18.100A Problem Set 0 due Fri. 9/8/17, by 1:05 in 4-163

Directions: Pset 0 is diagnostic, to get some idea of your "mathematical maturity" – how well you can read and write math, prove things, follow hints, find errors, etc. No other assignment will be due in just two days. It will be returned to you in class Monday. The numerical score on it will be recorded, but not count toward your total score. Therefore:

no collaboration, looking up solutions to problem sets from previous semesters, or outsider help (except from me–see below)**, is allowed on this first Pset.**

For subsequent Psets, you can collaborate, the rules given in the "18.100A Information and Rules" sheet on the website apply – reminders are on each of the early assignments.

Office hour help from me: 2-383 Thurs. 3:10-5:15 Tel:(617-25)3-4345 mattuck@mit.edu
It helps to have tried the problems first, so you know where you are having trouble.

E-mail help: I read it on a desk computer, so don't count on an instant response. I'll check e-mail a few times tonight (Wed. 10:30-11:30 PM), Thurs. 8-11PM, Fri. 8-9:30 AM.

Writing Style for P-sets:

1. Handwritten is preferred, but it must be dark enough amd with letters and symbols large enough to be easily readable. Subscripts and superscripts should be smaller sized, placed lower and higher, just as they appear in printed math.
2. Leave margins, left and right, for grading comments.
3. Use both sides of the paper, which should be heavy enough so what's written appears only on that side. Put successive problems on the same page until the next one won't fit, then start a new page: don't have a page break in the middle of a problem.
4. Imitate the textbook style, with equations and inequalities on successive lines, rather than on the same line, to make the clear the operations being used.
5. Use scratch paper first on problems you are unsure about, to avoid extensive crossing out on your answer sheet.
6. You can use computer writing (TeX, LaTeX, or something equivalent), if you can make it look like math books – especially 4. above.

Notation: To say $\{x_n\}$ has the limit L, write: $\lim_{n\to\infty} \{x_n\} = L$ or $\{x_n\} \to L$ as $n \to \infty$; often the $n \to \infty$ and braces $\{\ \}$ are understood and omitted.

In 100A, this use in limits is the **only** use for the single arrow $\to$. For implication, use only the double-stem arrow $\Rightarrow$, as in $A \Rightarrow B$ (A implies B).

If $a > 0$, by convention $\sqrt{a}$ always is understood to be its positive square root.

Problem 1. (2 points) **The A-G Mean Inequality** Read: 2.1 (Chapter 2, section 1).

Work P2-3 (Problem 2-3 at the end of Chapter 2) two ways, as follows:

a) **Algebraically:** In your answer, start from the inequality you want to prove and work backwards: keeping the $<$ going in the same direction on each successive line (3 or 4 lines should be enough), transform both sides using the inequality laws in 2.1, until it turns into an inequality which is obviously true. (In 100A this is called "backward reasoning".)

Then (side-by-side, or below) turn it into forward reasoning by copying the successive lines in reverse order, starting with the obvious final one and ending with the A-G Mean inequality. Check mentally that each forward step is legitimate.

b) **Geometrically:** Copy the book's figure onto your paper, giving the length of the two relevant line segments, and proof of the non-obvious length of one of them. (Add auxiliary line segments as needed, and use a similar-triangle argument.)

The figure then makes the inequality obvious.

Problem 2. (5 pts: 1.5, 1.5, .5, 1.5) (You can use any part – solved or not – in later parts.) Read: 1.1,1.2,1.3-skip proof; 1.6. (For (a), p.411; for (b) cf. 1.4,through p.6; Q1.4/1,pp.8,15.)

Let $a > 0$. Define $\{x_n\}$, $n \geq 0$ recursively by $\quad x_{n+1} = \frac{1}{2}\big(x_n + \frac{a}{x_n}\big), \quad x_0 > \sqrt{a}.$

Prove in the steps below that, given any such starting value $x_0 > \sqrt{a}$, the resulting sequence $\{x_n\}$ is bounded below, strictly decreasing, and has the limit $\sqrt{a}$.

(a) Prove using the A-G Mean inequality amd induction that $x_n > \sqrt{a}$ for all n.
(Include why a/x_n makes sense and why the conclusion has $>$ and not $\geq$.

(b) Prove $\{x_n\}$ is strictly decreasing, as follows.
As in Problem 1a, use backward reasoning and give both backward and forward reasonings on your solution sheet. Which step in the forward reasoning is a bit tricky?
(Cf. the reading on Q1.4/1. p.8. Signal backwards reasoning by a ? at the end of the first line of proof. Check mentally that each step in the backward reasoning is reversible; if necessary, justify it briefly.)

(c) Prove the sequence has a limit L, using only the ideas in Chapter 1.

