Directions: As before: list collaborators; illegal to consult previous semesters’ assignments.

The main topic for the next four classes will be continuity – the study of continuous functions: this week chapter 11 and half of 13 (which has the Extremal-value Theorem (aka the Maximum Theorem, and sometimes called the Fundamental Theorem of Analysis).

The two preliminary chapters 9 and 10 with the first few accompanying problems here are for background, and will be primarily a reading assignment.

Chapter 9 is about general functions and the elementary properties they can have; the assigned part should be mostly a review.

Chapter 10 is mostly about extending the language and properties of sequences to general functions – a sequence is actually just a special kind of function.

Chapter 11 then introduces two closely connected topics: continuous functions and limits of functions, the topics of the Mon. and Wed. classes and this problem set.

Reading Mon.
9.1-.3 Def’n and properties of functions. (The def’n “mapping” may be new to some.)
10.1-10.3 Estimation and approximation for functions; local and global properties.
11.1 $\epsilon$-form def’n of continuity, examples, types of discontinuities.

Problem 1. (4) a) Work 9.3/3
(b) Work 10.1/1 (to get familiar with the notation $f(I)$, where $I$ is an interval on which $f(x)$ is defined, and $f(x)$ is thought of as a map: $I \to \mathbb{R}$).
(c) Work 10.1/6b (needs 18.01 calculus)
(d) Work 10.1/7b (Express boundedness using absolute values, as in (2), section 10.1.)

Problem 2. (2) Work 10.3/2
(Use scratch paper to organize and write up the ideas in logical order.)
Do not use an indirect argument – argue directly.
Focus on the conclusion: i.e., start with an arbitrary $x_0$. What is it you wish to prove about $f(x_0)$? (And note the hint in Prob. 1d above.)

Problem 3. (2) Let $f(x) = \tan(1/x)$, $x \neq 0$; $f(0) = 0$.
Is $f(x)$ defined for $x \approx 0$? Prove your answer.

Problem 4. (2) Work 11.1/4: to prove $e^x$ is continuous at every point $x_0$, assuming it is continuous at 0.
(Write $x = x_0 + h$ and use the $\epsilon$-form for the definition of continuity given in 11.1.
For older printings, the exponential law is $e^{a+b} = e^a e^b$.)

Problem 5. (2) Work 11.1/6. Use the hint given.

Reading Wed: 11.2-.5 Limits and Limit theorems for functions, Limit form of Continuity, Discontinuity-types, Sequential Continuity

Problem 6. (1) Reprove problem 4, by using simple limit theorems.

(One of the two inequalities needed to apply the Squeeze Theorem is a bit less obvious than the other. Keep in mind what the variable will be when you do the squeezing.)
Problem 8. (3: 2,1) Work 11.4/2, changing \([a, b]\) in both of its occurrences to \([a, b]\).
This shortens the work without sacrificing any of the ideas. Do it in two steps:

a) Prove a weaker theorem: \(f(x)\) is increasing on \([a, b]\).
(Consider \(a < x < x_0 < b\); what involving \(x_0\) is it that you have to prove? Use a limit theorem to give a direct argument – no indirect proofs.)

b) Using part (a), prove the stronger result: \(f(x)\) is strictly increasing on \([a, b]\).
(A “buffer” on the \(x\)-axis is needed, analogous to the \(M\) in the proof of the ratio test for series.)

The last two problems are exercises in the use of the Sequential Continuity Theorem 11.5: in working them, use the limit form of continuity; don’t go back to the basic \(\epsilon\)-form definition of continuity.

Problem 9. (3: 1.5, 1.5)

a) Given any real number \(c\), prove there is a sequence of rational numbers \(a_n\) and a sequence of irrational numbers \(b_n\), both of which are increasing and have \(c\) as their limit.
Your proof must use Theorem 2.5 (aka Problem 4 on P-set 0), not infinite decimals.

b) Using part (a), prove that if two functions \(f(x)\) and \(g(x)\) are continuous on \(\mathbb{R}\) and agree on all rational points, i.e., \(f(a) = g(a)\) whenever \(a\) is a rational number, then \(f(x) = g(x)\) for all \(x \in \mathbb{R}\).

Problem 10. (3) Work P11-2 (Problem 11-2), assuming \(c > 0, c \neq 1\).
(Hint: if \(f(x)\) is a constant function, what must its constant value be?
Start with an arbitrary \(x\), and show \(f(x)\) has that value.)