

# 2

## Estimations and Approximations

### 2.1 Introduction. Inequalities.

Our work with bounded sequences  $\{a_n\}$  in Chapter 1 — the sequences whose limits were  $\epsilon$  and  $\gamma$  and the sequence of harmonic sums — depended in each case on estimating the size of  $a_n$ . In general, estimation is not formal or routine; it is an art, but one that can be learned by studying and imitating examples. It is at the core of analysis.

Two simple tools form the language of estimations: *inequalities* for making comparisons and *absolute values* for measuring size and distance. We begin in this section with inequalities.

First of all, in comparing numbers with 0 we will say:

$$\begin{array}{ll} a \text{ is } \textit{positive} \text{ if } a > 0, & a \text{ is } \textit{non-negative} \text{ if } a \geq 0, \\ a \text{ is } \textit{negative} \text{ if } a < 0, & a \text{ is } \textit{non-positive} \text{ if } a \leq 0. \end{array}$$

The terms “non-negative” and “non-positive” point in the wrong direction and are awkward; fortunately, they are not so often needed.

**INEQUALITY LAWS** The algebraic laws for working with inequalities are familiar, but we flag several which even good mathematicians sometimes trip over. We will use  $<$  in the statements; the laws using  $\leq$  are analogous.

**Addition** You can only add inequalities in the same direction:

$$\begin{array}{ll} a < b & a < b \\ \underline{c < d} & \underline{c > d} \\ a + c < b + d & a + c ? b + d \end{array}$$

**Subtraction** Don't even think of doing this; see discussion below.

**Multiplication** This is legal provided that the inequalities are in the same direction and the numbers are *positive*:

$$a < b, c < d \Rightarrow ac < bd, \quad \text{if } a, b, c, d > 0.$$

**Sign-change law** Changing signs reverses an inequality:

$$a < b \Rightarrow -a > -b; \quad a < b \Rightarrow ka > kb, \quad \text{if } k < 0.$$

**Reciprocal law** The inequality reverses, if both numbers are positive:

$$a < b \Rightarrow 1/a > 1/b, \quad \text{if } a, b > 0.$$

**Remarks.**

All of the preceding are common sources of error, but the worst is probably the sign-change law. The reason is that changing all signs in an *equality* (to get rid of excess minus signs) is legal, and so people tend to apply the same rule to inequalities, failing to reverse the inequality sign. Or they multiply both sides of an inequality by  $\sin x$ , failing to realize that the direction of the resulting inequality depends on whether  $\sin x > 0$  or  $\sin x < 0$ .

After the above, the most common sin is trying to subtract inequalities in the same direction. To see why you can't do this, change the subtraction to addition, by using the sign-change law:

$$(a < b) - (c < d) \quad \text{is the same as} \quad (a < b) + (-c > -d) :$$

you're trying to add inequalities in opposite directions.

**Questions 2.1** (Answers at end of chapter)

1. Give a numerical example to show you cannot subtract inequalities in the same direction:  $a < b$  and  $c < d \Rightarrow a - c < b - d$  is false.

2. Show the reciprocal law is also true if both numbers are negative.

(Call them  $-a$  and  $-b$ , and use the sign-change law.)

3. Prove if true; if false, give a counterexample, and if possible amend the statement so it becomes true and then prove it. (For some proofs, try proving the contrapositive instead. See Appendix A.3 for "counterexample", A.2 for "contrapositive".)

(i)  $a < b \Rightarrow \sqrt{a} < \sqrt{b}$  (assume the roots exist);

(ii)  $a^2 < b^2 \Rightarrow a < b$ ;

(iii)  $\{a_n\}$  increasing,  $a_n \neq 0$  for all  $n \Rightarrow \{1/a_n\}$  is decreasing.

4. If  $a_n > 0$ , under what hypotheses will  $\{1/a_n\}$  be bounded above?

**2.2 Estimations.**

One of the major uses of inequalities is to make estimations. These have a language of their own that you need to get used to.

**Definition 2.2** If  $c$  is a number we are estimating, and  $K < c < M$ , we say

$K$  is a **lower estimate** for  $c$ ;  $M$  is an **upper estimate** for  $c$ .

Sometimes the respective phrases *lower bound* or *upper bound* for  $c$  are used.

If two sets of upper and lower estimates satisfy the inequalities

$$K < K' < c < M' < M,$$

we say  $K', M'$  are **stronger** or **sharper** estimates for  $c$ , while  $K, M$  are **weaker** estimates.

Estimates are obtained in many ways; the inequality laws of the preceding section usually play an important role. Here are some examples to illustrate.

