18.100A Fall 2017 Problem Set 8 (complete) due **Mon. Nov. 13**

Directions: Next Friday is a holiday, so Pset 8 won't be due until the following Monday.

If you collaborate, list collaborators, and write up solutions independently. Do not consult solutions from previous semesters or on the internet.

Reading Fri: 20.6; (review 20.3-.4) Growth rate of functions; L'Hospital for ∞/∞.

Problem 1. (2) Work 20.3/2 (Using change of variable.)

Problem 2. (2) Work P20-4ab for the case $n = 1$ as described below.

Usually integration by parts is used to make an integral look better. In this somewhat bizarre application, it is used to make it look worse.

The idea is to prove Taylor's theorem (17.2), but just for the case $n = 1$: Theorem 16.1B, the Linearization Error Theorem. (The general case for higher values of n is proved by induction the same way, and no harder.)

a) Follow the directions in P20-4a, but just for $n = 1$, getting the remainder term $R_1(x)$ as a definite integral ("the integral remainder").

(The integration by parts is used in a strange way. As advised, watch the signs carefully, and remember that x is just an unspecifed constant, as far as the integral is concerned.)

b) Convert the remainder term to the form it has in Theorem 16.1B ("the Lagrange remainder" or "the derivative remainder") following the method described in P20-4b; use one of the formulas in P20-3.

Problem 3. (4: 1,1,2) Parts (a) and (b) might help with (c).

a) Do Q20.6/2, paying attention to the Note below (23) and (24).

b) If $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to \infty$ and f grows faster than g, prove $f - g \to \infty$.

c) Work 20.6/1a; prove your work, determining all four limits needed.

Problem 4 (2) Work 20.6/3ab .

Problem 5 (2) a) In an earlier P-set, the Prime Number Theorem (PNT) was stated in the sequence form: PNT: $p_n \sim n \ln n$, where $p_n =$ the $n-$th prime.

In 20.6/4, it is implicitly stated in two function forms: Euler's and Gauss':

PNT: $\pi(x) \sim \dfrac{x}{\ln x}$; $\quad \pi(x) \sim \mathrm{Li}(x)$; $\quad \pi(x) =$ the no. of primes $\le x$.

Show that the two PNT forms are equivalent, by showing that $\mathrm{Li}(x) \sim \dfrac{x}{\ln x}$.

b) The Riemann Hypothesis. Gauss' estimation of $\pi(x)$ is more accurate than Euler's: the graph of $x/\ln x$ is always below that of $\pi(x)$, whereas while the graph of $\mathrm{Li}(x)$ starts out above the graph of $\pi(x)$, the two graphs cross each other infinitely often afterwards.

In other words, if we write the estimation in the form $\pi(x) = \mathrm{Li}(x) + e(x)$, then the error term $e(x) < 0$ at the start, but is known today to first become > 0 at some $x < 1.4 \cdot 10^{317}$.

The Riemann Hypothesis (1859), is an estimate of the error term $e(x)$. Its truth is one of the great problems of number theory ($ 1,000,000 if you can solve it). It has two forms:

$$i) \ \ |e(x)| < C(\sqrt{x}\ln x) \text{ for some constant } C; \qquad ii) \ \ \lim_{x\to\infty} \frac{|e(x)|}{x^{\frac{1}{2}+\epsilon}} = 0 \ \text{ for all } \epsilon > 0.$$

Which of these is the stronger statement, i.e., which one implies the other? Prove it.

Reading Mon: 21.1-.2 Improper integrals with nonnegative integrands:; comparison tests for convergence; standard comparison integrals.

Problem 6. (2) Assume $f(t)$ is non-negative and integrable on $[0\,\infty]$. The analog for improper integrals of first-kind of the "nth term test for divergence" (7.2A) should be:

$$\int_0^\infty f(t)\,dt \text{ converges} \Longrightarrow \lim_{t\to\infty} f(t) = 0.$$

Give a counterexample where $f(t)$ is a continuous function (a clear sketch or description will be adequate; indicate why the integral converges).

(For half-credit, give a discontinuous counterexample, whose improper integral exists and converges.)

Problem 7. (2) Work 21.1/2; do it two different ways:

a) Change the variable to convert it to an integral of the first kind, then use comparison.

b) Do the integration, then evaluate the limit you get, either by changing variable, or converting it to a standard indeterminate form.

Problem 8. (3: 1.5, 1.5) In both of these, k is any real number. Show work or reasoning.

a) Work 21.2/1c, using asymptotic comparison at both ends.

b) Work 21.2/1e.

Reading Wed.: 20.5, 21.3 Geometric and analytic views of $n!$.

Stirling's asymptotic approximation and Euler's Gamma function.

Problem 9. (3: 1,2) a) Work 20.5/1a, Stirling's approximation for $\binom{2n}{n}$, which can be interpreted both as the the middle binomial coefficient in $(1+x)^{2n}$ and the number of ways of choosing n things from a set of $2n$ different things.

Simplify your answer as much as possible.

b) Work 20.5/2, except change the cube roots to square roots, and call the sum S_n.

Follow the same procedure and use the same notation as in the proof of Stirling's formula. Denote by B_n the sum of the little "bows" (or "slivers") on top of the trapezoids.

Get a little more accuracy by estimating L visually: about how much of the square over $[0,1]$ do the n bows take up?

Problem 10. (3: 1.5,1.5) Work 21.3/1ab.

a) You will need G3, G6, and the definition of $\lim_{x\to 0^+} f(x) = \infty$.

b) Get a lower bound for $\Gamma(x)$, $x \approx 0^+$ by changing the integrand to one that is simpler and lower, over the smaller interval $[0,1]$.