18.100A Fall 2017 Problem Set 3 (complete, due **Monday Oct. 2**)

Directions: Same as on previous problem sets: list collaborators on the left top page and write up solutions independently; cite significant theorems used by name or number; consulting P-set solutions from previous semesters is not allowed.

Friday hand-ins: Friday is the Career Fair holiday, so P-set 3 is due the following Monday.

But this won't give Minjae and Calvin enough time to get the papers back by Wednesday, unless enough students complete the P-set as usual by Friday.

If you can do this, bring it on Friday by 3 PM to **2-110** (the Math Academic Services Office, open 9-5, near the 4-story wire sculpture), where there will be a box just inside the entrance for the P-sets, and enjoy a P-set-3-free weekend. And thanks.

Reading Fri.: 6.5 (omit proof of Th'm 6.5) Set-language, Completeness Principle for sets.

Problem 1. (2) Work 6.5/1bd

Problem 2. (2) Work 6.5/3ag

(For example, to do (a), show $\sup B$ satisfies sup-1 for A. At each step of (a) or (g), indicate what property of sup or inf is being used, and for which set (e.g., "by inf-2 for B").

Problem 3. (4: 2,2) Work P6-2ab.

This is an important problem; hints and warnings are being given for both parts to help you. Try it first without the hints, checking your proofs using the warnings. If stuck, consult the hints; if still stuck, consult with others.

Warning: For part (a), the most common faults with the sequence $\{a_n\}$ constructed are that the a_n may not be in S, or that the sequence may not have $\sup(S)$ as its limi t.

Hints: The construction should describe what to do, say why it is possible to do it, and why the resulting sequence has the desired properties. (The Theorem 6.2 construction of a subsequence converging to a cluster point might give you ideas.)

For part (b), Problem 2 (3g) above does half the work for you. The other half can be done using sequences; part (a) will help. (There are other ways of doing this.)

Problem 4 (2) Theorem 6.5: The Completeness Property for Sets:

If a set $S \subset \mathcal{R}$ is non-empty and bounded above, $\bar{m} = \sup S$ exists.

The class Friday covered the first half of the proof, which constructs a collection of nested intervals $[a_n, b_n]$ having a single point inside, and that point is $\bar{m}$.

Ths construction is given on p. 87, and a summary of the significant properties of these nested intervals is given at the bottom of p.87 and the first four lines of p.88.

This is as far as Friday's class got.

The second half of the proof shows that $\bar{m} = \sup S$, by showing that $\bar{m}$ has the two properties $\sup -1$ and $\sup -2$ for S that characterize uniquely $\sup S$.

Treating this second half of the proof like a Question, prove using the book's summary of the properties of the nested intervals that $\bar{m} = \sup S$. It is a good application of the two location theorems 5.3A and 5.3B. The proof that $\bar{m}$ satisfies $\sup -1$ can be done directly; the $\sup -2$ proof uses an indirect argument.

As for a Question, if you get stuck, you can take a quick look at the second half of the book's proof for hints, but then do the rest of the work, including the write-up, by yourself.

Reading: Mon.: 7.1-.2, 7.4 Infinite series; convergence and divergence tests.

Problem 5. (2) In this problem and the next, since the series are not specified, the more sophisticated tests in 7.4-.5 are not applicable and you will have to go back to the more basic tests in section 7.2, or the definition of convergence. Cite the theorems being used.

a) Work 7.2/3b

b) Work 7.2/4

Problem 6. (2) Work 7.2/5, an exercise in subsequences and the definition of convergence.

Problem 7. (2)) Prove the n-th root test for convergence (Theorem 7.4B), for the case $L < 1$ only, with the added assumption that the series are non-negative (so the absolute-value signs are not needed in the statement or the proof).

Study the proof of the ratio test first (again assuming the series are non-negative).

Problem 8. (2: .5, .5, 1) In this problem, use any of the tests in section 7.4, but omit the absolute value signs, since the three series involved are all positive. Show enough work so a reader can see how the tests are being used. Read the first two lines of 7.4-.5/1.