(d) Prove $L = \sqrt{a}$. This is best done by using the familiar algebraic limit theorems from calculus, which we will get to in Chapter 5, but you can use this royal road ahead of time. Using the Notations for limits given earlier, the theorems say:

Let $\{a_n\} \to A, \quad \{b_n\} \to B. \qquad$ Then $\{a_n + b_n\} \to A + B, \quad \{a_n b_n\} \to AB,$

and $\{a_n/b_n\} \to A/B$, if $B \neq 0$ and $b_n \neq 0$ for all n.

Problem 3. (2 pts) Read: 1.4, p.7; E1.4/1 = Exercise 1.4 at the end of Chapter 1

a) Do E1.4/1, changing the numerators to $2, 3, 4, \ldots$ as in: $1/1+2/(1{\cdot}3)+3/(1{\cdot}3{\cdot}5)+\ldots$.
(The hint given is still valid for this new sequence, or do it some other way.)

b) Let m be a positive integer, and $x_n = \dfrac{m^n}{n!}$. Show x_n is bounded above, and find (in terms of m) its least upper bound (least = smallest, lowest).
(If stuck, try a special case: give m a small integer value, and study where the $\{x_n\}$ is increasing and where descreasing. "Show" means give a convincing argument, which can be somewhat informal, without a lot of symbols.

Problem 4. (2 pts.) Read: A.0-A.2 as needed, for notation and background.

Let $\mathcal{R}$ be the set of real numbers (think of them as the points on the real line). A subset S is said to be *dense* in $\mathcal{R}$ if every open interval (a, b), $a < b$ contains a point of S.

a) Prove that the set $\mathcal{Q}$ of rational numbers is dense in $\mathcal{R}$, without using the infinite decimal representation for real numbers; i.e., prove that

given (a, b), $a < b$, there are integers m, n such that $a < \dfrac{m}{n} < b$.

(Show how to find a suitable n, then an m.
Since any two successive integers k, $k + 1$, have spacing 1, you may use without proof the geometrically obvious fact:

if $c_2 - c_1 > 1$, then (c_1, c_2) contains an integer.)

b) Prove similarly that also the set of irrational numbers is dense in $\mathcal{R}$, by proving the stronger theorem that the numbers $r\sqrt{2}$, (r rational) are irrational and dense in $\mathcal{R}$.
(App. A proves $\sqrt{2}$ is irrational, so you can use this in your proof, but cite the page no.)

Problem 5. (4 pts.) Read: A.3-A.4 as needed.

The proof of Prop.1.4 begins with $(1+h)^2 \geq 1+2h$, for all h;
(the inequality $>$ is here being changed to $\geq$ to include the possibility $h = 0$).

A generalization of this is a statement we will denote by $P(n)$:

(1) $\qquad (1+h)^n \geq 1+nh, \quad \text{for all } n \geq 0 \text{ and all } h \qquad$ *(Berloony's Inequality)*

Proof of (1) by induction: (cf. App. A.4, Examples A and B)

The cases $P(0)$ and $P(1)$ are both trivial; either can be taken as the basis step.

The induction step $P(n) \Rightarrow P(n+1)$ is proved in the three inequalities below:
the first restates $P(n)$;
the second multiplies both sides by (1+h), preserving the inequality;
the third multiplies out the right-hand side and drops the positive nh^2 term, which preserves the inequality and converts it to $P(n+1)$, and this proves $P(n) \Rightarrow P(n+1)$.

$$\begin{aligned} (1+h)^n &\geq 1+nh \quad \text{for all } h; \\ (1+h)^{n+1} &\geq (1+nh)(1+h) \quad \text{for all } h; \\ (1+h)^{n+1} &\geq 1+(n+1)h \quad \text{for all } h, \text{ since } nh^2 \geq 0. \end{aligned}$$

a) (1) Berloony's inequality (1) is false as stated; see if you can locate and describe the error in its proof. If stuck after say ten minutes of effort, go on to (b) below (using scratch paper), and return to (a). If still stuck, there is another hint at the end of this problem.

b) (1) There are *counterexamples* (cf. A.3) to $P(5)$ which use integer values for h; find the one having h-value closest to 0, and prove it is a counterexample to $P(5)$ and the one with h-value closest to 0.

c) (1) Give the weakest hypothesis (i.e., condition) on h for which the attempted proof will actually be valid (cf. A.1 p. 405 for "weakest").

d) (1) Find an integer value h_0 for h which does not satisfy the condition in (c), but for which Berloony's inequality holds for all $n \geq 0$ and $h = h_0$; then prove it.

This last part (d) is significant, because it shows that **the failure of a method of proof** – induction here, for the case $h = h_0$ – **doesn't prove that a statement is false**: someone might find (as hopefully you just did) a different method which works. (But it does suggest the statement *might* be false and therefore it might be worth looking for a counterexample.)

Hint for Part (a):
After finding the counterexample, if you're still stuck on part (a), take the *trace* of your counterexample in the proof – i.e., substitute the numerical values of n and h used by the counterexample into each successive line of the proof, and see where it goes wrong; this should tell you in general where the error lies.