**Example 2.2A** Give upper and lower estimates for  $\frac{1}{a^4 + 3a^2 + 1}$ .

**Solution.** We are not told what  $a$  is, so our estimates will have to be valid no matter what  $a$  is, i.e., valid for all  $a$ .

We first estimate the denominator. Since  $a^2 \geq 0$ , the polynomial has its smallest value 1 when  $a = 0$ , and it has no upper bound since  $a$  can be arbitrarily large. We express this symbolically by the inequalities

$$1 \leq a^4 + 3a^2 + 1 < \infty .$$

Using the reciprocal law for inequalities (and writing symbolically  $1/\infty = 0$ ), the inequalities reverse and we get the upper and lower estimates

$$0 < \frac{1}{a^4 + 3a^2 + 1} \leq 1 .$$

These are the sharpest possible estimates valid for all  $a$  : the upper bound 1 is attained when  $a = 0$ ; the lower bound is 0 since the fraction can be made arbitrarily close to 0 by taking  $a$  sufficiently large.

**Example 2.2B** Give upper and lower estimates for  $\frac{1 + \sin^2 n}{1 + \cos^2 n}$ , for  $n \geq 0$ .

**Solution.** Both the numerator and the denominator lie between 1 and 2.

To make the quotient biggest, we use the *upper* bound for the top and the *lower* bound for the bottom; to make it smallest, we use the *lower* bound for the top and the *upper* bound for the bottom. Thus we get the estimates

$$\frac{1}{2} \leq \frac{1 + \sin^2 n}{1 + \cos^2 n} \leq 2 , \quad \text{for } n \geq 0 .$$

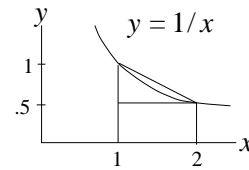
The above argument is very informal, but this is the way most people mentally answer the question. You should also be able to give a formal argument using the inequality laws: see Question 1 below.

**Example 2.2C** By interpreting the integral as the area under  $1/x$  and over the interval  $[1, 2]$ , estimate  $\ln 2 = \int_1^2 \frac{dx}{x}$ .

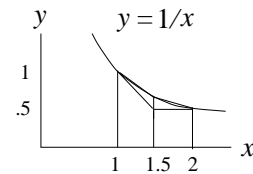
**Solution.** Referring to the picture, since the region under the curve lies between the rectangle (area  $1/2$ ) and the trapezoid (area  $3/4$ ), we get

$$.50 < \ln 2 < .75 .$$

This is pretty crude; can we sharpen the estimates?



Referring to the new picture, we divide the interval in half, use two trapezoids for the upper bound and a trapezoid and rectangle for the lower bound. (Note that the slope at  $x = 1$  is  $-1$ .) Calculating the area of the trapezoids and rectangle gives



$$\begin{aligned} 5/8 &< \ln 2 < 17/24 ; \\ .62 &< \ln 2 < .71 . \end{aligned}$$

Since  $\ln 2 \approx .69$ , the upper estimate is pretty good, while the lower one is weaker; the picture suggests this, in fact.

### Questions 2.2

1. In example 2.2B, estimates were given for a fraction of the form  $A/B$ ; the argument was rather informal however. Assuming the estimates  $1 \leq A \leq 2$  and  $1 \leq B \leq 2$ , complete the argument more formally, using the laws in 2.1.

2. Give upper and lower estimates for  $\frac{4 + \cos na}{3 + \sin na}$ , valid for all  $a$  and all integers  $n > 0$ . Then make a careful argument, using the inequality laws in 2.1.

3. In example 2.2C, if the lower estimate is made by halving the interval and using two rectangles, instead of the trapezoid and rectangle, is the resulting estimate stronger or weaker than the one given?

### 2.3 Proving boundedness.

Rather than an upper and lower estimate, often you want an estimate just in one direction. This was the case with our work with sequences in Sections 1.4 and 1.5. Using the language of estimations, we summarize the principles used there; for simplicity, we assume the sequences are non-negative:  $a_n \geq 0$ .

*To show  $\{a_n\}$  is bounded above, get one upper estimate:  $a_n \leq B$ , for all  $n$ ;*

*to show  $\{a_n\}$  is not bounded above, get a lower estimate for each term:  $a_n \geq B_n$ , such that  $B_n$  tends to  $\infty$  as  $n \rightarrow \infty$ .*

The method for showing unboundedness is rather restricted, but it's good enough most of the time. We will give a precise definition of "tend to  $\infty$ " in the next chapter; for now we will use it intuitively. The examples we gave in Sections 1.4 and 1.5 illustrate these principles.

**Example 1.4 (9)**  $b_k = \left(1 + \frac{1}{k}\right)^k$ .

We showed  $\{b_k\}$  bounded above by the upper estimate:  $b_k < 3$  for all  $k$ .

