18.100A Fall 2018: Problem Set 6 due Fri., Oct. 19

Directions: Same as before. This is complete now, due Oct. 19.. List collaborators on top left front page, if any, but write up independently. It is illegal to consult P-set solutions from previouos semesters or internet solutions. Cite significant theorems used.

Students are still losing credit by working the wrong problem from the book; notation is: Q14.1/2 (Question); 14.1/2 (Exercise) P14-2 (Problem).

Reading: Fri. 10.4, 14.1-.3 Local, global, and pointwise properties of functions.

Differentiation: Local Properties. Derivatives, formulas, Simple local-to-local theorems, critical and extremal points.

Problem 1 (1) Work 10.4/1 b,c,d,f .

Put your one-word answer in parentheses if it uses the Controversial Remark (p.147).

Problem 2 (2; .5, 1.5) Work P10-2 in two steps:

a) Let $f(x)$ be defined on the interval I. Express "*$f(x)$ is not a constant function on I*" as an equivalent affirmative statement (i.e., one not using the negative words " not, no, non-" or hidden synonyms (like "false" for "not true").

Use standard phrases like *"for all, given, there exist(s)"*, and make the affirmative statement as concise as possible. Begin it by: "*There exist points $a_1 < b_1$ such that ....*"

b) Give a concisely written indirect proof of P10-2 by assuming $f(x)$ is non-constant on I, expressed as in part (a), and using bisection rather than what's suggested in the book.

Aim for conciseness: four or five lines in normal-sized writing should be enough. Just tell what to do for this problem: at each successive step, which half-interval to pick and why it is possible to do so, and how the point x_0 in all the intervals gives a contradiction.

Use scratch paper, omit unnecessary English, use symbols (but please not the ones for *"for all, there exist"*).

Problem 3. (2: 1,.5,.5) Work the three parts of P14-1..

Include for part (a) a graph of $f(x)$ showing a, the two Δx lengths, and the geometric interpretation it gives for $F(\Delta x)$. Give a graph also of your example for part (b).

Problem 4. (1) Work 14.3/1, making the lattice diagram and following the directions. For any false statement, give a counterexample.

Problem 5. (1) Work 14.3/2a, using as interval $(-1, \infty)$ and the isolation principle of 14.3C for the reasoning. (Do not use other standard calculus tests as in Chapters 15, 16.)

Problem 6. (4) The functions $x^k \sin(1/x)$ for $k = 0, 1, 2, \ldots$ are a useful source of examples and counterexamples in elementary analysis.

They are defined, continuous, and differentiable when $x \neq 0$, according to Theorems 11.4A and D and the differentiation formulas 14.2A and B. So the interest centers on their properties at or near the point $x = 0$.

In what follows, complete the domain of $x^k \sin(1/x)$ and $x^k \cos(1/x)$ by defining their value at $x = 0$ to be 0.

Part of what follows is done in Examples in the book, but using the above definition for $x = 0$ simplifies somewhat the arguments in the book. Try to do them without referring to the book; if you do look, don't mindlessly copy what's there.

Prove in turn the following, by using limit theorems, the definition of derivative, the differentiation rules, sequential continuity, etc. In general, use if possible the limit form (11.4) for the definition of continuity at a, rather than the basic ϵ−form (11.1).

For each part you can use if needed the results in previous parts. In each case, draw a sketch of the function illustrating its behavior near 0 (cf. p. 154).

a) Prove $\sin(1/x)$ is discontinuous at 0, by using the sequential continuity theorem. (You can use also the same fact for $\cos(1/x)$ in the later parts of this problem.)

b) Prove $x\sin(1/x)$ is continuous at 0, but not differentiable at 0.

c) Prove $x^2\sin(1/x)$ is differentiable at 0, but its derivative is not continuous at 0.

d) Show $|x^2\sin(1/x)|$ is not differentiable in any neighborhood of 0. (Make a sketch, with the location of relevant x-points given and a statement about what they show.

Reading: (Mon.)

15.1-.2 Mean-value Theorem (MVT); geometric and approx'n forms, applications.

15.4 L'Hospital's rule for 0/0: Elementary case 15.4A, General case 15.4 B; ∞/∞; We're skipping the proof of 15.4B and section 15.3 (used only in the proof of 15.4B.)

Problem 7. (2) Work 15.2/1ab (uses the MVT in the approximation form)

Problem 8. (2: .5,.5,1)

(i) Using either the approximation form (6) or the geometric form –(1a), with b changed to x)–give the equation relating x and c and draw the sketch of $\ln(1+x)$ for $x \geq 0$ that illustrates it.

(ii) Work 15.2/2a, using the equation of part (i).

(iii) Work 15.2/2b, using the equation again; your sketch should illustrate the truth of the result.

Problem 9. (1) Work 15.4/1, to prove the elementary case of L'Hospital's rule.

Problem 10. (2: .5,1,.5)) Use L'Hospital's rule, show work; for (b), trig identities save differentiating toward the end.

$$a)\ \lim_{x\to 0}\frac{\sqrt{1+x}-1}{\sin x} \qquad b)\ \lim_{x\to 0}\frac{x-\tan x}{x-\sin x} \qquad c)\ \lim_{x\to\infty}\frac{\ln x}{x^\epsilon},\ \epsilon>0.$$

Problem 11. (2) Work P15-1ab.

Reading Wed. 16.1 Linearization error theorem and proof.

16.2 pp.225-6 (to statement of Thm.16.2C) App's to convexity and concavity of functions

Definition 16.2A of convexity and concavity is the usual one, but to save time and effort, we will assume $f(x)$ is differentiable, which gives an equivalent definition that uses Theorem 16.2C:

$f(x)$ is **convex** (rsp. **concave**) on I $\iff$ $f'(x)$ is increasing (rsp. decreasing) on I.

Problem 12. (2) Work 16.1b, which uses The Lin'zn error Thm. 16.1B to add an x^2 term to the estimates in Problem 8 above, which only use a linear x term for upper and lower estimates. The upper quadratic estimate should have the same form as the lower one this exercise asks for, and be the best possible using Theorem 16.1B .

(Some older printings have an error in this problem: it should read: $\ln(1+x) > x - x^2/2$.)

Problem 13. (2: .5, 1.5) Suppose $f'''(x)$ exists on $[a, b]$, and $f(a), f'(a), f(b), f'(b)$ all $= 0$.

Prove $f'''(c) = 0$ for some $c \epsilon (a, b)$.

a) What's wrong with just using Lemma 16.1 as stated to produce c_1, where $f''(c_1) = 0$, switchng the roles of a and b to get a similar point c_2, then using c_1, c_2 and $f''(x)$ to get c?

b) Give a correct proof.

Problem 14. (1)

a) Sketch the graph of $\ln(1 + x)$ over [0, 2] say, and draw in the secant line OP, where P is a point on the graph. Show by the geometric reasoning based on the Mean Value Theorem that the slope of this line is a strictly decreasing function of x. (We called this "geometically clear" in the first class of the semester.)

b) Assume $f(x)$ satisfies $f(0) = 0$, $f(x)$ is differentiable and strictly increasing and concave for $x \geq 0$, and let $P : (x, f(x))$ as above be a point on its graph. Prove analytically (i.e., by calculating with $f(x)$) that the slope of OP is a decreasing function of x.