18.100A Fall 2018: Assignment 10 (complete) due Fri. Nov.30

Directions: This is now complete. Directions same as on previous P-sets.
Reading Mon: 22.5 Term-by-term differentiation of sequences and series of functions.
Reading Wed.: 22.6 Application of theorems about term-by-term integration and differentation to power series and ODE's.

1. (2) a) Work 22.5/1 b) Work 22.5/2 (use Thm. 22.3.)

2. (2) Theorem 22.6 essentially proves for general power series that the series and its term-by-term derivative have the same radius R of convergence. The proof is a little tricky.

a) The proof is much simpler if you assume that R can be found for the original series by the ratio test. Prove that if this is so (as it is for the most used power series), then it will also be true for the differentiated power seres, which will turn out to have also R as its radius of convergence. (cf. 8.1/3, given on an earlier P-set.)

b) Theorem 22.6 proves the derived series $\sum n\, a_n x^{n-1}$ has a radius of convergence $\mathbf{R}'$ which is $\geq$ the radius of convergence R of $\sum a_n x^n$.
It is much easier to prove that $R' \leq \mathbf{R}$, which shows that $\mathbf{R}' = \mathbf{R}$. Do this, and begin by proving $\sum n\, a_n x^n$ also has $\mathbf{R}'$ as its radius of convergence. (Use Theorem 8.1: show that if one series is absolutely convergent for x_0, so is the other. Cite theorems used.)

The end result of applying the theorems of Chapter 22 to power series is this theorem, whose three parts can be found, with more detail, in sections 22.3, 22.4, and 22.6. You can use it in citations when it gives you what's needed,

Power Series Theorem. A power series $\sum a_n x^n = f(x)$ with radius of convergence R
a) converges uniformly on every $[-a, a]$, for $0 < a < R$;
b) defines, on the interval $(-R, R)$ a function $f(x)$ which is continuous, integrable, and differentiable on $(-R, R)$.
c) For a and x in $(-R, R)$, its integral $\int_a^x f(t)\, dt$ and derivative $f'(x)$ can be calculated by integrating and differentiating the series term-by-term.

Notes: If the series converges for $x = R$ or $-R$, the continuity or differentiability may be only one-sided there.
It's customary to do term-by-term integration of power series by finding the antiderivative of each term, lumping all the arbitrary constants from the antiderivatives together by writing the sum as $f(x) + c$, and evaluating c by substituting a numerical value for x into the series.

3. (1) Work 22.6/5a, using term-by-term integration. Either use a definite integral from 0 to x (using a dummy variable in the integrand), or use indefinite integration with an added constant of integration which you evaluate, as described above.

4. (2: .5, 1.5) In part (b), cite theorems; verify hypotheses.
a) Derive the series for $\ln(1 + x)$ by using differentiation, then term-by-term integration. On what x-interval does the series represent $\ln(1 + x)$ (i.e., are the two equal as functions)?
(Note how the power series theorem avoids calculation of Taylor series and having to estimate its remainder term to prove the function and its series are equal.)

b) Work Problem 22-1. Let $f(x)$ denote the sum of the series; what's the largest interval on which the series converges uniformly to $f(x)$?
To determine $f(x)$ explicitly, you will need the result in part (a), substituting $-x$ for x. Include a definition for $f'(0)$, with justification.

5. (3) Find an explicit formula for the function $f(x) = \sum_0^\infty \frac{x^{2n+1}}{(2n+1)!!}$,
where $(2n+1)!! = (2n+1)(2n-1)(2n-3)\cdot \ldots \cdot 3 \cdot 1$.

a) Use a dependent variable $y = f(x)$, and show using term-by-term differentiation that y satisfies the first-order differential equation $y' - xy = 1$. The calculations in Example 22.6B will help; pay careful attention to the leading terms of the series involved – write out the beginning of the series explicitly (i.e., without using summation notation) as a check on your calculations.

b) Then take a different tack and solve the ODE using the integrating factor $e^{-x^2/2}$ (see parentheses below) and an appropriate initial condition, and since a linear first order ODE with initial condition has only one solution, in this way determine the sum $f(x)$ as a non-elementary integral, i.e., don't try to do the integration!

Over what interval is the solution valid?

(To solve the ODE, multiply both sides of it by the integrating factor; then both sides become explicitly integrable, i.e., you can write down antiderivatives for them having the form of a known function, or a definite integral of a known function.)

6. (2) Work Problem 22-2, but follow the pattern of Example 22.6B – that is, start with Bessel's ODE (cancel an x from each term to make it look simpler), and follow the procedure in 22.6B for finding a power series solution $\sum_0^\infty a_n x^n$ satisfying the ODE and the initial conditions. Omit showing the solution is unique.

You must shift the indices so you can add the terms together to get a single power series. If unfamiliar with this, study Example 22.6B to see how it is done.

Make your solution look like the series for $J_0(x)$ given. You will need the Zero Theorem Cor. 26.2B to get the recursion formula for determining the a_n .

Read Mon.: 24.1-.2 omit proof of the B-W Thm 24.2C—see Problem 9 below
Read 24.3-.5 Norms, seq's and fcn's on $\mathbf{R}^2$; convergence and continuity th'ms in $\mathbf{R}^2$.