**Example 1.5A**  $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$ .

Here we showed  $\{a_n\}$  was not bounded above, by proving the lower estimate  $a_n > \ln(n+1)$ , and relying on the known fact that  $\ln n$  tends to  $\infty$  as  $n \rightarrow \infty$ .

These examples show that to estimate, you may have to first decide what kind of estimate you are looking for: upper or lower. If you aren't told this in advance, deciding may not be so easy. For example, by using the rectangles in Example 1.5A a bit differently, we could easily have obtained the *upper* estimate

$$a_n < 1 + \ln(n + 1)$$

for the terms of the sequence. But this would have been useless for showing the sequence was unbounded, since the estimate goes in the wrong direction. Nor does it show the sequence is bounded, since the estimate  $1 + \ln(n + 1)$  depends on  $n$  and tends to  $\infty$ , instead of being the same for all the terms.

As another example of the difficulty in deciding which type of estimate to look for, consider the sequence formed like the one above in Example 1.5, but using only the prime numbers in the denominators (here  $p_n$  denotes the  $n$ -th prime):

$$a_n = 1/2 + 1/3 + 1/5 + 1/7 + 1/11 + \dots + 1/p_n .$$

Should we look for an upper estimate that will show it is bounded, or a lower estimate that clearly tends to infinity? Anyone who can answer this without having studied number theory is a mathematical star of zeroth magnitude.

### Questions 2.3

1. Let  $a_n = 1 + \frac{1}{4} + \frac{1}{9} + \dots + \frac{1}{n^2}$ ; is it bounded or unbounded; i.e., should one look for an upper or a lower estimate for the  $a_n$ ? (cf. Example 1.5A)

## 2.4 Absolute values. Estimating size.

We turn now to the other basic tool of estimation, the absolute value. As with inequalities, its use requires both care and knowing some rules.

**Definition 2.4 Absolute value.** Define  $|a| = \begin{cases} a, & \text{if } a \geq 0; \\ -a, & \text{if } a < 0. \end{cases}$

The formal definition has been confusing students for centuries. Simply put:  $| \cdot |$  erases the  $-$  sign if a number is negative; it leaves positive numbers alone.

Here are two good ways to think about absolute value.

*Absolute value measures magnitude:  $|a|$  is the size of  $a$ .*

Negative numbers can be big too: a million dollar loss is a big sum for a small business. One says  $-N$  is “negatively large” or “large in absolute value”.

*Absolute value measures distance:  $|a - b|$  is the distance between  $a$  and  $b$ .*

This is often used to describe intervals on the  $x$ -axis. For instance the interval  $[2, 4]$  can be described as  $\{x : |x - 3| \leq 1\}$ , i.e., the set of points whose distance from the point 3 is at most 1.

We give two simple but common and important uses of the absolute value.

$$(1) \quad |a| \text{ is guaranteed to be non-negative.}$$

For instance,  $a \leq b \Rightarrow |c|a \leq |c|b$  is true regardless of the sign of  $c$ .

The absolute value is also an efficient way to give **symmetric bounds**: the inequality on the left below says the same thing as the two on the right,

$$(2) \quad |a| \leq M \Leftrightarrow -M \leq a \leq M,$$

and it is often more convenient to use. If the bounds are not symmetric to start with, they can be made so by using the following important fact:

$$(2a) \quad K \leq a \leq L \Rightarrow |a| \leq M, \quad \text{where } M = \max(|K|, |L|).$$

**Proof of (2a).** Regardless of whether  $a, K$ , and  $L$  are positive or negative,

$$-M \leq -|K| \leq K \leq a \leq L \leq |L| \leq M;$$

now apply (2) to the two ends.  $\square$

**ABSOLUTE VALUE LAWS** In working with absolute values we will make frequent use of the simple

$$(3) \quad \text{multiplication law} \quad |ab| = |a||b|; \quad |a/b| = |a|/|b|, \text{ if } b \neq 0;$$

and the law connecting absolute value with sums:

$$(4) \quad \text{triangle inequality} \quad |a + b| \leq |a| + |b|;$$

$$(4a) \quad \text{extended triangle inequality} \quad |a_1 + \dots + a_n| \leq |a_1| + \dots + |a_n|.$$

**Proof of (4).** We use (2) in the direction  $\Rightarrow$  to transform the inequalities  $|a| \leq |a|$  and  $|b| \leq |b|$ , and then we add them; the resulting inequalities are:

$$\begin{aligned} -|a| &\leq a \leq |a|; \\ -|b| &\leq b \leq |b|; \\ -|a| - |b| &\leq a + b \leq |a| + |b|; \end{aligned}$$

and according to (2) again, this last line is the same as the inequality (4).  $\square$

