18.100A Fall 2016: Assignment 2 due Mon. Sept. 12

You can collaborate, but must write up the solutions independently, i.e., in your own words, thinking them through yourself. List any collaborators in the upper-left corner of the top page; put your name on the upper-right corner, to make returning easier.

Consulting internet solutions or assignment solutions from previous years is not allowed.

Reading: Chapter 2.2-.6 Estimations, absolute values, approximations, “for n large”.
Section 3.1 Definition of limit; examples with proof.

Problem 1. (1.5: .5, 1) Proposition: “If the sequences $a_n$ and $b_n$ are bounded above, then $a_nb_n$ is bounded above.”
   a) Prove the Proposition is false by giving a counterexample.
   b) Strengthen the hypotheses and prove the amended Proposition.
      (Read top p. 405 for “stronger statement”: here the “statements” are the hypotheses on the two sequences in the Proposition. The answer will be judged partly on how weak the new hypotheses are – the weaker they are, the stronger the resulting Proposition.)

Problem 2. (2) Let $c_1, c_2, \ldots, c_N$ and $a$ be real numbers. Prove the implication:
   
   $|\sum_{1}^{N} c_n \cos na| \geq 1 + \ln N \Rightarrow |c_n| > 1/n$ for some $n \leq N$.

   (Prove the contrapositive: not B $\Rightarrow$ not A, but write it without using “not”: note that in mathspeak, the phrase “for some $n$” means “for at least one value of $n$”. You will also need some ideas in Chapter 1, section 1.5, suitably adapted.)

Problem 3 (1) Using the triangle inequality, prove the transitive law for $\approx$ (2.5, (8)) and use it to work 2.5/2 (p.32).

Problem 4. (2) (i) Work 2.5/4 (p.32) which uses an important idea to give a cleaner proof of Theorem 2.5(i) than the one in the book, though a less intuitive one. 
   (“Cleaner” because it avoids the use of infinite decimals. Follow the hints given in the Exercise. Use minimal English, and scratch paper first so you can edit your proof.)
   
   (ii) Prove Theorem 2.5(ii) using similar ideas, as follows: first find an irrational number $\alpha$ such that for any integer $n$ we have $n < n + \alpha < n + 1$, then finish the proof by modifying slightly the argument in part (i).

Problem 5. (2: 1,1) a) Prove $\{x_n\}$ defined by $x_{n+1} = \frac{n^2 + 10}{(n+1)(n+3)} x_n, \ x_0 > 0$ is monotone for $n \gg 1$.
   (Two ways to show a positive sequence $a_n$ is increasing are to show the ratio $a_{n+1}/a_n \geq 1$ or show the difference $a_{n+1} - a_n \geq 0$.) Similarly for decreasing.
   
   b) For what $n$ will $3n/n + 2 \approx \epsilon$, if (i) $\epsilon = .1$ (ii) $\epsilon = .01$?

Problem 6. (1.5: 1, .5) Work: (a) 3.1/1b (b) 3.1/1c (Exercises, p. 46)
   Do them directly from Definition 3.1 of limit; don’t use any limit theorems you know from calculus (in Chapter 5 here). You can often make the work easier by first simplifying the expression you want to show is $< \epsilon$ – “simplifying” by making it larger but easier to estimate as $< \epsilon$. 