Directions. Same as for previous assignment. This is an extension of Pset 10A. The whole P-set when posted will be due Dec. 1.

Read Mon.: 24.1-.2 
omit proof of the B-W Thm 24.2C—see Problem 4 below 
Read 24.3-.5 Norms, seq's and fcn's on \( \mathbb{R}^2 \); convergence and continuity th'ns in \( \mathbb{R}^2 \).

1. (2: 1,1) a) Work 24.1/3 (Equiv'ce of uniform norm \( || || \) and Euclidean norm \( | | \))
   b) Work 24.2/3, as a typical illustration of the significance of two norms being equivalent.

2. (2) An RAR (“right angle robot”) moves in an \( x \)-\( y \) coordinate system, but always parallel to one of the two axes, so it can only change its direction by a right angle to the left or right, or by reversing direction.
   a) Write down a suitable definition in \( x \)-\( y \) coordinates for the RAR-norm \( |||| \), measuring the minimal length of any of the paths it can use to get from \( 0 \) to \( x \).
   Using it, prove that \( |||| \) satisfies the triangle inequality.
   b) Draw a picture of the region \( C(0, r) = \{ x \in \mathbb{R}^2 : |||x||| < r \} \), analogous to the open disc for the Euclidean norm and the open box for the uniform norm.
   Show work; what are the equations of the boundary curves for \( C(0, r) \)?

3. (2) Work 24.2/2, as an exercise in convergence of sequences in the plane. Use coordinate-wise convergence.
   On each main section of the region \( D \), indicate with an arrow (or some other notation if necessary) what the limit of the points in that section is, as \( n \to \infty \).
   Be sure to include the boundary of the region: it has several sections, having different limits (or no limit: indicate that too).

4. (2: .5, 1.5) Proving the Bolzano-Weierstrass Theorem in \( \mathbb{R}^2 \) using coordinate-wise convergence. (The proof is in the book, so treat this as a Question, pecking (quickly) at the proof only for a hint if stuck.)
   a) Critique the following (false) proof often given by students:
   \[ \text{Proof:} \ 
   \text{Let } x_n = (x_n, y_n) \text{ be a bounded sequence in } \mathbb{R}^2. \text{ Then by the usual B-W Theorem in } \mathbb{R}, \text{ the bounded sequence } x_n \text{ has a convergent subsequence } x_{n_i} \to a \text{ and similarly, the bounded sequence } y_n \text{ has a convergent subsequence } y_{n_i} \to b.
   \]
   Then by coordinate-wise convergence, the subsequence \( (x_{n_i}, y_{n_i}) = x_{n_i} \to a = (a, b) \).
   b) Fix up the proof in part (a) so it becomes a real proof by coordinate-wise convergence.

5. (1.5: .5, 1) Work 24.5/1 as follows:
   a) First show \( f(x, y) \) is continuous on every vertical line \( x = a \) and every horizontal line \( y = b \), including the two such lines which go through the origin \( 0 \). (Use standard facts about the continuity of rational functions.)
   b) Then work 24.5/1 as written.

Reading Wed.: 24.6-7 Theorems about continuous functions on a compact set in \( \mathbb{R}^2 \),

6. (2) a) Work 24.7/1 (assume \( S \) is non-empty, and Euclidean norm for distance)
   b) Work Q24.7/2 (use book sol’n only if stuck, and only for hints, not copying)
More reading Wed: 25.1: pp.364,365 only) Cluster points and closed sets in $\mathbb{R}^2$

Problems 7 and 8 below are about closed sets; both are important theory problems and ask for proofs. The proofs give a good application of the definitions of cluster points and closed sets, and of theorems in Chapter 24 about continuous functions on $\mathbb{R}^2$.

**Hint:** For problems involving cluster points of a set $S$, in Def'n 25.1A try first to use the limit definition (1c in the book): it often it is the best choice.

(Its use is so frequent that many books call such a point a "limit point of $S" instead of "cluster point of $S", since these are the points which are limits of sequences in $S").

**Note:**
1. A cluster point $a$ need not be in $S$.
2. In the sequence $x_n \to a$, we require $x_n \neq a$ for all $n$.
   (Otherwise, if $a$ were any point in $S$, $\lim_{n \to \infty} a, a, a, \ldots = a$; thus every point of $S$ would be a cluster point, and when everyone is somebody, then no one’s anybody.

7. (1.5: .5,1) Thm. 25.1B: If $f(x)$ is continuous on $\mathbb{R}^2$, then
   \[ \overline{S}_{\overline{f}} = \{ x : f(x) = 0 \}, \quad \overline{S}_{\overline{f}^+} = \{ x : f(x) \geq 0 \}, \quad \overline{S}_{\overline{f}^-} = \{ x : f(x) \leq 0 \}. \]
   are closed sets.