The extended triangle inequality (4a) follows by applying (4) repeatedly (i.e., using mathematical induction, Appendix A.4). We leave it for the Exercises.  $\square\square$

The triangle inequality is used to get an *upper* estimate for a sum. Two equivalent but less familiar forms of (4) are sometimes useful for getting *lower* estimates of sums and differences:

$$(4b) \quad \text{difference forms of the triangle inequality} \quad |a - b| \geq |a| - |b|; \quad |a + b| \geq |a| - |b|.$$

The first form follows from (4) by changing  $a$  to  $a - b$ ; the second form is just the first with  $b$  changed to  $-b$ . In using either form,  $a$  should be the larger-sized number (i.e.,  $|a| > |b|$ ), otherwise the inequality gives no information.

Most of the estimates in this book and in introductory analysis are estimates of *size*: they will use the absolute value laws combined with the inequality laws. In the hope of forestalling the two most common errors:

*To estimate size, you have to use  $|\cdot|$ , and the estimations have to be in the right direction:*

*to show  $a$  is small in size, show  $|a| < a$  small number;*

*to show  $a$  is large in size, show  $|a| > a$  large number.*

This hardly sounds like the sort of advice one would pay money for, yet one often sees on student papers inequalities obtained at great effort, but going in the wrong direction and therefore useless for estimating.

Even more common is the failure, largely through sloppiness, to use  $|\cdot|$  in size estimations. If you want to show  $a_n$  is small, it does no good to show that

$$a_n < 1/n$$

unless you know in advance that  $a_n$  is non-negative, for otherwise  $a_n$  could be negatively large and still satisfy the above inequality; instead, show

$$|a_n| < 1/n .$$

**Example 2.4A** In Fourier analysis, one uses trigonometric sums of the form

$$S_n = c_1 \cos t + c_2 \cos 2t + \dots + c_n \cos nt .$$

If  $c_i = 1/2^i$ , give an upper estimate for the size of  $S_n$ .

**Solution.** This is a typical use of the extended triangle inequality (4a), whose *raison d'être* is to give upper estimates for sums. Note first that

$$|\cos A| \leq 1 \quad \text{for all } A ;$$

thus by the extended triangle inequality and the multiplication law (3),

$$\begin{aligned} |S_n| &\leq |c_1| |\cos t| + |c_2| |\cos 2t| + \dots + |c_n| |\cos nt| ; \\ &\leq \frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 1 + \dots + \frac{1}{2^n} \cdot 1 < 1 . \end{aligned}$$

Thus  $|S_n| < 1$  for all  $n$ . □

**Example 2.4B** If  $|a| \geq 2$  and  $|b| \leq 1/2$ , estimate  $|a + b|$  from below.

**Solution** Since we want a lower estimate for the size of a sum, the triangle inequality in the difference form (4b) is the thing to use. It gives

$$|a + b| \geq |a| - |b| \geq 3/2 ,$$

since the first term is at least 2, and we subtract at most 1/2. (One can also argue more formally using the inequality laws; see question 4 below.) □

**Important Warning 2.4** You can't use the triangle inequality mechanically. For example, the above reasoning gives a silly lower estimate for  $|\sin n - \cos n|$ :

$$|\sin n - \cos n| \geq |\sin n| - |\cos n| \geq 0 - 1 = -1.$$

As another example (one of the most common misuses, actually), if  $a$  and  $b$  are close, you won't see this using the triangle inequality:

$$|\pi - 3.14| \leq |\pi| + |3.14| \leq 6.3 .$$

Even worse, many try to do this estimation by using

$$|a - b| \leq |a| - |b|, \quad (\text{the fool's inequality}),$$

which is obvious nonsense (what if  $|a| < |b|$ ?)—if you highlight your text, highlight that one in black.

As another example of size estimation, we show that  $||$  lets us prove boundedness by using just one inequality, instead of two. This proposition, though simple, is important; always formulate boundedness this way, unless there's some obvious reason not to (for instance, if  $a_n$  are known to be positive).

**Proposition 2.4**

(5)  $\{a_n\}$  is bounded  $\Leftrightarrow$  there is a  $B$  such that  $|a_n| \leq B$  for all  $n$  .

