18.100A Fall 2017 Problem Set 2 (complete) due Mon.. Sept. 24

Directions: You can collaborate, but should list those you worked with and write up the solutions independently (i.e., not copying but thinking them through by yourself). List collaborators on the top left, put your name on the top right.

Consulting solutions to problem sets of previous semesters is not allowed.

Read: Fri. Chap. 5.4, 6.2 Subsequences; non-existence of limits; Cluster Points.

Friday Lecture Note:

The two lines above (18) in the proof of the Subsequence Theorem are a simple but important fact about subsequences, and turned into a Lemma:

Subsequence Lemma $\{x_{n_i}\}$ is a subsequence of $\{x_n\}$ $\implies$ $n_i \geq i$ for all i.

Proof by induction: Assume $i \geq 0$ and call the above statement $P(i)$.

$P(0)$ is trivial; $P(i) \Rightarrow P(i+1)$: $n_{i+1} > n_i \geq i \ \Rightarrow \ n_{i+1} \geq i+1$;

the 3 inequalities use the definition of subsequence, $P(i)$, and the strictness of $<$.

Problem 1. (3: 1, 1, 1) Part (a) is a special case of (b), but uses simpler notation and is a good preliminary exercise.

a) Work 5.4/1a;

b) Work 5.4/1b..

c) Work 5.5/1, an application of part (b).

Problem 2. (2) Work 6.2/1b . The sequence is $x_n = \sin[(n + \frac{1}{n})\frac{\pi}{2}]$.

Problem 3. (3)

a)Work 5.4/2, following the ideas in section 5.4; (the primes are 2,3,5,7,11,13,17, ...).

b) Prove that all the numbers $1/p$ (p prime) are cluster points of this sequence, and find two more cluster points (with proof).

Read: Mon. Chap. 6.1, 6.3 Other forms of the Completeness Principle: Nested Intervals and Bisection, Bolzano-Weierstrass Theorem

Problem 4 (2)

A sequence of nested intervals is constructed by taking $[0, 1]$ as the starting interval $[a_0, b_0]$, bisecting it, and choosing the left half as the next interval $[a_1, b_1]$; then bisecting this in turn and choosing its right half as $[a_2, b_2]$; then continuing in this way, alternately choosing the left half and the right half of the bisected previous interval.

What unique number lies inside all these intervals? Prove it.

(Suggestion: obtain it as the limit of a_n; express a_n not as a single rational number, but rather as the partial sum of an infinite series.)

Problem 5. (3) Work 6.3/1, for the following sequences $\{b_n\}$; assume in each case that the sequences a_n are limited to those for which b_n is defined for all n.

$$(a)\ b_n = \sin(a_n) \qquad (b)\ b_n = \tan(a_n) \qquad (c)\ b_n = \tan^{-1}(a_n) \qquad (d)\ b_n = \frac{1}{1+a_n^3}$$

Problem 6. (2) Suppose all the terms of the sequence x_n lie in the interval $[a, b]$. Prove it has a subsequence converging to a point in $[a, b]$. Cite theorems being used from Ch. 1-6.

Reading Wed. Chap. 6.4 Cauchy sequences.

Problem 7. (3: 1,2)

a) Work 6.4/1

b) Work 6.4/3 . How do you know it is not a Cauchy sequence?
(To estimate the difference between two successive terms, the identity

$$(A - B)(A + B) = A^2 - B^2$$

is helpful; the estimation it gives can be further simplified.)

Problem 8 (2) Work 6.4/2 (this will be used in Problem 9).

Problem 9 (3) Work P6-1ab (Problem 6-1 on page 91) taking $a = 0,\ b = 1$.

For part (a) of P6-1, you can use Example 6.4 (9) and (10), and the result in Problem 8. The calculations are then applicable to part (b) of P6-1.

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