18.100A Fall 2017: Problem Set 9 due Mon. Nov. 20

Directions: If you collaborate, list collaborators and write up independently. It is illegal to consult solutionss from previous semesters..

Reading Mon.: 22.1, 22.2 (through Example 22.2C only)

Pointwise vs. uniform convergence of sequences $\{f_n(x)\}$ and series $\sum_0^\infty u_k(x)$ on an x-interval I. Weierstrass M-test for uniform convergence of series.

1. (4: 1,2,1) a) 22.1/1a, b) like 22.1/1c, but use $\dfrac{n^2x}{1+n^2x^2}$, $[a,\infty)$, $a>0$.
c) 22.1/3 (omit the proof of non-uniform convergence on $(-\infty,\infty)$)

2. (2) 22.2/2b (estimate $|\cos(x/n)-\cos(0)|$)

3. (2) Using the M-test, work: a) 22.2/2d b) 22.2/4

4. (1) Read just the statement of Th'm 22.C; then prove it, i.e. treat it like a Question.

5. (2) Prove 7.3/1a – it is needed in the proof of the M-test – as follows:

Starting from the finite triangle inequality, prove (in order) that as $n\to\infty$, the right hand side and left-hand side have as their respective limits the right and left sides of the infinite tringle inequality. (Justify each step by citing the relevant theorem.)

Then show the two limits are connected by $\le$, again citing relevant theorems.

This proof revisits convergence theorems for numerical series, and theorems about continuity and limits.

Reading Wed.: 22.3, 22.4 (omitting Cor. 22.3)

6. (1) Treating it as a Question: read the statement of Corollary 22.3 and prove it.

7. (6: 2,2,1,1) a) Work 22.3/2 b) Work 22.3/3 c) Work 22.3/4a,b

4a) You don't have to copy the proof of Theorem 22.3 onto your paper; just indicate what modifications are necessary to the "chain of approximations" in the third displayed line, and prove the modifications lead to the desired conclusion.

4b) Include a graph of a typical approximating function $f_n(x)$.

There are two hypotheses and one conclusion. For the conclusion, review section 13.5; you can give an intuitive argument.

8. (2) Work 22.4/3. For the last step, Chapter 17 gives some standard Taylor series around the point 0 (in other words, taking $a=0$).

Reading: Fri. 22.5, 22.6 (Theorem 22.6, through Examples 22.6A and B)

Differentiation of sequences and series term-by-term. Power series.

9. (2) Work 22.5/1

10. (2) Work 22.5/2

(continued on p.2)

11. (3) Find an explicit formula for the function $f(x) = \sum_{0}^{\infty} \frac{x^{2n+1}}{(2n+1)!!}$,
where $(2n+1)!! = (2n+1)(2n-1)(2n-3) \cdot \ldots \cdot 3 \cdot 1$.

Use a dependent variable $y = f(x)$, and show using term-by-term differentiation that y satisfies the first-order differential equation $y' - xy = 1$. The calculations in Example 22.6B will help; pay careful attention to the leading terms of the series involved – write out the beginning of the series explicitly (i.e., without using summation notation) as a check on your calculations.

Then take a different tack and solve the ODE using the integrating factor $e^{-x^2/2}$ (see parentheses below) and an appropriate initial condition, and since a linear first order ODE with initial condition has only one solution, in this way determine the sum $f(x)$ as a non-elementary integral (i.e., don't try to do the integration!)

Over what interval is the solution valid?

(To solve the ODE, multiply both sides of it by the integrating factor; then both sides become explicitly integrable, i.e., you can write down antiderivatives for them having the form of a known function, or a definite integral of a known function.)

12. (3) Work Problem 22-3b (Uses and reviews 22.4 mostly.)

Background: The function $J_0(x)$ is the function whose graph over an interval $[-a, a]$ gives at a single moment in time the cross-sectional shape along a diameter of the vibrating membrane of a bass drum in a marching band, shortly after the drummer hits it exactly in the middle. (The number a can be thought of as the radius of the drumhead.)

From this, one can derive a differential equation that this cross-sectional shape must satisfy – Bessel's ODE, given in Problem 22-2 – and by using undetermined coefficients (as in Example 22.5B), find a series solution to this ODE which after putting in the initial or boundary conditions will be the power series representing $J_0(x)$.

Actually, Bessel was an astronomer, and he discovered the function in a different context, and in the strange-looking form given in P22-3b. So the purpose of this problem is to show that the two representations of $J_0(x)$ – as a series, and as a definite integral – are equivalent.

Procedure: You will need the power series form of $J_0(x)$, as given in P22-2, and also the standard definite integrals in 22-3a. (These are used in 18.01 and 18.02 for evaluating various multiple integrals – moments of inertia of solids having an axis of symmetry, for instance.)

Start with the power series for $\cos u$ (cf. 17.1). After $x \sin\theta$ is substituted for u, it is no longer a power series, but the theorems in 22.3,.4 are still usable, since they are phrased in terms of general functions $u_k(x)$.

The main confusion is likely to come from the notation. This problem uses the standard notation in the literature for the integral form of the Bessel function, but that makes it an exception to the general rule followed in the textbook that "uniform whatever" is always with respect to the variable x. If you take this warning to heart, mind your θ's, x's, and u's and don't make any errors in calculation, it should all work out in the end, despite some possible dark moments in the middle.

Show your understanding of Chapter 22 by citing at the right points in the proof the key theorems you are using; verify that their hypotheses are satisfied here – this is an important step (i.e, it's at least -1 if you fail to do it.)