18.100A Fall 2018: Problem Set 11 due Fri. Dec. 7

Directions: List collaborators; illegal to consult solutions from previous semesters .

Reading Fri.: 25.3 Open Sets: def'n, ways to construct and recognize them.
Notes on Open and Closed Sets – see p.3 here) Used in Probs. 3,4 below.
A way to define them that clarifies "*S open $\iff$ S' closed*" and helps recognition.

1. (2) Work 25.1/4a,b (These use Thms. 25.1A,B and 25.3A,B).
Make a sketch of both sets and give your reasoning.

2. (3: 1,2) Work 25.2/5, one of the important facts about compact sets.

Let $S \subseteq U$ be a compact subset of an open set U in $\mathbf{R}^2$, and $f(\mathbf{x})$ be a function continuous on U mapping U to $\mathbf{R}^1$. Then its image $f(S)$ is a compact subset of R^1.

a) Compact sets in $\mathbf{R}^2$ are characterized by Theorem 25.2 as the sets which aree closed and bounded. This suggests trying to prove the above theorem by "divide and conquer": proving separately that using the above notation,

$$S \text{ bounded} \Rightarrow f(S) \text{ bounded} \qquad \text{and} \qquad S \text{ closed} \Rightarrow f(S) \text{ closed} \;.$$

Alas, show by two counterexamples using the functions $f(\mathbf{x}) = x$ and $f(\mathbf{x}) = 1/x$ on suitable open sets U that both statements are false. (Problem 1 may help.)

b) Prove the theorem instead by going back to the definition of compact set.

Focus on the conclusion of the Theorem: to prove $f(S)$ is compact, what you are trying to show about a given sequence $\mathbf{x}'_n$ of points in $f(S)$? How can the hypotheses about f and S help you do this?

3. (3; .5 for each) Work 25.3/1a,d,g,h,i,j, in conjunction with Problem 25-1 and the Notes on Open and Closed sets.

For each of these six sets, draw a sketch of the set, describe its boundary points, tell whether it is open, closed, compact, or none of these, and give a brief reason (you don't have to go into as much detail as in Problem 1, for example..

4. (1) The Notes on p. 3 show how the definition given there for open sets is equivalent to the definition given in Section 25.3 of the book.

Do the same for closed sets: looking at the definitions of closed set in the Notes and in Section 25.1. What has to be proved is: (use Definition 1a of cluster point)

S contains all its cluster points $\iff$ S contains all its boundary points;

which will follow from the even stronger statement

If $\mathbf{x}$ is a point not in S, $\mathbf{x}$ is a cluster point $\iff$ $\mathbf{x}$ is a boundary point .

Reading: Mon. 26.1-.2, to top p.379
Functions defined by integrals with a parameter: continuity, differentiabiliy

5. (2) Work 26.1/1b, in reverse order:
(i) first find the largest x-interval on which the Continuity Theorem 26.1 predicts the integral will be continuous;
(ii) then use this information to determine the limit asked for. Cite relevant theorem(s) for the limit.

6. (3: 1,2) Let $\phi(x) = \int_0^\pi \sin(xt)\,dt$.
a) The Continuity Theorem 26.1 says $\phi(x)$ is continuous for all x; verify this.
First, calculate $\phi(x)$ explicitly for all x, including $x = 0$ (no limits needed for this).
Then verify it is continuous for all x; use the limit form (11.4A) when $x = 0$.

b) The Derivative Theorem 26.2A says $\phi'(x)$ exists for all x and gives a formula for it.

(i) Use the formula to calculate $\phi'(x)$ explicitly for all x, including $x = 0$, using standard integration techniques; (no limits are needed for this).
Verify your calculations for $x \neq 0$ by using part (a) to calculate $\phi'(x)$, $x \neq 0$.

(ii) The Continuity Theorem predicts $\phi'(x)$ will be continuous for all x: why?
Verify this using (i), again using (11.4A) to prove continuity at $x = 0$.

7. (3) Work P26-1, showing that $J_0(x) = \frac{1}{\pi}\int_0^\pi \cos(x\,\sin\theta)\,d\theta$ is the solution to Bessel's ODE satisfying the given initial conditions.
(Since the IVP has a unique solution, this gives another proof that $J_0(x)$ in its integral form and in its power series form represent the same function.)

