18.100A Fall 2017: Problem Set 5 (complete) due Fri. Oct.13

Directions: These are the same as before – list collaborators; write up independently; consulting assignments from previous semesters is illegal.

Reading Fri: 13.1-.3
Sequential compactness, the Boundedness and Extremal-Value (Maximum) Theorems:

A continuous function on a sequentially compact interval $I = [a, b]$ *is bounded and has extremum points (i.e., maximum and minimum points) on* I.

Comment: A maximum point $\bar{x}$ is one satisfying $f(x) \leq f(\bar{x})$ for all $x \in I$; similarly for a minimum point. There can be more than one – for the extreme case of a constant function, every point is a both a maximum point and a minimum point for the function on I.

Distinguish carefully between: a maximum point $\bar{x}$ (which lies in I on the x-axis), the maximum m of the function on I (which is $f(\bar{x})$), and a "local" maximum point x_0 (often called "relative" maximum point in calculus), which is a maximum point for $f(x)$ in some δ-neighborhood of x_0.

Problem 1. (2) Work 13.1/1a.
Try to use a direct argument throughout: – start with an arbitrary sequence of points in S, and show that it has a subsequence converging to a point in one of the finite closed intervals that make up S.

Problem 2. (3: 1,1,1) (i), (ii): Work 13.2/1a,b and (iii): 13.3/3a
These give a geometric interpretation and application of the two theorems.

Problem 3. (3: 2,1)
Let $I = [0, \infty)$. Assume $f(x)$ is continuous and positive ($f(x) > 0$) on I, and $\lim_{x\to\infty} f(x) = 0$.
Prove that $f(x)$ has a maximum point on I.

The point of this problem is that I is not sequentially compact, so the Extremal Value (Maximum) Theorem is not immediately applicable. You have to get around that somehow.

a) Use $f(0)$ to divide I into two pieces: a compact interval $[0, a]$, and an infinite "tail" interval $[a, \infty]$ on which the maximum point $\bar{x}$ you are seeking cannot lie, then finish the proof.

b) How would you modify the argument if $f(0) = 0$, but the other hypotheses were unchanged? Give a brief reason why your modification will work; you don't have to repeat the whole argument.

Problem 4. (3) You can deduce the Maximum Theorem from the Boundedness Theorem by a different argument than the one given in the textbook as the proof of Theorem 13.3. This alternative argument is indirect, but gives a shorter proof.

Assume $f(x)$ is continuous on the compact interval I but has no maximum point $\bar{x}$. Derive a contradiction by proving that if this were so, the following function would then be a counterexample to the Boundedness Theorem: $\dfrac{1}{\overline{m} - f(x)}$, where $\overline{m} = \sup_I f(x)$.

(continued $\longrightarrow$

Reading Wed.: 12.1, 12.2 (omit Example 12.2B) Bolzano's Theorem, Intermediate Value Theorem, Intersection Principle

Problem 5. (3: 2,1)

a) Using properties of $f(x) = x^3$, as well as the Intermediate Value Theorem, prove that every $a \geq 0$ has a cube root $\sqrt[3]{a}$.

(Cite the properties explictly, Do not use inverse functions, e.g. $\sqrt[3]{x}$. Note that the IVT does not recognize ∞ as a number.

b) Prove that $a \geq 0$ has only one cube root. (Argue indirectly.)

Problem 6. (3: 2,1)

a) Prove that the function $x - \tan x$ has a strictly increasing sequence of positive zeros $x_1, x_2, \ldots, x_n, \ldots$.

(Use the Intersection Principle; in verifying that its hypotheses are satisfied you can treat ∞ informally like a real number. Include a reasonably careful sketch of the relevant graphs.)

b) If N is large, approximately how many positive zeros are there $\leq N$? Indicate your reasoning briefly.

Problem 7. (3) For what positive value of the parameter A will $f(x) = x - A \sin x$ have exactly 4 positive zeros?

Express this value of A in terms of the x_n of Problem 6.

(No formal proof is required, but show your reasoning and calculations.)

Problem 8. (4) Work P12-3 (Problem 12-3), proving the existence of a (horisontal) chord for the graph of a function satisfying certain conditions specified in the problem.

(Try making some sketches on scatch paper. if stuck, see the hint below.

Hint: A chord is a horizonal line segment satisfying three conditions. Try three experiments, where in each you drop one of the conditions, and see what all the horisontal line segments satisfyinging the remaining two conditions look like.