**Proof.**  $\Leftarrow$  : Using (2),  $|a_n| \leq B \Rightarrow -B \leq a_n \leq B$ , so  $\{a_n\}$  is bounded.

$\Rightarrow$  : Using (2a),  $K \leq a_n \leq L$  for all  $n \Rightarrow |a_n| \leq \max(|K|, |L|)$  for all  $n$ .  $\square$

**Questions 2.4**

1. Proof or counterexample: (i)  $a < b \Rightarrow |a| < |b|$  (ii)  $|a| < |b| \Rightarrow a < b$ .
  2. (a) Without looking it up, write the proof of the triangle inequality (4).  
(b) Using (4), prove the extended triangle inequality (4a) when  $n = 3$ .
  3. In example 2.4A, estimate  $|S_n|$  if  $c_i = 1/i!$
  4. Write out the end of example 2.4B formally, using the inequality laws.
  5. Express the boundedness of the sequence  $\{\frac{1}{n} \cos n\pi\}$ ,  $n \geq 1$  by  
(a) giving upper and lower bounds (the sharpest possible);  
(b) one inequality (the sharpest possible), as in (5).
- (Remember that to show boundedness, the bounds must be independent of  $n$ .)

**2.5 Approximations.**

A common way of estimating some quantity is to say that it is approximately equal to some number. In scientific work one often writes  $a \approx b$  to mean that  $a$  and  $b$  are approximately equal. This statement has no exact mathematical meaning however, since it says that  $a$  and  $b$  are close, but not how close. After all, a short distance for a snail might seem an intolerable distance to an amoeba. We must be more precise.

Suppose that  $a$  and  $b$  are within  $\epsilon$  of each other, where  $\epsilon$  is some positive number (invariably thought of as small). The standard way of writing this says that the distance between  $a$  and  $b$  is less than  $\epsilon$ :

(6)  $|a - b| < \epsilon .$



While we shall use this notation frequently (and your teacher may insist that you use it always), we shall also use the non-standard notation

$$(6a) \quad a \underset{\epsilon}{\approx} b$$

to express the same meaning as (6), when we particularly want to focus on the approximation point of view.

Often a mathematical argument which depends on making different sorts of approximations can be worked out roughly by using  $\approx$  everywhere, and then it can be made precise by putting in the  $\epsilon$ 's underneath. Lazy folk can even skip this last step and still feel they understand the argument more or less.

As an example of approximation, let  $a > 0$  be written as an infinite decimal (if it terminates, assume it ends with all 0's, rather than all 9's):

$$a = a_0 . a_1 a_2 a_3 \dots a_n \dots; \quad \begin{cases} a_0 = \text{integer part,} \\ a_i = i\text{-th decimal place,} \end{cases} \quad a > 0 .$$

We define its  $n$ -th *truncation* to be:

$$a^{(n)} = a_0 . a_1 a_2 a_3 \dots a_n 00 \dots, \quad a_i = 0, \quad i > n.$$

Like any terminating decimal,  $a^{(n)}$  is a rational number (e.g.,  $3.14=314/100$ ). Thus it gives an approximation to  $a$  by a rational number; how close are they?

$$(7) \quad a \underset{\epsilon}{\approx} a^{(n)}, \quad \text{where } \epsilon = \frac{1}{10^n}.$$

We leave the simple proof for Question 1b. In practice, the approximation may be much closer than what (7) guarantees. For example, since  $\pi = 3.14159\dots$ ,

$$\pi \underset{.1}{\approx} 3.1, \quad \pi \underset{.01}{\approx} 3.14 \quad \text{by (7),} \quad \text{whereas actually } \pi \underset{.05}{\approx} 3.1, \quad \pi \underset{.002}{\approx} 3.14 .$$

As an example of the use of the truncation approximation in analysis, we prove a theorem we will use occasionally. Some of the details of the proof are left as Questions, to give you practice with these ideas and with inequalities.

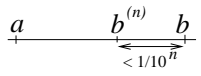
In what follows, the set of rational numbers is denoted by  $\mathbb{Q}$ ; the sum, difference, product, and quotient of rational numbers is again a rational number. A number which is not rational is called *irrational*; for example, in Appendix A.2 it is proved that  $\sqrt{2}$  is irrational, a fact we shall make use of in the proof.

(Appendix A.0 describes the notation and number facts that we use.)

**Theorem 2.5** For any two real numbers  $a < b$ , there is

- (i) a rational number  $r \in \mathbb{Q}$  between them:  $a < r < b$ ;
- (ii) an irrational number  $s \notin \mathbb{Q}$  between them:  $a < s < b$ .

**Proof.** (i) We prove (i); assume  $b > 0$  and make two cases.

If  $b$  is rational choose  $n$  so large that  $a < b - \frac{1}{10^n} < b$ ;  then we are done since the middle number is also rational.

If  $b$  is not rational, choose  $n$  so large that  $a < b^{(n)} < b$ . (see the picture). To do this, by (7) we need only choose  $n$  so that  $\frac{1}{10^n} < b - a$ . (See Question 2a.)

