18.100A Fall 2017: Problem Set 10 (complete) due Fri. Dec. 1

Directions. Same as for previous assignment. This is an extension of Pset 10A. The whole P-set when posted will be due Dec. 1.

Read Mon.: 24.1-.2 omit proof of the B-W Thm 24.2C—see Problem 4 below
Read 24.3-.5 Norms, seq's and fcn's on $\mathbf{R}^2$; convergence and continuity th'ms in $\mathbf{R}^2$.

1. (2: 1,1) a) Work 24.1/3 (Equiv'ce of uniform norm $|| \ ||$ and Euclidean norm $| \ |$)

b) Work 24.2/3, as a typical illustration of the significance of two norms being equivalent.

2. (2) An RAR ("right angle robot") moves in an x-y coordinate system, but always parallel to one of the two axes, so it can only change its direction by a right angle to the left or right, or by reversing direction.

(a) Write down a suitable definition in x-y coordinates for the RAR-norm $||| \ |||$, measuring the minimal length of any of the paths it can use to get from $\mathbf{0}$ to $\mathbf{x}$.
Using it, prove that $||| \ |||$ satisfies the triangle inequality.

(b) Draw a picture of the region $\mathbf{C}(\mathbf{0}, r) = \{\mathbf{x} \ \epsilon \ \mathbf{R}^2 : |||\mathbf{x}||| < r\}$, analogous to the open disc for the Euclidean norm and the open box for the uniform norm.
Show work; what are the equations of the boundary curves for $\mathbf{C}(\mathbf{0}, r)$?

3. (2) Work 24.2/2, as an exercise in convergence of sequences in the plane. Use coordinate-wise convergence.
On each main section of the region $\mathcal{D}$, indicate with an arrow (or some other notation if necessary) what the limit of the points in that section is, as $n \to \infty$.
Be sure to include the boundary of the region: it has several sections, having different limits (or no limit: indicate that too).

4. (2: .5, 1.5) Proving the Bolzano-Weierstrass Theorem in $\mathbf{R}^2$ using coordinate-wise convergence. (The proof is in the book, so treat this as a Question, peeking (quickly) at the proof only for a hint if stuck.)

a) Critique the following (false) proof often given by students:
Proof: Let $\mathbf{x}_n = (x_n, y_n)$ be a bounded sequence in $\mathbf{R}^2$. Then by the usual B-W Theorem in $\mathbf{R}$, the bounded sequence x_n has a convergent subsequence $x_{n_i} \to a$ and similarly, the bounded sequence y_n has a convergent subsequence $y_{n_i} \to b$.
Then by coordinate-wise convergence, the subsequence $(x_{n_i}, y_{n_i}) = \mathbf{x}_{n_i} \to \mathbf{a} = (a, b)$.

b) Fix up the proof in part (a) so it becomes a real proof by coordinate-wise convergence.

5. (1.5: .5, 1) Work 24.5/1 as follows:

a) First show $f(x, y)$ is continuous on every vertical line $x = a$ and every horizontal line $y = b$, including the two such lines which go through the origin $\mathbf{0}$. (Use standard facts about the continuity of rational functions.)

b) Then work 24.5/1 as written.

Reading Wed.: 24.6-7 Theorems about continuous functions on a compact set in $\mathbf{R}^2$,

6. (2) a) Work 24.7/1 (assume S is non-empty, and Euclidean norm for disance)

b) Work Q24.7/2 (use book sol'n only if stuck, and only for hints, not copying)

More reading Wed: 25.1: pp.364,365 only) Cluster points and closed sets in $\mathbf{R}^2$

Problems 7 and 8 below are about closed sets; both are importnt theory problems and ask for proofs. The proofs give a good application of the definitions of cluster points and closed sets, and of theorems in Chapter 24 about continuous functions on $\mathbf{R}^2$.

Hint: For problems involving cluster points of a set S, in Def'n 25.1A try first to use the limit definition (1c in the book): it often it is the best choice.

(Its use is so frequent that many books call such a point $\mathbf{a}$ "limit point of S" instead of "cluster point of S", since these are the points which are limits of sequences in S.)

Note: 1. A cluster point $\mathbf{a}$ need not be in S.

2. In the sequence $\mathbf{x}_n \to \mathbf{a}$, we require $\mathbf{x}_n \neq \mathbf{a}$ for all n.

(Otherwise, if $\mathbf{a}$ were any point in S, $\lim\limits_{n\to\infty} \mathbf{a}, \mathbf{a}, \mathbf{a}, \ldots = \mathbf{a}$; thus every point of S would be a cluster point, and when everyone is somebody, then no one's anybody.

7. (1.5: .5,1) Thm. 25.1B: *If $f(\mathbf{x})$ is continuous on $\mathbf{R}^2$, then*
$\overline{S}_f = \{\mathbf{x} : f(\mathbf{x}) = 0\}$, $\quad \overline{S}_f^+ = \{\mathbf{x} : f(\mathbf{x}) \geq 0\}$, $\quad \overline{S}_f^- = \{\mathbf{x} : f(\mathbf{x}) \leq 0\}$. are closed sets.

