18.100A Fall 2018: Problem Set 1 due Fri. Sept. 14

Directions: This is the first 2/3 of Pset 1, covering the Friday and Monday classes.

You can collaborate, but must write up the solutions independently, i.e., in your own words, thinking them through yourself.

To make returning papers easier, list any **collaborators** in the **upper-left** corner of the top page; put **your name** in the **upper-right corner**.

Consulting internet solutions or problem set solutions from previous semesters or years is not allowed.

Reading Fri.: Chapter 2.2-.6 Estimations, absolute values, approximations, "for n large".

Problem 1. (1) Proposition:

If the sequences a_n and b_n are strictly increasing, then $a_n b_n$ is strictly increasing.

Is this T or F (true or false)? If T, give a proof. If F, give a counterexample (cf. App. A), and add a hypothesis which will make it true.

Problem 2. (2) Let $p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$. and $|x_0| \leq 1$. Prove that

$$|p(x_0)| > n+1 \implies \text{ for some } i, \quad |a_i| > 1.$$

(Use contraposition (cf. A.2): not B $\Rightarrow$ not A, but write the contrapositive statement avoiding all negative words like "not","no" and symbols for them. Note that in Mathspeak, the phrase '*for some i*' means '*for at least one value of i*'.

Problem 3. (2: 1,1) a) Prove $\{x_n\}$ defined by $x_{n+1} = \dfrac{n^2+10}{(n+1)(n+3)} x_n$, $x_0 > 0$ is monotone for $n \gg 1$.

(Two ways to show a positive sequence a_n is increasing are to show the ratio $a_{n+1}/a_n \geq 1$ or show the difference $a_{n+1} - a_n \geq 0$.) Analogously for decreasing: use ≤ 1, ≤ 0.

b) For what n will $\dfrac{3n}{n+2} \underset{\epsilon}{\approx} 3$, if (i) $\epsilon = .1$ (ii) $\epsilon = .01$?

Problem 4. (2) This problem gives an alternative and cleaner approach to proving Theorem 2.5, though what's in the book is probably what would occur to you first if you think of a real number as an infinite decimal.

Read: A.0-A.2 as needed, for notation and background.

Let $\mathcal{R}$ be the set of real numbers (think of them as the points on the real line). A subset S is said to be *dense* in $\mathcal{R}$ if every open interval (a,b), $a < b$ contains a point of S.

a) Prove that the set $\mathcal{Q}$ of rational numbers is dense in $\mathcal{R}$, without using the infinite decimal representation for real numbers; i.e., prove that

given (a,b), $a < b$, there are integers m, n such that $a < \dfrac{m}{n} < b$.

(Show how to find a suitable n, then an m.

Since any two successive integers k, $k+1$, have spacing 1, you may use without proof the geometrically obvious fact:

if $c_2 - c_1 > 1$, then (c_1, c_2) contains an integer.)

b) Prove similarly that also the set of irrational numbers is dense in $\mathcal{R}$, by proving the stronger theorem that the numbers $r\sqrt{2}$, (r rational) are irrational and dense in $\mathcal{R}$.

(App. A proves $\sqrt{2}$ is irrational, so you can use this in your proof, but cite the page no.)

Read: Mon. Ch. 3.1-6
Definition of limit,; K-ϵ principle, $\lim = \infty$, $\lim a^n$. Examples.

Problem 5. (2)) Work: (a) E3.1/1b (b) E3.1/1c (Exercises, p. 46)
Do them directly from Definition 3.1 of limit; don't use any limit theorems you know from calculus (in Chapter 5 here).
You can often make the work easier if you first 'simplify' the expression you want to show is $< \epsilon$. Here 'simplify' means change the expression into a one which is clearly larger, but but easier to estimate as $< \epsilon$.

Problem 6. (2 pts: 1, 1))
a) Prove that if $\{x_n\}$ converges (i.e., has a limit), it is bounded for $n \gg 1$.
b) Then prove that it is bounded (i.e., for all n)
(For part (a), you'll need an ϵ; tell who gets to choose it, and the reason for your answer. Also, the equivalent form of $a_n \underset{\epsilon}{\approx} L$ suggested by [2.4, (2)] is helpful.)

Problem 7. (2) Work 3.4/3 (at the end of the chapter).
(Follow the hint given; note that the proof of Theorem 3.4 could be simplified slightly by dropping the 1 in addition to the other terms dropped.)

Problem 8. (2 pts.) Work 3.6/1b (use the hint given).

Read Wed. Chapter 4.1 Error form for a limit. (Continued on page 3)

Problem 9. (2) Prove the **Product Theorem** for limits:
$$a_n \to L, \quad b_n \to M \;\Rightarrow\; a_n b_n \to LM\ .$$

This is proved in 5.1. Treat it as a Question: don't look at the book's proof, instead follow the suggestions below.. If you get stuck for more than a few minutes, you can take a quick look at the book's proof for a hint.
Write limits in the error form (cf. 4.1), $a_n = L + e_n$; let e'_n be the error term for b_n.
From these get the error form for $a_n b_n$; group the terms together to get the error term e''_n for $a_n b_n \to LM$.
Then show $e''_n \to 0$ by using the definition of limit, the Triangle Inequality, and the $K - \epsilon$ principle. Watch out that in the proof you do not use the product theorem to evaluate $\lim e_n e'_n$ – circular reasoning! But you may assume $\epsilon < 1$, since limit demons only give out small epsilons.

(Continued on page 3.)

Read Wed: Chapter 5.1-.3, 5.5

Algebraic theorems (review); Inequality theorems: Squeeze Theorem, Location Theorems.

Problem 9 was about using the error form to give a natural proof of a typical algebraic limit theorem; these last two are problems giving typical uses of the squeeze and location theorems respectively.

Problem 10. (2: .5, 1.5))

(a) Do Q5.2/1c (at the end of section 5.2) as a preliminary;

(b) Work 5.2/3 (Exercise 5.2/3 at end of Chapter 5) as follows:

Interpret this sum as the total area of n rectangles of width 1 lying over $[0, n]$ on the x-axis, and having the successivde heights given by this sum.

Estimate the area by comparing the area under the curve $y = \sqrt{x}$ and over the related x-interval both to the area of the rectangles, and to the area of the rectangles all shifted one unit to the right.

Finally, use the squeeze theorem to prove the ratio given in the problem has the limit 1 as $n \to \infty$.

Problem 11 (2)

The Completeness Principle in Chapter 1 essentially defines the limit (call it B for this problem) of an increasing sequence bounded above to be its *least upper bound* – B is an upper bound and there is no lower upper bound for the sequence.

Chapter 3 gives a different definition of the limit L which applies to all sequences, not just increasing ones.

Prove that if a sequence has a limit L and is increasing, then $L = B$, i.e., L is also its least upper bound – its limit in the sense of Chapter 1 – as follows:

a) Prove that L is an upper bound for x_n, indirectly: suppose it is not an upper bound, how does this contradict the definition of L as the limit?

b) Use a location theorem to prove indirectly that L is the least (i.e., lowest) upper bound.