Work in 7.4-7.5/1 the Exercises: i) 1b ii) 1e (iii) 1h

Reading Wed: 7.3, 7.5-6, 8.1-2 through p.117 Abs. and Cond. convergence, 3 more tests; power series.

Focus mainly on the statements and use of the Theorems, not the proofs: cf. alternative proofs below.

Problem 9. (2: .5,1.5) Another proof of the Absolute Convergence Theorem 7.3

a) What condition on the terms of a series $\sum_0^\infty a_n$ is equivalent to saying its partial sums $\{s_n\}$ form a Cauchy sequence? Show that it is the following condition, called the "Cauchy criterion for series convergence":

$$\text{given } \epsilon > 0, \quad |a_{n+1} + \ldots + a_m| < \epsilon \quad \text{for } m > n \gg 1\ . \tag{1}$$

and prove that a series converges $\iff$ it satisfies the Cauchy criterion (1). You can use:

A sequence converges $\iff$ *it is a Cauchy sequence.* (Theorem 6.4 and Exercise 6.4/1)

b) Work P7-5, which uses part (a) to provee the Absolute Convergence theorem 7.3 .

Problem 10. (3: 1,2)

(a) Prove the Asymptotic Comparison test (Theorem 7.5B), assuming the series are non-negative (so you can drop the | | signs – compare one series with twice the other.)

(b) In using the Asyptotic Comparison test, the most important comparison series are the ones in Example 7.5A, whose convergence is established by the integral test. (The more commonly used ratio test fails for all of these.)

Test the following for convergence: (show work or reasoning)

(i) 7.4/1a (ii) 7.4/1g (iii) $\displaystyle\sum_1 \frac{n-5}{\sqrt{10n^4+n^5}}$ (iv) $\displaystyle\sum_1 \cos(1/n)$

Problem 11. (2: .5, 1.5) This is about the history of two famous problems concerning prime numbers; it uses convergence tests.

Around 1740 Euler proposed the problem of determining whether the series of prime reciprocals converges or diverges:

$$\sum_1^\infty \frac{1}{p_n} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots + \frac{1}{p_n} + \ldots\ ; \qquad p_n = \text{the } n\text{-th prime}$$

If primes are sparsely distributed among large integers, the terms of the series should be very small for $n \gg 1$ and the series should converge; if there are "enough" primes far out, the series should diverge.

About 70 years later, Gauss and Dirichlet independently conjectured the most accurate version of what is called the Prime Number Theorem (PNT), but it was not proved until around 1900, after almost a century of strenuous efforts by many mathematicians.

There are various equivalent forms in which te PNT can be stated, but the most convenient one to use here is

$$\textit{Prime Number Theorem:} \qquad p_n \sim n \ln n\ ,$$

where the symbol $\sim$ (read: "is asymptotic to" or "twiddles") has the meaning given in the Asymptotic Comparison test 7.5B.)

a) For $n = 20$, find to the nearest integer the numerical value of the left and right sides of the PNT.

b) Use the PNT to determine (with proof, and citing theorems) whether the series of reciprocal primes converges or diverges.

Problem 12. (1) Work 7.6/1a,b
(For conditional convergence, you have to show two things.)

Problem 13. (3: 1, .5, 1.5) For each of the following power series,
find its radius of convergence R, showing work;
determine with proof whether it converges or diverges at the endpoints $x = \pm R$;
(for part (c) do this for just one of the endpoints).
Identify the test being used.

$$a)\ \sum_1^\infty \frac{x^n}{\sqrt{2^n n}} \qquad b)\ \sum_0^\infty \frac{(-1)^n 3^{2n} x^n}{n!} \qquad c)\ \sum_1^\infty \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!}\, x^n$$

Problem 14. (1) Work 8.1/3ab.
This gives a proof of the Radius of Convergence Theorem 8.1 for practically all commonly occurring power series.