18.100A Fall 2018 Problem Set 4 (due Fri. Oct. 5)

Directions: Write dark enough, put your name on the top right. List collaborators on the top left; write up solutions independently; cite significant theorems used by name or number. Consulting P-set solutions from previous semesters is not allowed.

Reading Fri.: 6.5 Set-language, Completeness Principle for sets.

Problem 1. (2) Work 6.5/1a,c

Problem 2. (2) Work (i) 6.5/3b (ii) 6.5/3g.

Arrange the work on successive lines, providing a reason in parentheses for each, if needed, like: sup-2 for A (or an quivalent phrase using lub (least upper bound) for A).

Conventions : capitals A, S, etc. denote sets; small letters a, s, etc. elements of the corresponding set: $a \epsilon A$, $s \epsilon S$, etc.; the phrases "each a, every a, all a" have the same meaning. As an example, a proof of 6.5/3a could go

every $a \epsilon B \Rightarrow a \le \sup(B)$	(sup-1 for B) or (sup(B) is an u.b. for A);
$\Rightarrow \sup(A) \le \sup(B)$	(sup-2 for A) or (sup(A) is the l.u.b. for A).

This will help with (i); for (ii), show first that $\sup(A) + \sup(B)$ is an upper bound for all $a + b$ – four lines with the $\le$ signs lined up in a neat column are all that's needed (with the reason for each on the right side).

Problem 3. (1) Medieval War in Lineland: Work 6.5/4.

Think of S and T as opposing armies on the real line, each led by their respective generals: $\sup(S)$ and $\inf(T)$. To start you off, make the first line (it needs no further reason):

each $s <$ all $t \Rightarrow s$ is a l.b. for T.

Finish the proof in three more lines, two with reasons in parentheses.

Problem 4. (2) Work P6-2a. (This is an important problem).

Warning: The most common faults with the sequence s_n being constructed are that the sequence may not be in S, or that it may not have $\sup(S)$ as its limit.

The construction should describe what to do, say why it is possible —–to do it, and why the resulting sequence has the desired properties.

(The Theorem 6.2 construction of a subsequence converging to a cluster point of the sequence should give you the right ideas. Here you will be constructing a sequence, but it is not a subsequence of a given sequence; this changes the method of construction a little and the proof that it is actually possible.)

Problem 5. (1) Work P6-2b, by using the results in Problem 4 and Problem 2 (ii). You only need to prove the reverse inequality: $\sup(A + B) \ge \sup(A) + \sup(B)$.

Do this by using combining the sequences $a_n \to \sup(A)$ and $b_n \to \sup(B)$ given by Problem 4, and applying the Limit Location Theorem (make clear how it is applied).

The topic for the next four classes will be continuity – the study of continuous functions in Chapters 11, 12, and 13, which are central to analysis. Sequences will not be abandoned, since they are important in the continuity proofs

. Two preliminary Chapaters 9 and 10 are mostly a reading assignment, reviewing some facts about functions, and establishing the notation used for limits of functions, which is like that for limits of sequences.

Reading Mon.

9.1-.3 Def'n and properties of functions, (The def'n "mapping" may be new to some.)

10.1-10.2 Estimation and approximation for functions.

11.2-.4 Limit Definition and Limit Theorems for functions (like the ones for sequences). The Limit Definition of Continuity. (We take Theorem 11.4A as a definition of continuity, so there is nothing to prove.)

Problem 6. (3: .5,,5,1,1) a) Work 9.3/3

(b) Work 10.1/1 (to get familiar with the notation $f(I)$, where I is an interval on which $f(x)$ is defined, and $f(x)$ is thought of as a map: $I \to \mathbf{R}$).

(c) Work 10.1/6b (needs 18.01 calculus)

(d) Work 10.1/7b (Express boundedness using absolute values, as in (2), section 10.1.)

Problem 7. (2) Work 10.3/2

(Use scratch paper to organize and write up the ideas in logical order.)

Do not use an indirect argument – argue directly.

Focus on the conclusion: i.e., start with an arbitrary x_0. What is it you wish to prove about $f(x_0)$? (And note the hint in Prob. 1d above.)

Problem 8. (2) Let $f(x) = \tan(1/x),\ x \neq 0; f(0) = 0$. Is $f(x)$ defined for $x \approx 0$? Prove your answer.

Problem 9 (1)

Work 11.1/4, using the limit definition of continuity (in section 11.4) and a simple algebraic limit theorem, to prove e^x is continuous at every point x_0, assuming it is continuous at 0. (Write $x = x_0 + h$ and use the exponential law)

Problem 10 (2: 1, .5, .5) Work 11.3/1a; 1b (assume $\cos x$ is continuous at 0), and 11.3/2 .

Problem 11. (1) Assume $f(x)$ is increasing on (a, b) and continuous at b. Prove it is increasing on $(a, b]$.

(Let $a < x < b$; what has to be proved about $f(x)$? Which of the limit theorems will prove it? Make clear how it applies.)

Reading Wed. 11.1, 11.5 The Basic Definition of Continuity. Sequential Continuiity Theorems.

The Basic Definition given in 11.1 is the original one; it is equivalent to the Limit Definition given in 11.4A – the proof of equivalence takes only one line. Unfortunately, using it as the definition often produces awkward or incorrect proofs.

Use instead the Limit Definition and the limit theorems if possible.

Problem 12. (2) Work 11.1/6.

This is an example of where the limit definition of continuity cannot be used, since x is inside the integral sign. You have to use the basic definition of continuity, as in the 11.1 example.

Follow the hint given, or use the result in Problem 10 above, part c) 11.3/2.)

Problem 13. (2: 1,1) These last two problems use sequential continuity and the limit definition of continuity.

a) Given any real number c, prove there is a sequence of rational numbers a_n and a sequence of irrational numbers b_n, both of which are increasing and have c as their limit.
Your proof must use Theorem 2.5, not infinite decimals.

b) Using part (a), prove that if two functions $f(x)$ and $g(x)$ are continuous on $\mathbf{R}$ and agree on all rational points, i.e., $f(a) = g(a)$ whenever a is a rational number, then $f(x) = g(x)$ for all $x \in \mathbf{R}$.

Problem 14. (2) Work P11-2 (Problem 11-2).

Hint: if $f(x)$ is a constant function, wh0at must its constant value be?
Start with an arbitrary x, and show $f(x)$ has that value. Assume that $c < 1$ for the minimal positive period c.