18.100A Fall 2016: Assignment 5 due Mon. Sept. 26

Assignment 5 is split into 5A and 5B, with different graders; label and staple the two parts separately, and put your name on each. (Office hour as usual: Thurs. 3:10 - 5)

The directions are the same as on previous problem sets.

Assignment 5A

Reading: Mon. 5.4, 6.2 (read the two Examples 6.2A, 6.2B, and just the statement (not the proof) of Theorem 6.2) Subsequences; Cluster points

Problem 1A. (1.5: .5, 1) Work 5.4/1ab
Part (a) is a special case of part (b), but (a) uses simpler notation and needs less reasoning, so it is a good warmup exercise for (b).

Problem 2A. (1.5) For the integer \( n = 2, 3, 4, \ldots \), let \( h(n) \) be its highest prime factor, and \( s(n) \) the sum of its prime factors, as illustrated by:
\[
\begin{align*}
  s(5) &= 5, \\
  h(18) &= 3, \\
  s(18) &= 8 \\
  h(8) &= 2, \\
  s(8) &= 6
\end{align*}
\]
Let \( x_n = \frac{h(n)}{s(n)} \). Prove that \( \lim_{n \to \infty} \{x_n\} \) does not exist.
(Use the Subsequence Theorem, and if needed the result in Exercise 3.4/4.)

Problem 3A. (1.5) For the sequence \( \cos\left[ (n + \frac{1}{n}) \frac{\pi}{2} \right] \), find its cluster points, and for each, give a subsequence which converges to it.
(No proofs required, but show work; assume: \( \lim a_n = c \Rightarrow \lim \cos(a_n) = \cos(c) \).)

Problem 4A. (1.5: 1, .5)
(a) Prove \( \{x_n\} \) in Problem 2A has the numbers \( 1/k, k = 1, 2, 3, \ldots \) as cluster points.
(b) Prove 0 is also a cluster point.

Problem 5A. (1.5) (Treat this as a Question: try to do it without looking at the book’s argument; if stuck, you can look at it for hints, but then think it through in your own words.)
Prove the forward direction of Theorem 6.2: if \( c \) is a cluster point of \( \{x_n\} \), it has a subsequence \( \{x_{n_i}\} \) converging to \( c \).
(Show how to construct the subsequence by picking out terms of \( \{x_n\} \); make sure the terms you pick out really form a subsequence – no repetitions, and in the original order.)

Assignment 5B

5B Reading: 6.1, 6.3, 6.4 Bolzano-Weierstrass Theorem; Cauchy sequences.

Problem 1B. (1) As a simple numerical example of The Nested Intervals Theorem, a sequence of intervals is constructed by taking \( [0, 1] \) as the starting interval \( [a_0, b_0] \), bisecting it at its midpoint, and choosing the right half as the next interval \( [a_1, b_1] \); then bisecting this in turn at its midpoint and choosing its left half as \( [a_2, b_2] \); then continuing in this way, alternately choosing the right half and the left half of the bisected previous interval.
According to Theorem 6.1, there will be a unique number \( c \) lying inside all of the intervals \( [a_n, b_n], n = 0, 1, \ldots \); why is this, and what is the value of \( c \)? (Show work.)
(Suggestion: obtain \( c \) as the limit of \( \{a_n\} \); express \( a_n \) not as a single rational number, but rather as the sum of \( 2n \) alternating forward and backward steps along the line.)
Problem 2B. (2: .5,.5,1) Work 6.3a,b,c’, where (c’) is \( \frac{a_n}{1+a_n^2} \). (Show reasoning.)

(The last one needs different estimations according to whether \(|a_n|\) is large or small.)

Problem 3B. (1) Suppose all the terms of the sequence \( \{x_n\} \) are in the interval \([a, b]\).
Prove that the sequence has a subsequence converging to a point \(c\) in \([a, b]\).

Cite the theorems you are using, by name or number.


Problem 5B. (2.5: 1; .5,1) (i) Work 6.4/2 (ii) Work P6-1ab (p. 91), for \(a = 0, \ b = 1\).

(In P6-1a, use Example 6.4 (9) and (10), and the result in 6.4/2; the calculations are then applicable to P6-1b.)