18.100A Fall 2017 Problem Set 2 due Fri. Sept. 22

Directions: You can collaborate, but should list those you worked with and write up the solutions independently (i.e., not copying but thinking them through by yourself). List collaborators on the top left, put your name on the top right.

Consulting solutions to problem sets of previous semesters is not allowed.

Completion: of Fri. class proof of:

c a cluster point of $\{x_n\}$ $\Rightarrow$ $\{x_n\}$ has a subsequence $\{x_{n_i}\}$ $\to c$.

Proof that the subsequence $\{x_{n_i}\}$ constructed in class converges to c :

We know $|x_{n_i} - c| < 1/i$ for all i.
Given $\epsilon > 0$, choose an integer I_ϵ so that $1/I_\epsilon < \epsilon$.
Then, for all $i \geq I_\epsilon$, we have $|x_{n_i} - c| < 1/i < 1/I_\epsilon < \epsilon$.
By the definition of limit, this shows $x_{n_i} \to c$.

Reading: Fri. Chap. 5.4, 6.2 Subsequences; non-existence of limits; Cluster Points.

Problem 1. Read p. 411 (1 pt.) Most students think of what is called simple induction as the basis step, usually $P(0)$, and the induction step $P(n) \Rightarrow P(n+1)$.

While this is all right for some problems, the more general boxed form of induction given on p. 411, which starts from and focuses on $P(n+1)$, rather than $P(n)$, is often a better way of looking at induction.

In the boxed formulation, simple induction is where you just use $P(n)$ in the proof of $P(n+1)$; strong induction − less often needed –is where you use some or all of the $P(i)$, $i = 1, 2, \ldots, n$ in the proof of $P(n+1)$.

Prove the Lemma below by simple induction but use the boxed form of it on p. 411.

The Subsequence Lemma: Let $\{n\} = 0, 1, 2, 3, \ldots, n, \ldots$, and $\{n_i\} = n_0, n_1, n_2, \ldots$ be a subsequence. Then $n_i \geq i$ for all i.

Problem 2. (3: 1, 1, 1) Part (a) is a special case of (b), but uses simpler notation and is a good preliminary exercise.

a) Work 5.4/1a;
b) Work 5.4/1b..
c) Work 5.5/1, an application of part (b).

Problem 3. (2) Work 5.4/2, following the ideas in section 5.4; you can use Exer. 3.4/4.

Problem 4. (2) Work 6.2/1b

Problem 5 (2) For the sequence $\{\dfrac{s(n)}{n}\}$ in Problem 3, prove that all the rational numbers $1/n$, $n \geq 1$ are cluster points of the sequence.

(Easier: for (1) point, prove the numbers $1/p$, p prime, are cluster points.)

Reading: Mon. Chap. 6.1, 6.3 Other forms of the Completeness Principle: Nested Intervals and Bisection, Bolzano-Weierstrass Theorem

Problem 6 (2: 1.5, .5)

a) A sequence of nested intervals is constructed by taking $[0, 1]$ as the starting interval $[a_0, b_0]$, bisecting it, and choosing the left half as the next interval $[a_1, b_1]$; then bisecting this in turn and choosing its right half as $[a_2, b_2]$; then continuing in this way, alternately choosing the left half and the right half of the bisected previous interval.

What unique number lies inside all these intervals? Prove it.

(Suggestion: obtain it as the limit of a_n; express a_n not as a single rational number, but rather as the partial sum of an infinite series.)

b) Without repeating any of the work of part (a), give the answer to the same problem, starting instead with an arbitrary interval $[a, b]$. Give in a few words your reasoning.

Problem 7. (3) Work 6.3/1abc

Problem 8. (2) Suppose all the terms of the sequence x_n lie in the interval $[a, b]$. Prove it has a subsequence converging to a point in $[a, b]$.

Reading Wed. Chap. 6.4 Cauchy sequences.

Problem 9. (3: 1,2)

a) Work 6.4/1

b) Work 6.4/3 using the sequence $\ln n,\ n > 0$ as the example. You may assume that $x_n \to a \ \Rightarrow\ \ln(x_n) \to \ln(a)$.

(Show it satisfies the condition given; how do you know it is not a Cauchy sequence?)

Problem 10 (2) Work 6.4/2 (this will be used in Problem 11).

Problem 11 (3) Work P6-1ab (Problem 6-1 on page 91) taking $a = 0,\ b = 1$.

For part (a) of P6-1, you can use Example 6.4 (9) and (10), and the result in Problem 10—; the calculations are then applicable to part (b) of P6-1.

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