18.100A Fall 2017 Problem Set 4 (complete) due Fri. Oct. 9

Directions: As before: list collaborators; illegal to consult previous semesters' assignments.

The main topic for the next four classes will be continuity – the study of continuous functions: this week chapter 11 and half of 13 (which has the Extremal-value Theorem (aka the Maximum Theorem, and sometimes called the Fundamental Theorem of Analysis).

The two preliminary chapters 9 and 10 with the first few accompanying problems here are for background, and will be primarily a reading assignment.
Chapter 9 is about general functions and the elementary properties they can have; the assigned part should be mostly a review.
Chapter 10 is mostly about extending the language and properties of sequences to general functions – a sequence is actually just a special kind of function.
Chapter 11 then introduces two closely connected topics: continuous functions and limits of functions, the topics of the Mon. and Wed. classes and this problem set.

Reading Mon.
9.1-.3 Def'n and properties of functions, (The def'n "mapping" may be new to some.)
10.1-10.3 Estimation and approximation for functions; local and global properties.
11.1 ϵ-form def'n of continuity, examples, types of discontinuities.

Problem 1. (4) a) Work 9.3/3

(b) Work 10.1/1 (to get familiar with the notation $f(I)$, where I is an interval on which $f(x)$ is defined, and $f(x)$ is thought of as a map: $I \to \mathbf{R}$).

(c) Work 10.1/6b (needs 18.01 calculus)

(d) Work 10.1/7b (Express boundedness using absolute values, as in (2), section 10.1.)

Problem 2. (2) Work 10.3/2
(Use scratch paper to organize and write up the ideas in logical order.)
Do not use an indirect argument – argue directly.
Focus on the conclusion: i.e., start with an arbitrary x_0. What is it you wish to prove about $f(x_0)$? (And note the hint in Prob. 1d above.)

Problem 3. (2) Let $f(x) = \tan(1/x),\ x \neq 0; f(0) = 0$.
Is $f(x)$ defined for $x \approx 0$? Prove your answer.

Problem 4. (2)
Work 11.1/4: to prove e^x is continuous at every point x_0, assuming it is continuous at 0.

(Write $x = x_0 + h$ and use the ϵ-form for the definition of continuity given in 11.1.
For older printings, the exponential law is $e^{a+b} = e^a e^b$.)

Problem 5. (2) Work 11.1/6. Use the hint given.

Reading Wed: 11.2-.5 Limits and Limit theorems for functions, Limit form of Continuity, Discontinuity-types, Sequential Continuity

Problem 6. (1) Reprove problem 4, by using simple limit theorems.

Problem 7. (3) Work 11.3/3a.
(One of the two inequalities needed to apply the Squeeze Theorem is a bit less obvious than the other. Keep in mind what the variable will be when you do the squeezing.)

Problem 8. (3: 2,1) Work 11.4/2, changing $[a, b]$ in both of its occurrences to $[a, b)$.
This shortens the work without sacrificing any of the ideas. Do it in two steps:

a) Prove a weaker theorem: $f(x)$ is increasing on $[a, b)$.
(Consider $a < x < x_0 < b$; what involving x_0 is it that you have to prove? Use a limit theorem to give a direct argument – no indirect proofs.)

b) Using part (a), prove the stronger result: $f(x)$ is strictly increasing on $[a, b)$.
(A "buffer" on the x-axis is needed, analogous to the M in the proof of the ratio test for series.)

The last two problems are exercises in the use of the Sequential Continuity Theorem 11.5: in working them, use the limit form of continuity; don't go back to the basic ϵ−form definition of continuity.

Problem 9. (3: 1.5, 1.5)

a) Given any real number c, prove there is a sequence of rational numbers a_n and a sequence of irrational numbers b_n, both of which are increasing and have c as their limit.
Your proof must use Theorem 2.5 (aka Problem 4 on P-set 0), not infinite decimals.

b) Using part (a), prove that if two functions $f(x)$ and $g(x)$ are continuous on $\mathbf{R}$ and agree on all rational points, i.e., $f(a) = g(a)$ whenever a is a rational number, then $f(x) = g(x)$ for all $x \in \mathbf{R}$.

Problem 10. (3) Work P11-2 (Problem 11-2), assuming $c > 0,\ c \neq 1$.

(Hint: if $f(x)$ is a constant function, what must its constant value be?
Start with an arbitrary x, and show $f(x)$ has that value.)