18.100A Fall 2018 Problem Set 3 (due Friday, Sept. 28)

Directions: Same as on previous problem sets: list collaborators on the left top page and write up solutions independently; cite significant theorems used by name or number; consulting P-set solutions from previous semesters is not allowed.

Class Note: Thm. 6.4: $\{a_n\}$ *is a Cauchy sequence* $\Rightarrow$ $\{a_n\}$ *converges.*

More detailed proof of Part(C) in book:

(1) $\quad a_m \underset{\epsilon}{\approx} a_n$, say for $m, n \geq N$; (def'n of Cauchy seq.);

(2) $\quad a_{n_i} \underset{\epsilon}{\approx} L$, say for $i >$ an integer $I > N$; (def'n of limit);

Then $a_n \underset{\epsilon}{\approx} a_{n_I} \underset{\epsilon}{\approx} L$ by (1) and (2), showing $a_n \underset{2\epsilon}{\approx} L$ for $n >> 1$, since $I > N \Rightarrow n_I \geq I > N$ (by the Subsequence Lemma: $n_i \geq i$ in the P-set 2 Note.)

Reading: Mon.: 7.1-.2, 7.4 Infinite series; convergence and divergence tests for non-negative series.

Problem 1. (1) Work 7.2/1. (The sum is a famous discovery by Euler.) Cite Theorems used.

Problem 2. (2) Work 7.2/2. Cite Theorems, Definitions used.

Problem 3. (2) Work 7.2/5, a series theorem involving sequences, subsequences and convergence.

Problem 4. (2; 1.5, .5)

(a)Prove the n-th root test for convergence (Theorem 7.4B), for the case $L < 1$ only, with the added assumption that the series are non-negative (so the absolute-value signs are not needed in the statement or the proof).

Study the proof of the ratio test first (again assuming the series are non-negative).

(b) Prove the test for the case $L > 1$ without using the "buffer" M.

Problem 5. (2: .5, .5, 1) In this problem, use any of the tests in section 7.4, but omit the absolute value signs, since the three series involved are all positive. Show enough work so a reader can see how the tests are being used. Read the first two lines of 7.4-.5/1.

Work in 7.4-7.5/1 the Exercises: i) 1b ii) 1e (iii) 1h

Reading Wed: 7.3, 7.5-6, 8.1-2 to end of p.117; Abs. and Cond. convergence, 3 more tests; power series.

Focus mainly on the statements and use of the Theorems, not the proofs; cf. alternative proofs below. Omit proof of theorem 8.1 for now.

Problem 6. (2: .5,1.5) Another proof of the Absolute Convergence Theorem 7.3

a) What condition on the terms of a series $\sum_0^\infty a_n$ is equivalent to saying its partial sums $\{s_n\}$ form a Cauchy sequence? Show that it is the following condition, called the "Cauchy criterion for series convergence":

(1) $$\text{given } \epsilon > 0, \quad |a_{n+1} + \ldots + a_m| < \epsilon \quad \text{for } m > n \gg 1 .$$

and prove that a series converges $\iff$ it satisfies the Cauchy criterion (1). You can use:

A sequence converges $\iff$ *it is a Cauchy sequence.* (Theorem 6.4 and Exercise 6.4/1)

b) Work P7-5, which uses part (a) to provee the Absolute Convergence theorem 7.3

Problem 7. (2: 1,1)

(a) Prove the Asymptotic Comparison test (Theorem 7.5B), assuming the series are non-negative (so you can drop the | | signs – compare one series with twice the other.)

(b) In using the Asyptotic Comparison test, the most important comparison series are the ones in Example 7.5A, whose convergence is established by the integral test. (The more commonly used ratio test fails for all of these.)

Test the following for convergence: (show work or reasoning)

$$\text{(i)}\ \ 7.4/1\text{a} \qquad \text{(ii)}\ \sum_1 \frac{n-5}{\sqrt{10n^4+n^5}}$$

Problem 8. (2: .5, 1.5)

Around 1740 Euler proposed the problem of determining whether the series of prime reciprocals converges or diverges:

$$\sum_1^\infty \frac{1}{p_n} = \frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ldots+\frac{1}{p_n}+\ldots\ ; \qquad p_n = \text{the } n\text{-th prime}$$

Prime Number Theorem (PNT): $\qquad p_n \sim n \ln n$,

where the symbol $\sim$ (read: “is asymptotic to” or “twiddles”) has the meaning given in the Asymptotic Comparison test 7.5B.)

a) For $n = 10$, find to the nearest integer the numerical value of the left and right sides of the PNT. (It was not proved until around 1900, around 150 years after Euler.)

b) Use the PNT to determine (with proof, and citing theorems) whether the series of reciprocal primes converges or diverges.

Problem 9. (1) Work 7.6/1a,b

(For conditional convergence, you have to show two things.)

Problem 10. (2) For each of the following power series,

find its radius of convergence R, showing work;

determine with proof whether it converges or diverges at the endpoints $x = \pm R$;

(for (b) do this for just one of the endpoints).

Identify the test being used.

$$a)\ \sum_1^\infty \frac{x^n}{\sqrt{2^n n}} \qquad b)\ \sum_1^\infty \frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!}\,x^n$$

Problem 11. (1) Work 8.1/3ab.

This gives a proof of the Radius of Convergence Theorem 8.1 for practically all commonly occurring power series.

Problem 12. (1) Work P8-1 (Problem 8-1 p.124), by determining its convergence for $|x| < 1$ and for $x = 1$, and applying Theorem 8.1 to the results. Further hints are also given in P8-1, if needed.