You will need to differentiate $J_0(x)$ twice; verify hypotheses for Thm.26.2A each time, and verify the initial conditions are satisfied.

After substituting into Bessel's ODE and combining the integrals, you get a very big definite integral, which is supposed to have the value 0.
There are different ways to show this, using theorems in Chapter 20. If stuck, try sleeping on it and seeing if what to do or try is clearer in the morning.

Reading Wed. 26.2, 26.3 Leibniz formula; Fubini's theorem.

8. (3: 1.5,1.5) Work 26.2/2 two ways:
a) using the Leibniz formula (26.2B) (verify hypotheses);
b) in a more elementary way by setting $u = x - t$; cite theorems; verify hypotheses.

9. (2) Work 26.2/5.

10. (3) Work 26.3/2, which gives another way to prove the Derivative Theorem 26.2A, by using Fubini's theorem.
Only a few lines and a few theorems are needed; cite them as you use them.

Notes on Open Sets and Closed Sets.

On p. 348 are defined the two types of δ-disks having center $\mathbf{a}$ and radius δ:

the *open* disk $D(\mathbf{a}, \delta)$, which *excludes* its boundary circle;

the *closed* disk $\overline{D}(\mathbf{a}, \delta)$, which *includes* its boundary circle .

(The open disk is referred to as a *δ-neighborhood of* $\mathbf{a}$.)

We now want to extend these notions of open and closed to more general sets $S \subset \mathbf{R}^2$. We denote by S' the *complement* of S in $\mathbf{R}^2$ — the set of all points of $\mathbf{R}^2$ not in S.

Definition. The point $\mathbf{a}$ is a **boundary point** of S if every δ-neighborhood of $\mathbf{a}$ contains an $\mathbf{x}_1 \in S$ and an $\mathbf{x}_2 \in S'$.

We will denote by $\partial(S)$ the set of boundary points of S.

Since the definition treats S and S' symmetrically, it follows that S and S' have the same boundary points:

$$\partial(S) = \partial(S') .$$

Definition. $S \subseteq R^2$ is **open** if $\partial(S) \subseteq S'$; it is **closed** if $\partial(S) \subseteq S$.

Thus, an open set is one containing *none* of its boundary points; a closed set is one containing *all* of its boundary points. (This checks out for the open and closed disks above.)

Theorem. S is open $\iff$ S' is closed.

Proof. Since $\partial(S) = \partial(S')$, it follows that $\partial(S) \subseteq S' \iff \partial(S') \subseteq S'$.

Most sets are neither open nor closed – they contain some, but not all, of their boundary points. So you can't prove a set is closed by showing it is not open. The above theorem is the only useful relation between open and closed sets.

An equivalent way of stating the definition of open set is this: a set is open if every point $\mathbf{a} \in S$ is *not* a boundary point, i.e., $\mathbf{a}$ has a δ-neighborhood not containing any points of S':

Definition. $S \subseteq R^2$ is **open** if for every $\mathbf{a} \in S$, there is a δ-neighborhood $D(\mathbf{a}, \delta) \subset S$.

This doesn't mention boundary points and is often easier to verify.

Examples. In the Euclidean plane $\mathbf{R}^2$,

1. any finite set S of points is closed, since S' is open: any point $\mathbf{a} \in S'$ has a minimum distance r to the points of S, so if $\delta < r$, $D(\mathbf{a}, \delta) \subset S'$;
2. a line segment I (including its endpoints) is closed, since its complement I' is open, by similar reasoning to that in 1.;
3. an infinite line in $\mathbf{R}^2$ with one point removed is neither open nor closed.

There are two subsets of $\mathbf{R}^2$ which are special by virtue of having *no* boundary points: $\mathbf{R}^2$ itself, and the **null** (or empty) set $\emptyset$. According to the definitions, each of these is both closed and open, or as they say nowadays, "clopen".

If you find this weird, you are not alone. See "Hitler learns topology" on Youtube.

Three Warnings:

This is PG13 because of the many four letter words.

It's followed by several ripoffs, not as good, and some downright offensive. Skip them and watch the topology one again.

Midway, there's a wrong word in the subtitles, at 1:56 minutes – "close point" should be "cluster point".