If  $b \leq 0$ , then  $b + N > 0$  for some integer  $N$ ; this reduces it to the previous case, since  $a + N < r < b + N \Rightarrow a < r - N < b$ , and  $r - N$  is still rational.  $\square$

(ii) To prove this, from (i) we have  $a < r < b$ , where  $r$  is rational. Then if  $n$  is a sufficiently large integer, we have

$$a < r < r + \frac{\sqrt{2}}{n} < b,$$

and  $r + \sqrt{2}/n$  is irrational (see Question 2b).  $\square\square$

There are a set of laws for calculating with approximations, of which we single out the two most important; others are in the Questions and Exercises.

$$(8) \quad \text{transitive law} \quad a \underset{\epsilon}{\approx} b \text{ and } b \underset{\epsilon'}{\approx} c \Rightarrow a \underset{\epsilon+\epsilon'}{\approx} c;$$

$$(9) \quad \text{addition law} \quad a \underset{\epsilon}{\approx} a' \text{ and } b \underset{\epsilon'}{\approx} b' \Rightarrow a + b \underset{\epsilon+\epsilon'}{\approx} a' + b'.$$

Both are proved by the triangle inequality; for example, to prove (9),

$$|(a + b) - (a' + b')| \leq |a - a'| + |b - b'| < \epsilon + \epsilon'. \quad \square$$

If we think of  $\epsilon$  as being the possible error in a measurement, the addition law (9) tells how errors propagate under addition. Their behavior under multiplication and taking reciprocals is a bit less simple, and left for the exercises.

Because the notation  $\approx$  suggests a sort of rough equality, it ought to obey the transitive law:  $a \approx b$ ,  $b \approx c \Rightarrow a \approx c$ . While this is suggestive, it has no precise meaning. In arguments, it must be used in the form (8) with the  $\epsilon$ 's put in. Forget this and your teacher won't let you use  $\approx$  (he or she might not, regardless).

### Questions 2.5

1. (a) We have  $e = 2.718\dots$ . If we write  $e \underset{\epsilon}{\approx} 2.72$ , what can we take for  $\epsilon$ ?  
 (b) Prove (7).
2. Fill in the details in Theorem 2.5 by
  - (a) in part (i), proving the inequalities without referring to a picture;
  - (b) in part (ii), giving an explicit expression for how large  $n$  must be, and proving  $r + \sqrt{2}/n$  is irrational.
3. (a) Prove the transitive law (8).  
 (b) If  $a_1 \underset{\epsilon}{\approx} a_2$ ,  $a_2 \underset{\epsilon}{\approx} a_3$ ,  $\dots$ ,  $a_{n-1} \underset{\epsilon}{\approx} a_n$ , what accuracy can you give for the approximation  $a_1 \approx a_n$ ? Prove it, either using (4a), or using (8) and induction (Appendix A.4.)
4. If  $a \underset{\epsilon}{\approx} b$ , what is the accuracy of
  - (i)  $a^2 \approx b^2$ ?
  - (ii)  $1/a \approx 1/b$ ? (assume  $a, b \neq 0$ )

## 2.6 The terminology “for $n$ large”.

In estimating or approximating the terms of a sequence  $\{a_n\}$ , sometimes the estimate is not valid for all terms of the sequence; for example, it might fail for the first few terms, but be valid for the later terms. In such a case, one has to specify the values of  $n$  for which the estimate holds.

**Example 2.6A** Let  $a_n = \frac{5n}{n^2 - 2}$ ,  $n \geq 2$ ; for what  $n$  is  $|a_n| < 1$ ?

**Solution.** For  $n = 1$ , the estimate is not valid; if  $n > 1$ , then  $a_n$  is positive, so we can drop the absolute value. Then we have

$$\frac{5n}{n^2 - 2} < 1 \Leftrightarrow 5n < n^2 - 2 \Leftrightarrow 5 < n - \frac{2}{n},$$

and by inspection, one sees this last inequality holds for all  $n \geq 6$ .  $\square$

**Example 2.6B** In the sequence  $a_n = \frac{n^2 + 2n}{n^2 - 2}$ , for what  $n$  is  $a_n \underset{.1}{\approx} 1$ ?

**Solution.**  $\left| \frac{n^2 + 2n}{n^2 - 2} - 1 \right| = \frac{2n + 2}{n^2 - 2}$ ,

$$\text{which is } < \frac{1}{10} \Leftrightarrow n(n - 20) > 22,$$

and by inspection, this last inequality holds for all  $n \geq 22$ .  $\square$

The above is a good illustration of Warning 2.4. To show  $a_n$  and 1 are close, we get a small upper estimate of the difference by first transforming it algebraically; trying instead to use the triangle inequality, in either the sum or difference form, would produce nothing useful.