7. (2: 1,1) a) Work 24.1/3 (Equiv'ce of uniform norm $||\ \ ||$ and Euclidean norm $|\ \ |$).

b) Work 24.2/3 as a typical illustration of the significance of two norms being equivalent.

8. (2) Work 24.2/2, as an exercise in convergence of sequences in the plane. Use coordinate-wise convergence.

On each main section of the region $\mathcal{D}$, indicate with an arrow (or some other notation if necessary) what the limit of the points in that section is, as $n \to \infty$.

Be sureto include the boundary of the region: it has several sections, having different limits (or no limit: indicate that too).

9. (1.5: .5, 1) Proving the Bolzano-Weierstrass Theorem in $\mathbf{R}^2$ using coordinate-wise convergence. (The proof is in the book, so treat this as a Question, peeking (quickly) at the proof only for a hint if stuck.) (continued on next page)

a) Critique the following (false) proof often given by students:

Proof: Let $\mathbf{x}_n = (x_n, y_n)$ be a bounded sequence in $\mathbf{R}^2$. Then by the usual B-W Theorem in $\mathbf{R}$, the bounded sequence x_n has a convergent subsequence $x_{n_i} \to a$ and similarly, the bounded sequence y_n has a convergent subsequence $y_{n_i} \to b$.

Then by coordinate-wise convergence, the subsequence $(x_{n_i}, y_{n_i}) = \mathbf{x}_{n_i} \to \mathbf{a} = (a, b)$.

b) Fix the proof in part (a) so it becomes a real proof by coordinate-wise convergence.

10. (1.5: .5, 1) a) Work 24.5/1 as follows:

First show $f(x, y)$ is continuous on every vertical line $x = a$ and every horizontal line $y = b$, including the two such lines which go through the origin $\mathbf{0}$. (Use standard facts about the continuity of rational functions.)

Then work 24.5/1 (using sequential continhuity)..

11. (1) Read in 24.6 the definition of a sequentially compact set in $\mathbf{R}^2$, including the example of a closed rectangle with sides parallel to the x and y axes.

Then read in 24.7 the statement of the Extremal Value (Maximum) Theorem for $\mathbf{R}^2$ – you can skip the proof which is the same as the one for $\mathbf{R}^1$ in Chapter 13 with all the x changed to $\mathbf{x}$. (It should include the hypothesis S is non-empty.)

Then do Q24.7/2, as usual looking at the solution in the book briefly only if stuck.

Reading Wed: 25.1; Thm. 25.2 (statement only) Cluster points and closed sets in $\mathbf{R}^2$; Theorem: Compact = closed and bounded.

Hint: For problems involving cluster points of a set S: in Def'n 25.1A try first to use the limit definition (1c in the book): it often it is the best choice.

(Its use is so frequent that many books call such a point $\mathbf{a}$ "limit point of S" instead of "cluster point of S", since these are the points which are limits of sequences in S.)

Note: 1. A cluster point $\mathbf{a}$ of a set S is not necessarily in S.
2. In the sequence $\mathbf{x}_n \to \mathbf{a}$, we require $\mathbf{x}_n \neq \mathbf{a}$ for all n. (Otherwise, every point $\mathbf{a}$ in S would be a cluster point of S, since $\lim_{n\to\infty} \mathbf{a}, \mathbf{a}, \mathbf{a}, \ldots = \mathbf{a}$ – and when everyone is somebody, then no one's anybody.

12. (2) Work Question Q25.1/1bcde for easy practice in using Theorems 25.1A and B.

13. (4: 1, 1.5, 1.5)

a) Work 24.7/1; assume S is non-empty and the origin $\mathbf{0}$ is not in S.

b) Work 25.2/2, continuing the assumptions in part (a). (Make the compact set so it is sure to contain the minimal-distance point you want. Prove eyesverything – Thms. 25.1A and 25.2 should help to prove your set is compact.)

c) Let $\mathbf{x}_n = (\cos^2 n, \sin^2 n)$. Prove it has a subsequence that converges to a point on the line $x + y - 1 = 0$, even though the line is not compact. (The idea in part (b) is helpful.)

14. (2) We can think of a function $w = f(\mathbf{x})$ defined for all $\mathbf{x} \in \mathbf{R}^2$ as giving a map $f : \mathbf{R}^2 \to \mathbf{R}^1$. If $S \subset \mathbf{R}^1$, we define the *inverse image of S under f* to be

$$f^{-1}(S) = \{\mathbf{x} \in \mathbf{R}^2 : f(\mathbf{x}) \in S\} \ .$$

Assume $f(\mathbf{x})$ is continuous; prove that if S is closed in $\mathbf{R}^1$, then $f^{-1}(S)$ is closed in $\mathbf{R}^2$.

(Focus on what you have to prove about $f^{-1}(S)$; observe the *Note* and *Hint* given above.)

Comment: The above is true for any map $f(\mathbf{x}) : \mathbf{R}^n \to \mathbf{R}$ defined by a continuous function on $\mathbf{R}^n$. Conversely, if $f^{-1}(S)$ is closed for all closed subsets S of R, then the function $f(\mathbf{x})$ is continuous on $\mathbf{R}^n$. This gives an alternative definition of a continuous function on $\mathbf{R}^n$.