18.100A Problem Set 0 due Fri. 9/7/18, by 1:05 in 4-163

Directions: Pset 0 is diagnostic, to get some idea of your "mathematical maturity" – how well you can read and write math, prove things, follow hints, find errors, etc. No other assignment will be due in just two days. I will look at them and return them in class Monday . The numerical score on it will be recorded, but not count toward your total score.

No collaboration with other students, looking up solutions to problem sets from previous semesters, or help from others (including me) is allowed on this zeroth Pset.

For subsequent Psets, you can collaborate, the rules given in the "18.100A Information and Rules" sheet on the website apply – reminders will be on each of the earlier Psets..

Writing Style for Psets:

1. Handwritten is preferred, but it must be dark enough, and with letters and symbols large enough to be easily readable. Subscripts and superscripts should be smaller-sized, placed lower and higher, just as they appear in printed math.
2. Leave margins, left and right, for grading comments.
3. The paper should be heavy enough for writing on both sides, without the writing showing through. Try not to have a page break in the middle of a problem, unless it separates two sections of the problem.
4. Imitate the textbook style, with equations and inequalities lined up on successive lines, rather than on the same line, to make clear what was done to each line to get the next one.
5. Use scratch paper first on problems you are unsure about, to avoid extensive crossing out on your answer sheet.
6. You can use computer writing (TeX, LaTeX, or something equivalent), if you can make it look like math books – especially 4. above.

Notation: To say $\{x_n\}$ has the limit L, write: $\lim_{n\to\infty} \{x_n\} = L$ or $\{x_n\} \to L$ as $n \to \infty$; often the $n \to \infty$ and braces { } are understood and omitted.

In 100A, this use in limits is the **only** use for the single arrow $\to$. For implication, use only the double-stem arrow $\Rightarrow$, as in $A \Rightarrow B$ (A implies B).

If $a > 0$, by convention $\sqrt{a}$ always is understood to be its positive square root.

Problem 1. P(n): The General Arithmetic/Geometric Mean Inequality of order n

$$P(n): \qquad (a_1 a_2 \dots \mathbf{a}_n)^{1/n} \quad \le \quad \frac{a_1 + \cdots + a_n}{n}; \qquad a_i > 0,\ i = 1, \dots, n, \quad n = 2, 3, 4, \dots .$$

The aim of this problem is to prove the first 3 cases $P(2), P(3), P(4)$. From this, one can see how to prove it in general.

a) The classic **A-G Mean Inequality** is the case P(2). **Read 2.1** (Chapter 2, section 1) Work P2-3 (Problem 2-3 at the end of Chapter 2) two ways, as follows:

(i) **Algebraically:** In your answer, start from the inequality you want to prove and work backwards: keeping the $\le$ going in the same direction on each successive line (3 or 4 lines should be enough), transform both sides using the inequality laws in 2.1, until it turns into an inequality which is obviously true. (In 100A this is called "backward reasoning".)

(This proof's logic has to be read in the opposite order, i.e., from the bottom up, but it should not be written (or rewritten) this way, since it makes the steps unmotivated, and therefore harder to read and follow. The only stumbling block in backwards reasoning is that you have to check mentally (or explicitly, if not obvious) that each step is reversible – each line implies the line just above it.)

(ii) **Geometrically:** Copy the book's figure onto your paper, giving the length of the two relevant line segments, and prove it for the length of the one whose length is not obvious (e.g., you can give the high-school proof using similar triangles and auxiliary line segments.)
The figure then makes the inequality obvious.

(b) Prove $P(2) \Rightarrow P(4)$. Use the following method.
Let $a = (a_1a_2)^{1/2}$, $b = (b_1b_2)^{1/2}$, and apply $P(2)$ to $(ab)^{1/2}$.

(A similar argument proves: For any $n \geq 2$, $P(n)$ and $P(2) \Rightarrow P(2n)$.)

(c) **Algebraic Magic** deserving the adjective "awesome" in its original meaning before its demotion to "good". Prove $P(4) \Rightarrow P(3)$ as follows:
(i) For clarity, let $a, b, c > 0$ be the three numbers in $P(3)$.
What x-value (in terms of a, b, c) turns $(abcx)^{1/4}$ into $(abc)^{1/3}$? Show work or reason.
(ii) Using this x-alue, show that $P(4)$ using a, b, c, x is actually $P(3)$.

(A similar proof shows that for any $n \geq 3$, $P(n) \Rightarrow P(n-1)$.)

The two "similar" (or analogous) proofs in (b) and (c), along with $P(2)$, show by forward and backward induction that $P(n)$ is true for all $n \geq 2$.

