18.100A Fall 2017 Assignment 11 (complete) due Fri. Dec. 8

Directions: Same as before: list collaborators, write up independently, illegal to use ass't solutions from previous semesters

Reading: Fri. 26.1-.2, to top p.379
Functions defined by integrals with a parameter: continuity, differentiabiliy

1. (2) Work 26.1/1b, in reverse order:
(i) first find the largest x-interval on which the Continuity Theorem 26.1 predicts the integral will be continuous;
(ii) then use this information to determine the limit asked for. Cite relevant theorem(s) for the limit.

2. (3: 1,2) Let $\phi(x) = \int_0^\pi \sin(xt)\,dt$.
a) The Continuity Theorem 26.1 says $\phi(x)$ is continuous for all x; verify this.
First, calculate $\phi(x)$ explicitly for all x, including $x = 0$ (no limits needed for this).
Then verify it is continuous for all x; use the limit form (11.4A) when $x = 0$.

b) The Derivative Theorem 26.2A says $\phi'(x)$ exists for all x and gives a formula for it.

(i) Use the formula to calculate $\phi'(x)$ explicitly for all x, including $x = 0$, using standard integration techniques; (no limits are needed for this).
Verify your calculations for $x \neq 0$ by using part (a) to calculate $\phi'(x)$, $x \neq 0$.

(ii) The Continuity Theorem predicts $\phi'(x)$ will be continuous for all x: why?
Verify this using (i), again using (11.4A) to prove continuity at $x = 0$.

3. (3) Work P26-1, showing that $J_0(x) = \frac{1}{\pi}\int_0^\pi \cos(x\sin\theta)\,d\theta$ is the solution to Bessel's ODE satisfying the given initial conditions.
(Since the IVP has a unique solution, this gives another proof that $J_0(x)$ in its integral form and in its power series form represent the same function.)

You will need to differentiate $J_0(x)$ twice; verify hypotheses for Thm.26.2A each time, and verify the initial conditions are satisfied.

After substituting into Bessel's ODE and combining the integrals, you get a very big definite integral, which is supposed to have the value 0.
There are different ways to show this, using theorems in Chapter 20. If stuck, try sleeping on it and seeing if what to do or try is clearer in the morning.

Reading Mon. 26.2, 26.3 Leibniz formula; Fubini's theorem.

4. (3: 1.5,1.5) Work 26.2/2 two ways:
a) using the Leibniz formula (26.2B) (verify hypotheses);
b) in a more elementary way by setting $u = x - t$; cite theorems; verify hypotheses.

5. (2) Work 26.2/5.

6. (3) Work 26.3/2, which gives another way to prove the Derivative Theorem 26.2A, by using Fubini's theorem.
Only a few lines and a few theorems are needed; cite them as you use them.

Reading Wed: 27.1,-.2; 27.3 (Thm. 27.3: statement only) Improper integrals with a parameter: Uniform convergence, M-test analog, Laplace transform, continuity theorem.

7. (2: 1,1)) a) Work Q27.2/1 and Q27.3/1 (Usual Question treatment).
b) Work Q 27.2/2ab. Use the standard definition of "exponential type": there are constants C and k such that $|h(t)| \leq Ce^{kt}$, for $t \gg 1$. (It's a bit weaker than 27.1 (8).)

8. (4) a) Work 27.2/2; b) Work 27.3/1; where is $\phi(x)$ continuous, give reasoning.)

9. (2: .5,.5,1) Everyone knows that a ripe apple falling on his head gave young Isaac Newton the idea of gravitational force and the inspiration for inventing calculus to solve the problems in physics that gravitation posed.,

The Laplace Transform had a similar but less-well-known genesis in a daydream of the 19-year old Pierre-Simon de Laplace from a poor French farm family (his later promotion to "Marquis" was still far off). In it he dreamt of life as a famous mathematician and astronomer/physicist, immortal and rich.

The only real problem will be the money – financial questions do not interest me, they will interrupt my work and I do not wish to be bothered by them.

I will hoard it until the pile has grown big enough, then I will deposit all of it one day with the bank and tell them how much and when to pay my income from it for the rest of my life, or perhaps forever since I will have descendants and they in turn will have descendants.

I will also have to tell them what constant interest rate (compounded continuously, of course) I shall require to guarantee my initial deposit will exactly cover all the the payments forever.

After returning to reality, he amused himself by writing down the formula connecting the amount F of the one-time initial deposit, the desired income function $f(t)$ for all time t, and the constant interest rate x that would guarantee enough money at all times to cover the payouts:

$$F(x) = \int_0^\infty e^{-xt} f(t)\, dt \qquad \textit{The Laplace Transform}$$

Derive it yourself, by following these steps.

a) Find $\lim_{h\to 0}(1+h)^{\frac{1}{h}}$. (Use the ln and exp functions; show work.)

b) An amount A is deposited in the bank at time $t = 0$ (years), at the interest rate r (e.g., $r = .05$ if the rate is 5%). If the time interval $[0, t]$ is divided into n equal subintervals, and the successive amounts A has grown to at the end of each interval are calculated, the resulting amount at time t is

$$A_n = A\Big(1 + r\frac{t}{n}\Big)^n \qquad \text{the amount after } n \text{ compoundings.}$$

Find the amount $A_\infty = \lim_{n\to\infty} A_n$: the amount at time t if the interest is, as one says, 'compounded continuously' using the rate r. (Use part (a), show work.)

c) As usual in setting up integrals, saying $f(t)$ is the income function means that over a small time interval $[t,\ t + \Delta t]$, the amount paid out will be approximately $f(t)\Delta t$.

(In what follows, use your answer to part (b), but change the variable r representing the interest rate to x, to make it more compatible with the Laplace transform notation.)

Assuming $f(t)$ is integrable, what amount must be deposited at time $t = 0$ for this amount to grow into the amount of payout required

over $[t,\ t + \Delta t]$? over $[0,\ T]$? over $[0,\ \infty]$?