18.100A Fall 2018: Problem Set 5 due Fri. Oct.12

Directions: As previously. Put your name on top right; list any collaborators on top left; write up independently. Consulting P-sets from previous semesters is illegal.

Reading Fri: 13.1-.3 Compactness; Boundedness and Extremal-Value Theorems:
If $f(x)$ is a function continuous on a compact interval $I = [a, b]$, then it is bounded and has extremal points (i.e., maximum and minimum points) on I.

Note: A maximum point $\bar{x}$ is one satisfying $f(x) \leq f(\bar{x})$ for all $x \in I$; for a minimum point $\underline{x}$, use $\geq$ instead. As examples, for $f(x) = \sin x$, all $\bar{x} = \frac{\pi}{2} \pm 2n\pi$ are maximum points; all $\underline{x} = \frac{3\pi}{2} \pm 2n\pi$ are minimum points. For a constant function $f(x) = k$, every x is both a maximum and a minimum point.

Distinguish carefully between: a maximum point $\bar{x}$ (which lies in I on the x-axis), the corresponding maximum value $\bar{m} = f(\bar{x})$, and a "local maximum" point x_0 (or "relative maximum" point), which is a maximum point for $f(x)$ in some δ-neighborhood of x_0.

Problem 1. (2: 1.5, .5) Work 13.1/1a,b.
Dirichlet's "drawer principle" says that if you distribute $n+1$ balls among n drawers (or "pigeon-holes" or "cubby-holes"), at least one drawer must contain two balls.
It has analogous formulations for $N > n$ balls, or for an infinite number of balls.
Use one of these variants to give a simple direct proof of 1a: – start with an arbitrary sequence $\{x_n\}$ of points in S, and show that it has a subsequence converging to a point in one of the finite closed intervals that make up S.

Problem 2. (2: .5,.5, 1) (i), (ii): Work 13.2/1a,b and (iii): 13.3/3a
These give a geometric interpretation and application of the two theorems.

Problem 3. (3: 2,1)
Let $I = [0, \infty)$. Assume $f(x)$ is continuous and > 0 on I, and $\lim\limits_{x\to\infty} f(x) = 0$.
Prove that $f(x)$ has a maximum point on I.

The point of this problem is that I is not sequentially compact, so the Extremal Value (Maximum) Theorem is not immediately applicable. You have to get around that somehow.

a) Use $f(0)$ to divide I into two pieces: a compact interval $[0, a]$, and an infinite "tail" interval $[a, \infty)$ on which the maximum point $\bar{x}$ cannot lie, then finish the proof.

b) How would you modify the argument if $f(0) = 0$, but the other hypotheses were unchanged? Give a brief reason why your modification will work; you don't have to repeat the whole argument.

Problem 4. (3) You can deduce the Maximum Theorem from the Boundedness Theorem by a different argument than the one given in the textbook as the proof of Theorem 13.3. This alternative argument is indirect, but gives a shorter proof.

Assume $f(x)$ is continuous on the compact interval I but has no maximum point $\bar{x}$. Derive a contradiction by proving that if this were so, the following function would then be a counterexample to the Boundedness Theore m: $\dfrac{1}{\overline{m} - f(x)}$, where $\overline{m} = \sup\limits_I f(x)$.

Reading Wed.: 12.1, 12.2 (omit Example 12.2B) Bolzano's Theorem, Intermediate Value Theorem, Intersection Principle

Problem 5. (2: .5, 1.5)

a) Prove that if $f(x)$ is continuous and strictly monotone (i.e., increasing or decreasing) on $[a, b]$, the equation $f(x) = k$ has exactly one solution for k in $[f(a), f(b]$ (or $[f(b), f(a)]$).

b) Let $f(x) = \dfrac{\ln x}{x}$; find with proof the number of solutions to $f(x) = k$ for every $k \geq 0$. Assume $\lim_{x\to\infty} f(x) = 0$; use calculus and part (a).

Problem 6. (3: 2,1)

a) Prove that the function $f(x) = x - \tan x$ has a strictly increasing sequence of positive zeros $x_1, x_2, \ldots, x_n, \ldots$.

(Use the Intersection Principle; in verifying that its hypotheses are satisfied you can treat ∞ informally like a real number. Include a reasonably careful sketch of the relevant graphs.)

b) If N is large, approximately how many positive zeros $\leq N$ does $f(x)$ have? Indicate your reasoning briefly.

Problem 7. (3) For what positive value of the parameter A will $f(x) = x - A\sin x$ have exactly 4 positive zeros?

Express this value of A in terms of the x_n of Problem 6.

(No formal proof is required, but show your reasoning and calculations.)

Problem 8. (2) Work P12-3 (Problem 12-3), proving the existence of a (horizontal) chord for the graph of a function satisfying certain conditions specified in the problem.

(Try making some sketches on scratch paper. If stuck, see the suggestion below.

A chord is a horizonal line segment that must satisfy several conditions.

A good way of proceeding when trying to prove that such a mathematical object exists is to weaken the hypotheses by dropping in turn just one of the conditions it must satisfy and seeing whether the resulting object exists and what it looks like.

Doing this for each condition in turn can reveal if there is something that can satisfy all of the conditions simultaneously.