Prove the first two sets are closed.

(Since the proof is in the book, treat this problem like a Question. Use sequential continuity in R^2 and the usual limit location theorem for sequences in R if needed.

8. (2) We can think of a function $w = f(\mathbf{x})$ defined for all $\mathbf{x} \in \mathbf{R}^2$ as giving a map $f : \mathbf{R}^2 \to \mathbf{R}^1$. If $S \subset \mathbf{R}^1$, we define the *inverse image of S under f* to be

$$f^{-1}(S) = \{\mathbf{x} \in \mathbf{R}^2 : f(\mathbf{x}) \in S\} .$$

Assume $f(\mathbf{x})$ is continuous; prove that if S is closed in $\mathbf{R}^1$, then $f^{-1}(S)$ is closed in $\mathbf{R}^2$.

(Focus on what you have to prove about $f^{-1}(S)$; observe the *Note* and *Hint* given above.)

Comment: The above is true for any map $f(\mathbf{x}) : \mathbf{R}^n \to \mathbf{R}$ defined by a continuous function on $\mathbf{R}^n$. Conversely, if $f^{-1}(S)$ is closed for all closed subsets S of R, then the function $f(\mathbf{x})$ is continuous on $\mathbf{R}^n$. This gives an alternative definition of a continuous function on $\mathbf{R}^n$.

Read Mon.: 25.2-.3 Compactness Theorem; Open Sets. (You can skip the proof of the Complementation Thorem 25.3C – a more intuitive approach will be given in Notes for the next class.)

9. (2) Work Q25.1/bcde for easy practice in using Theorems 25.1A and B.

10. (3: 1.5; .5,1) Both of these problems are about using theorems about compact sets to prove things about sets which are not compact. Use the theorems in 25.1 and 25.2.

a) Work 25.2/2, an extension of Problem 6a above to a set S which is not assumed to be compact. Assume only that S is closed and not empty. Use the ordinary notion of distance (i.e. the Euclidean norm) to interpret the word "nearest".

b) (i) Let S be the graph in $\mathbf{R}^2$ of the parabola $y = 2x^2 - 1$. Using 25.1 and 25.2, tell (with proof) whether it is closed, compact, or neither.

(ii) Consider the sequence $\mathbf{x}_n = (\cos n, \cos 2n)$, $n = 0, 1, 2, \ldots$ in $\mathbf{R}^2$.

Using the theorems in Chapter 25, prove the sequence has a subsequence which converges to a point $\mathbf{a}$ on the parabola in part (i).

11. (2: 1,1) a) Work 25.1/4a b) Work 25.1/4b
These use 25.1A and B, and 25.3A and B. Give reasoning and make a sketch of both sets.

12. (3: 1,2) Work 25.2/5 — prove the following theorem, one of the important facts about compact sets:

Let $f(\mathbf{x}) : \mathbf{R}^2 \to \mathbf{R}^1$ be a continuous function mapping $\mathbf{R}^2$ to $\mathbf{R}^1$. Then if S is a compact set in $\mathbf{R}^2$, its image $f(\mathbf{S})$ is a compact set in $\mathbf{R}^1$.

a) Compact sets in $\mathbf{R}^n$ are characterized two ways: as the closed and bounded sets, or – using sequences $\{\mathbf{x}_n\}$ – as the sets satisfying the sequential compactness condition.

One could try proving the above theorem by "divide and conquer": proving separately that

$$S \text{ bounded} \Rightarrow f(S) \text{ bounded} \qquad \text{and} \qquad S \text{ closed} \Rightarrow f(S) \text{ closed} \quad .$$

Prove by counterexamples that both statements are false. (Problem 11 helps.)

b) Instead, prove the theorem by using sequences: show $f(S)$ satisfies the sequential compactness definition, if S does.

Focus on the theorem's conclusion: what are you trying to show about $f(S)$? How can the hypotheses about f and S help you do this?

13. (1) Is the domain D of the function $\tan(1/x)$ an open subset of $\mathbf{R}$, a closed subset, or neither? Indicate reason.

Read Wed: Notes on Open and Closed Sets (one page, sent by e-mail attachment).

14. (3; .5 for each part) Using the Notes, work 25.3/1a,d,g,h,i,j, in conjunction with Problem 25-1.

For each of these six sets, draw a sketch of the set, describe its boundary points, tell whether it is open, closed, compact, or none of these, and give a brief reason that shows you are not guessing.

15. (1) Let $\mathbf{a}$ be a cluster point of S. Prove: if $\mathbf{a}$ is not in S, then $\mathbf{a}$ is in $\partial(S)$.
(Use the first definition of cluster point: 25.1(1a).