18.100A Fall 2018: Problem Set 12 optional: not graded or handed in

Directions: This problem set and reading covers the last three classes, and is optional. One problem on the final will assume you have studied it, but there will be a lot of choice on the final so you can avoid the problem. Solutions will be e-mailed.

Reading: Chapter 23.1-.3 Countable and uncountable sets, cardinal numbers, Cantor set.
Sets in **R** of measure zero; characterization of Riemann-integrable functions; application to sequences of discontinuous functions.

Definition: A map $f(x): S \to T$ between two subsets S and T of **R** is said to be:

injective, or a map of S **into** T, if $s_1 \neq s_2 \Rightarrow f(s_1) \neq f(s_2$;

surjective, or a map of S **onto** T, if for each t there is an s such that $f(s) = t$;

bijective, or a one-to-one map of S to T, if it is both injective and surjective;
sometimes called a "one-one correspondence of S with T".

These terminologies will be used in the solutions; occasionally ESL speakers will try "onjective" with a rising inflection – look thoughtful and nod imperceptibly.

1. Work 23.1/4b

2. Work 23.1/7ab

3. Work 23.1/10

4. Work 23.2/5b

5. Let $x_r = a/2^k$ for all $k \geq 0$, and $a = 1, 3, 5, \ldots < 2^k$.
Thus x_r represents all rational numbers in the interval [0, 1] having only a power of 2 in the denominator, and a numerator not divisible by 2 — i.e., "written in lowest terms".

The ruler function $f(x)$ is defined for $0 \leq x \leq 1$ by

$$f(x) = \begin{cases} 1/2^k, & \text{for } x = x_r, \text{ as defined above,} \\ 0, & \text{otherwise.} \end{cases}$$

The function $f_n(x)$ on [0, 1] is defined similarly, but limits the rational numbers x_r :
denominators must use only $k \leq n$ — i.e., requiring $2^k \leq 2^n$.
Thus each successive $f_n(x)$ keeps the non-zero $f(x)$-values of the previous ones, but promotes the new values $f_n(a/2^n)$ from 0 to $1/2^n$, off the x-axis.

a) Draw the graph of $f_3(x)$, using not just points for the non-zero values, but connecting them to the x-axis by vertical lines of the correct relative length to make them more conspicuous, more like the rulings on a ruler. (The scales on the x and y axes need not be the same.)
On the same graph, draw the graph of $f_4(x)$ by adding the new points – indicating them by tiny circles instead of black dots.

b) On the interval [0, 1], clearly $f_n(x) \longrightarrow f(x)$ by pointwise convergence, but $f(x)$ can only be imagined, not graphed.
Using theorems in 23.2, 23.3, and 22.4, prove that $f(x)$ is Riemann-integrable on $[0, 1]$ and calulate its integral over 0, 1].
(You can also use the result in 18.3/1 on one of the problem sets.)