Problem 2. (3 pts: 1, 1.5, .5, 1) (You can use any part – solved or not – in later parts.)
Read: 1.1,1.2,1.3-skip proof; **1.6. (For (a), p.411; for (b) cf. 1.4,through p.6; Q1.4/1,pp.8,15.**

Define $\{x_n\}$, $n \geq 0$ recursively by $x_0 > 2$, $x_{n+1} = \sqrt{2 + x_n^2/2}$.
The steps below prove that x_n is a strictly decreasing sequence, whose limit is 2.

(a) Prove by induction that $x_n > 2$ for all $n \geq 0$.

(b) As a preliminary step, prove by contraposition that $a > b > 0 \Rightarrow \sqrt{a} > \sqrt{b}$.

Then prove $\{x_n\}$ is strictly decreasing. Use backwards reasoning (as in Problem 1a). Where is the preliminary step used to justify the backwards reasoning?
(Cf. the reading on Q1.4/1. p.8. Signal backwards reasoning by a ? at the end of the first line of proof, since this line tells the reader what you are trying to prove.

(c) Prove the sequence has a limit L, using only the ideas in Chapter 1.

(d) Prove $L = 2$. This is best done by using the familiar algebraic limit theorems from calculus, which we will get to in Chapter 5. For use in this problem they say (using the notation for limits given earlier)

$$\lim_{n\to\infty} x_n = L \implies \lim_{n\to\infty} \sqrt{2 + x_n^2/2} = \sqrt{2 + L^2/2}$$

Problem 3. (2 pts) **Read: 1.4, p.7; E1.4/1 = Exercise 1.4 at the end of Chapter 1**

a) Do E1.4/1, changing the successive numerators in the expresion for a_n into:

$1\sin^2(1),\ 2\sin^2(2),\ 3\sin^2(3), \ldots, n\sin^2(n)$. (The hint is still valid for this new $\{a_n\}$.)

b) Let m be a positive integer, and $x_n = \dfrac{m^n}{n!}$. Show x_n is bounded above, and find (in terms of m) its least upper bound (least = smallest, lowest).
(If stuck, try a special case: give m a small integer value, and study where the $\{x_n\}$ is increasing and where descreasing. "Show" means give a convincing argument, which can be somewhat informal, without a lot of symbols.

Problem 4. (4 pts.) **Read: A.3-A.4 as needed.**

The proof of Prop.1.4 begins with $(1+h)^2 \geq 1+2h$, for all h;
(the inequality $>$ is here being changed to $\geq$ to include the possibility $h=0$).

A generalization of this is a statement we will denote by $B(n)$:

(1) $$(1+h)^n \geq 1+nh, \quad \text{for all } n \geq 0 \text{ and all } h \qquad \textit{(Burnoily's Inequality)}$$

Proof of (1) by induction: (cf. App. A.4, Examples A and B)

The cases $B(0)$ and $B(1)$ are both trivial; either can be taken as the basis step.

The induction step $B(n) \Rightarrow B(n+1)$ is proved in the three inequalities below:
the first restates $B(n)$;
the second multiplies both sides by (1+h), preserving the inequality;
the third multiplies out the right-hand side and drops the positive nh^2 term, which preserves the inequality and converts it to $B(n+1)$, and this proves $B(n) \Rightarrow B(n+1)$.

$$(1+h)^n \geq 1+nh \quad \text{for all } h;$$
$$(1+h)^{n+1} \geq (1+nh)(1+h) \quad \text{for all } h;$$
$$(1+h)^{n+1} \geq 1+(n+1)h \quad \text{for all } h, \text{ since } nh^2 \geq 0.$$

a) (1) Burnoily's inequality (1) is false as stated; see if you can locate and describe the error in its proof. If stuck after say ten minutes of effort, go on to (b) below (using scratch paper), and return to (a). If still stuck, there is another hint at the end of this problem.

b) (1) There are *counterexamples* (cf. A.3) to $B(3)$ which use integer values for h; find the one having h-value closest to 0, and prove it is a counterexample to $B(3)$ and the one with h-value closest to 0.

c) (1) Give the weakest hypothesis (i.e., condition) on h for which the attempted proof will actually be valid (cf. A.1 p. 405 for "weakest").

d) (1) Find an integer value h_0 for h which does not satisfy the condition in (c), but for which Burnoily's inequality holds for all $n \geq 0$ and $h = h_0$; then prove it.

This last part (d) is significant, because it shows that **the failure of a method of proof** – induction here, for the case $h = h_0$ – **doesn't prove that a statement is false**: someone might find (as hopefully you just did) a different method which works. (But it does suggest the statement *might* be false and therefore it might be worth looking for a counterexample.)

Hint for Part (a):
After finding the counterexample, if you're still stuck on part (a), take the *trace* of your counterexample in the proof – i.e., substitute the numerical values of n and h used by the counterexample into each successive line of the proof, and see where it goes wrong; this should tell you in general where the error lies.