18.100A Fall 2017 Problem Set 7 (Complete) due **Fri. Nov. 3**

Directions: If you collaborate, list collaborators, and write up solutions independently. Do not consult solutions from previous semesters or on the internet.

Reading Wed.: 13.5 Uniform Continuity of $f(x)$ on an interval I: definition, equivalence with regular continuity if $I = [a, b]$, i.e., a sequetially compact interval.

Most of the problems below deal with continuous functions on non-compact intervals I which are nonetheless uniformly continuous on I.

Problem 1. (2)

Let $f(x)$ be a function defined on a given interval I of some unspecified type.

Assume that $f(x)$ has *bounded secant slope* on I. By definition, this means there is a fixed constant K such that, given any two points $x' < x$" in I, the slope λ of the secant line joining the points $(x', f(x'))$ and $(x'', f(x''))$ on the graph of $f(x)$ satisfies $|\lambda| < K$.

(Try sketching a few graphs to get a feel for what this means.)

Prove that $f(x)$ is uniformly continuous on I.

Problem 2. (2)

a) Prove using Problem 1 that if $f(x)$ has a bounded derivative $f'(x)$ on some interval I, then $f(x)$ is uniformly coninuous on I.

b) Prove $\ln x$ is uniformly continuous on the non-compact interval $[1, \infty)$.

Problem 3. (2) In what follows, you may use the

Theorem: *Assume the interval I is the union of two intervals I_1 and I_2 which intersect in a single common endpoint c. Then:*

$f(x)$ uniformly continuous on I_1 and $I_2 \Rightarrow f(x)$ uniformly continuous on I.

(The proof has to fuss a little, since the two intervals use different δ_ϵ, and the x' and x'' may lie in different inervals.)

a) Consider $f(x) = \sqrt{x}$ on the non-compact interval $I = [0, \infty)$. Give separate proofs that it doesn't satisfy either of the two conditions in Problems 1 and 2:

(i) $f'(x)$ bounded on I (ii) $f(x)$ has bounded secant slope on I. (cf. Example 3.1)

b) Prove that nonetheless, $f(x)$ is uniformly continuous on I.

Problem 4. (2: 1.5, .5) Work P13-6ab. This ties together uniform continuity with the earlier notion of Cauchy sequence in Section 6.4 .

For part (b), you can use an earlier P-set problem: if a sequence $\{x_n\}$ converges, then it is a Cauchy sequence.)

Reading Fri.:

18.1-.2 Partitions, Riemann-integrability definition.

18.3: read Theorem 18.3B only (skip Theorem 18.3A, which is Problem 5, here being treated as a Question.)

18.4 Properties of integrability: read the statements, skip the proofs.

Problem 5. (2)

a) Prove: (Theorem 18.3A): If $f(x)$ is decreasing on $[a,b]$, it is integrable on $[a,b]$.

(Try to do this without consulting the proof in the book. Using the graph of $f(x)$, represent the difference $U(P)-L(P)$ as a sum of rectangles, and show they can all be fitted without overlapping into a rectangle of width ϵ and fixed height, if the mesh $|P| < \epsilon$ and the function $f(x)$ is decreasing on $[a,b]$.

If you need to consult the book's proof for a further hint, let some time elapse before you write up your own version of it, in your own words.

Problem 6. (2) Work: 18.3/1, which is an example of a function with k discontinuities that is integrable.

(Some older printings of the book have n where the current printing has k; but you need n for the n-partitions in your proof. Be careful with the argument – note that a point can lie in more than one subinterval of a partition.)

Problem 7. (1) Using the result of Problem 5, give an example of an integrable function having an infinity of jump discontinuities on $[0,1]$.

Reading: (Mon.) 19.1-.3: The Riemann integral: def'n, calculation by Riemann sums.

Problem 8. (2) Work 19.2/1, using lower sums (not upper sums, as in older printings).

You may assume e^x is continuous for all x.

To make the calculation of the limit needed at the end more transparent, you can substitute the continuous variable h for $1/n$; what theorems guarantee that

$$\lim_{h\to 0} f(h) = \lim_{n\to\infty} f(1/n) \ ?$$

(Note that the left side is a function, the right side a sequence.)

Problem 9. (2) Work 19.3/3ab.

The trapezoidal rule is a standard numerical method used for evaluating integrals, in all calculus textbooks; it's meant to provide motivation for the problem, but all you need to do the problem is the formula given in the problem.

Problem 10. (2) Both (a) and (b) below should be treated like Questions – they are two very important properties of the integral we have been using earlier in the semester, now proven as Theorems in section 19.4. Just read the Theorem statements in 19.4, and prove them as suggested below.

a) Prove Theorem 19.4B, the Comparison Theorem for integrals, by using Riemann sums to calculate the integrals, and one of the limit theorems for sequences to prove the inequality.

b) Prove Theorem 19.4C, the Absolute Value Theorem for integrals, by using part (a), and the absolute value law (2) p. 22 (as in the proof of (4) below it).

The proof requires something you can assume to be true – what is it? (It's proved elsewhere in the book – where?)

Problem 11. (2: 1.5,.5) Work 19.4/2ab

(Part (a) and its immediate corollary Part (b) are important results; Part (b) sharpens for continuous functions the simpler comparison theorem in Problem 6.)

You have to find and interpose a "buffer" $g(x)$ – not a number this time, but rather a function such that $\int_a^b g(x) > 0$ and $f(x) \geq g(x) \geq 0$.

Use $>$ and $\geq$ carefully; and no circular reasoning – justifying a step by using as the reason for it the very theorem you're trying to prove! The function $g(x)$ has to be one whose integral is known without calculation.

Reading Wed.: 20.1 - 20.4 The two Fundamental Theorems of Calculus; Integration techniques; $\ln x$ and e^x.

Problem 12. (1.5) Let $F(x) = \displaystyle\int_a^x f(t)\,dt$, where $f(t)$ is integrable on I and $a, x \,\epsilon\, I$.

To see that the Second Fundamental Theorem in general is not applicable if $f(x)$ is not continuous:

Let $f(x) = 1$ on $[0,1]$, and $= -1$ on $(1,2]$.
Taking $a = 0$ and $I = [0,2]$, calculate $F(x)$ explicitly and show that $F'(x) \neq f(x)$ on I .
(You can assume that the value of $\displaystyle\int_a^b f(x)\,dx$ is not changed if you alter the value of $f(x)$ at a single point.)

Problem 13. (2) Prove that if $f(t)$ is integrable on any finite interval, then
$$F(x) = \int_a^x f(t)\,dt \quad \text{is continuous for all } x.$$

(The truth of this is suggested by the result in Problem 1 above. The proof isn't difficult if you follow the suggestions below, but it requires using (and citing) several theorems in Chapter 19, as well as some earlier definitions.
Use the $\epsilon - \delta$ definition of continuity, in the form using absolute values, and the definition of boundedness using absolute values (p. 139).

Problem 14. (4: .5,1,1,1.5))

(a) Work Problem 20-1 part (b)
(b) Work Problem 20-1 part (c)
(c) Work Problem 20-1 part (d)
(d) Evaluate $\displaystyle\int_0^1 \frac{u\,du}{a^4 + u^4}$ in terms of values of $T(x)$, by making a change of variable. Verify hypotheses.

Problem 15. (2) Work 20.3/3. This is an exercise involving integration by parts.