus textbooks; it's meant to provide motivation for the problem, but all you need to do the problem is the formula given in the problem, and the definition of Riemann sum.

Problem 9. (2: 1.5,.5) Work 19.4/2ab

Part (a) and its immediate corollary Part (b) are important results; Part (b) sharpens for continuous functions the simpler Positivity Theorem 11.4B.

You have to find and interpose a "

buffer" $g(x)$ – a function such that $\int_a^b g(x) > 0$ and $f(x) \geq g(x) \geq 0$.

Use $>$ and $\geq$ carefully; and no circular reasoning – justifying a step by using as the reason for it the very theorem you're trying to prove! The function $g(x)$ has to be one whose integral is known without calculation.

Reading Wed.: 19.3-.4; 20.1-.4 Riemann Sum and Inequality Theorems for integrals.

The two Fundamental Theorems of Calculus; Integration techniques; $\ln x$ and e^x.

Problem 10. (2: .5, 1.5) Prove the following statement:

If $f(t)$ is defined for all t and integrable on every compact interval, then $F(x) = \displaystyle\int_a^x f(t)\,dt$ is continuous for all x.

a) What's wrong with: By the Second Fundamental Theorem, $dF/dx = f(x)$, and a differentiable function is continuous. (Thm. 14.1)

b) Prove $F(x)$ is right continuous at every point x_0.
(Use a drawing like that for the 2nd F.T., and the basic $\delta - \epsilon$ def'nn of continuity in 11.1.)

Problem 11. (2: .5 each) Just some quick Calculus: the first two by change of variable.

a) Evaluate $\displaystyle\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\,dx.$

b) Using the definition $\ln x = \int_1^x 1/x\,dx$, prove $\ln(ab) = \ln a + \ln b$.

c) Evaluate $\displaystyle\int_0^1 \tan^{-1} x\,dx$ (write $(\tan^{-1} x)\cdot 1$, use integration by parts).

d) Get the estimate $\displaystyle\int_0^x \frac{dt}{1+t^2} < 2$ for all $x > 1$, by breaking the interval at t=1 snd estimating the integral over the two pieces separately.

Problem 12. (2) Work 20.3/3. This is an exercise involving integration by parts.