As the above examples illustrate, sometimes a property of a sequence  $a_n$  is not true for the first few terms, but only starts to hold after a certain place in the sequence. In this case, a special terminology is used.

**Definition 2.6** The sequence  $\{a_n\}$  has property  $\mathcal{P}$  **for  $n$  large** if

(10) there is a number  $N$  such that  $a_n$  has property  $\mathcal{P}$  for all  $n \geq N$ .

One can say instead *for large  $n$* , *for  $n$  sufficiently large*, etc. Most of the time we will use the symbolic notation *for  $n \gg 1$* , which can be read in any of the above ways.

Note that in the definition,  $N$  need not be an integer.

To illustrate the definition, Examples 2.6A and 2.6B show, respectively:

$$\begin{aligned} \text{if } a_n &= \frac{5n}{n^2 - 2}, \quad \text{then } |a_n| < 1 \quad \text{for } n \gg 1; \\ \text{if } a_n &= \frac{n^2 + 2n}{n^2 - 2}, \quad \text{then } |a_n| \underset{.1}{\approx} 1 \quad \text{for } n \gg 1. \end{aligned}$$

In both examples, we gave the smallest integer value of  $N$  (it was 6 and 22, respectively), such that the stated property of  $a_n$  held for all  $n \geq N$ . In general, this is overkill: to show something is true for  $n$  large, one only has to give *some* number  $N$  which works, not the “best possible”  $N$ . And as we said, it need not be an integer.

The symbols  $a \gg b$ , with  $a, b > 0$ , have the meaning “ $a$  is relatively large compared with  $b$ ”, that is,  $a/b$  is large. Thus we do not write “for  $n \gg 0$ ”; intuitively, every positive integer is relatively large compared with 0.

**Example 2.6C** Show the sequence  $\{\sin 10/n\}$  is decreasing for large  $n$ .

**Proof.** The function  $\sin x$  is increasing on the interval  $0 < x < \pi/2$ , i.e.,

$$a < b \Rightarrow \sin a < \sin b, \quad \text{for } 0 < a < b < \pi/2 .$$

Thus

$$\sin \frac{10}{n+1} < \sin \frac{10}{n}, \quad \text{if } \frac{10}{n} < \frac{\pi}{2}, \quad \text{i.e., if } n > \frac{20}{\pi}. \quad \square$$

**Remarks.** This completes the argument, since it shows we can use  $N = 20/\pi$ . If you prefer an integer value, take  $N = 7$ , the first integer after  $20/\pi$ .

In this example, it was no trouble to find the exact integer value  $N = 7$  at which the sequence starts to be decreasing. However this is in general not necessary. Any value for  $N$  greater than 7 would do just as well in showing the sequence is decreasing for large  $n$ .

Thus, for example, if it turned out for some sequence  $a_n$  that

$$a_{n+1} < a_n \quad \text{if } n^2 + n > 100 ,$$

to show the sequence is decreasing for large  $n$ , it is a waste of time to solve the quadratic equation  $n^2 + n = 100$ ; by inspection one sees that if  $n \geq 10$ , then  $n^2 + n > 100$ , so that one can take  $N = 10$ .

The final two examples illustrate how to use “for  $n \gg 1$ ” in a formal argument; we give proofs as model arguments you can imitate. Watch carefully how the  $N$  is handled.

**Example 2.6D** If  $\{a_n\}$  is bounded above for  $n \gg 1$ , it is bounded above.

**Proof.** There is a  $B$  and an  $N$  (we may take it to be an integer) such that

$$a_n \leq B \quad \text{for } n \geq N .$$

Let  $B'$  be a number greater than  $a_0, a_1, \dots, a_N$  and  $B$ . Then

$$a_n \leq B' \quad \text{for } n = 0, 1, \dots, N ;$$

$$a_n \leq B < B' \quad \text{for } n \geq N ;$$

therefore  $a_n \leq B'$  for all  $n$ . □

**Example 2.6E** Suppose that  $\{a_n\}$  and  $\{b_n\}$  are increasing for  $n \gg 1$ .

Prove that  $\{a_n + b_n\}$  is increasing for  $n \gg 1$ .