   Prove the first two sets are closed.

   (Since the proof is in the book, treat this problem like a Question. Use sequential continuity in $\mathbb{R}^2$ and the usual limit location theorem for sequences in $R$ if needed.

8. (2) We can think of a function $w = f(x)$ defined for all $x \in \mathbb{R}^2$ as giving a map $f : \mathbb{R}^2 \to \mathbb{R}^1$. If $S \subset \mathbb{R}^1$, we define the inverse image of $S$ under $f$ to be
   \[ f^{-1}(S) = \{ x \in \mathbb{R}^2 : f(x) \in S \}. \]
   Assume $f(x)$ is continuous; prove that if $S$ is closed in $\mathbb{R}^1$, then $f^{-1}(S)$ is closed in $\mathbb{R}^2$.

   (Focus on what you have to prove about $f^{-1}(S)$; observe the Note and Hint given above.)

Comment: The above is true for any map $f(x) : \mathbb{R}^n \to \mathbb{R}$ defined by a continuous function on $\mathbb{R}^n$. Conversely, if $f^{-1}(S)$ is closed for all closed subsets $S$ of $R$, then the function $f(x)$ is continuous on $\mathbb{R}^n$. This gives an alternative definition of a continuous function on $\mathbb{R}^n$.

Read Mon.: 25.2-.3 Compactness Theorem; Open Sets. (You can skip the proof of the Complementation Theorem 25.3C – a more intuitive approach will be given in Notes for the next class.)

9. (2) Work Q25.1/bcede for easy practice in using Theorems 25.1A and B.

10. (3: 1.5; .5,1) Both of these problems are about using theorems about compact sets to prove things about sets which are not compact. Use the theorems in 25.1 and 25.2.

   a) Work 25.2/2, an extension of Problem 6a above to a set $S$ which is not assumed to be compact. Assume only that $S$ is closed and not empty. Use the ordinary notion of distance (i.e. the Euclidean norm) to interpret the word “nearest”.

   b) (i) Let $S$ be the graph in $\mathbb{R}^2$ of the parabola $y = 2x^2 - 1$. Using 25.1 and 25.2, tell (with proof) whether it is closed, compact, or neither.

   (ii) Consider the sequence $x_n = (\cos n, \cos 2n), \quad n = 0, 1, 2, \ldots$ in $\mathbb{R}^2$.

   Using the theorems in Chapter 25, prove the sequence has a subsequence which converges to a point $a$ on the parabola in part (i).
11. (2: 1,1) a) Work 25.1/4a  b) Work 25.1/4b
These use 25.1A and B, and 25.3A and B. Give reasoning and make a sketch of both sets.

12. (3: 1,2) Work 25.2/5 — prove the following theorem, one of the important facts about compact sets:
Let \( f(x) : \mathbb{R}^2 \rightarrow \mathbb{R}^1 \) be a continuous function mapping \( \mathbb{R}^2 \) to \( \mathbb{R}^1 \). Then if \( S \) is a compact set in \( \mathbb{R}^2 \), its image \( f(S) \) is a compact set in \( \mathbb{R}^1 \).

a) Compact sets in \( \mathbb{R}^n \) are characterized two ways: as the closed and bounded sets, or – using sequences \( \{x_n\} \) – as the sets satisfying the sequential compactness condition.
One could try proving the above theorem by “divide and conquer”: proving separately that
\[
S \text{ bounded } \Rightarrow f(S) \text{ bounded } \quad \text{and} \quad S \text{ closed } \Rightarrow f(S) \text{ closed }.
\]
Prove by counterexamples that both statements are false. (Problem 11 helps.)

b) Instead, prove the theorem by using sequences: show \( f(S) \) satisfies the sequential compactness definition, if \( S \) does.
Focus on the theorem’s conclusion: what are you trying to show about \( f(S) \)? How can the hypotheses about \( f \) and \( S \) help you do this?

13. (1) Is the domain \( D \) of the function \( \tan(1/x) \) an open subset of \( \mathbb{R} \), a closed subset, or neither? Indicate reason.

Read Wed: Notes on Open and Closed Sets (one page, sent by e-mail attachment).

14. (3; .5 for each part) Using the Notes, work 25.3/1a,d,g,h,i,j, in conjunction with Problem 25-1.
For each of these six sets, draw a sketch of the set, describe its boundary points, tell whether it is open, closed, compact, or none of these, and give a brief reason that shows you are not guessing.

15. (1) Let \( a \) be a cluster point of \( S \). Prove: if \( a \) is not in \( S \), then \( a \) is in \( \partial(S) \).
(Use the first definition of cluster point: 25.1(1a).)