**Proof.** By hypothesis, there are numbers  $N_1$  and  $N_2$  such that

$$a_n \leq a_{n+1} \quad \text{for } n \geq N_1 ,$$

$$b_n \leq b_{n+1} \quad \text{for } n \geq N_2 .$$

Choose any  $N \geq N_1, N_2$ ; for example, let  $N = \max(N_1, N_2)$ . Then

$$a_n \leq a_{n+1} \quad \text{for } n \geq N, \text{ since } N \geq N_1 ;$$

$$b_n \leq b_{n+1} \quad \text{for } n \geq N, \text{ since } N \geq N_2 ; \quad \text{therefore}$$

$$a_n + b_n \leq a_{n+1} + b_{n+1} \quad \text{for } n \geq N . \quad \square$$

You should be very clear about why the proof does not consist simply of the last three lines (with the two “since” clauses deleted). When we start out, each sequence has its private  $N_1$ ; the point of the proof is to get them to agree to use the same  $N$ , so that the inequalities can be added.

**Remarks.** A couple of further remarks about “for  $n \gg 1$ ” might be useful. In this phrase, an  $N$  is concealed. Why is it hidden, and when do we have to make it explicit?

We have to make the  $N$  explicit whenever we have to *prove* that some property holds for large  $n$ . To make such a proof, we have to go back to the definition of “for  $n \gg 1$ ” and exhibit explicitly the  $N$  which works. We saw this in Examples 2.6A and 2.6B.

Most of the time however the  $N$  remains hidden, and hiding it is the main function of the phrase “for  $n \gg 1$ ”. We use it whenever we want to say a sequence behaves in a certain way when  $n \geq N$ , but we don’t want to have to specify exactly what  $N$  is. Maybe it would be awkward to calculate a value for  $N$ . More likely, the value of  $N$  is of no importance, and we don’t want to clutter up the definition or argument with a lot of irrelevant numbers or letters, just as a novelist does not give names to the characters who appear for a moment in the story, but will not be seen again.\*

There will be many examples of “for  $n$  large” in the next chapter. For now, let’s conclude by seeing how we can use the terminology to weaken slightly the hypothesis of the Completeness Property formulated at the end of the preceding chapter.

Suppose  $\{a_n\}$  is increasing and bounded above. In finding its limit, we see that how the sequence behaves near its beginning is not important; the early terms

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\* There are, of course, exceptions: “With only two minutes left to reach the Bursar’s office with the check for my first tuition installment, I sprinted the four blocks, took the five flights of stairs three steps at a time, and arrived gasping at the window just as Algernon Finch, Payments, with a thin-lipped smile, closed it.”

could be changed, but the limit would stay the same. Indeed, if the sequence is bounded, but is increasing only after some term  $a_N$  :

$$a_n \leq a_{n+1}, \quad \text{for } n \geq N,$$

it still will have a limit, and exactly the same procedure we used before (watching each decimal place until it no longer changes) will produce it. The same observation applies to decreasing bounded sequences, and we are led to a slightly more general form of the **Completeness Property**:

*A sequence which is bounded and monotone for  $n \gg 1$  has a limit.*

### Questions 2.6

1. Show that each of the following sequences has the indicated property for  $n \gg 1$ . (This means you have to specify some number  $N$  and show that the sequence has the property when  $n \geq N$  or  $n > N$ .)

(a)  $a_n = \cos 20\pi/n$  (increasing);    (b)  $a_n = n^4 - 100n^3 - 1$  (positive).

2. Proof or counterexample:  $a_n < k$  for  $n \gg 1 \Rightarrow a_n/n < k$  for  $n \gg 1$ .

3. Prove  $\frac{2n}{n-1} \approx 2$ , for large  $n$ .

4. Prove:  $\{a_n\}$  increasing and not bounded above  $\Rightarrow a_n > 0$  for  $n \gg 1$ .

### Exercises

#### 2.1

1. Let  $\{a_n\}$  and  $\{b_n\}$  be increasing; are the following increasing? Proof or counterexample. (i)  $\{a_n + b_n\}$  (ii)  $\{a_n - b_n\}$

2. Let  $\{a_n\}$  and  $\{b_n\}$  be bounded above. State hypotheses which guarantee  $\{a_n b_n\}$  will also be bounded above, and prove it.

3. "If  $\{a_n\}$  and  $\{b_n\}$  are increasing, then  $\{a_n b_n\}$  is increasing". Show this is false, make the hypotheses stronger, and prove the amended statement (cf. p.405, Example A.1E for the meaning of "stronger").

4. Define a sequence by  $a_{n+1} = -a_n^2$ , with  $-1 \leq a_0 < 0$ . Prove that  $\{a_n\}$  is increasing and bounded above.

5. Prove  $\{\sqrt{n+1} - \sqrt{n}\}$ ,  $n \geq 0$ , is monotone, using just algebra. (You can use the true results in Question 2.1/3; cf. Ans. 1.4/1 for the write-up.)

#### 2.2

1. Give upper and lower bounds for the sequences ( $n \geq 0$ ):

(a)  $\frac{1}{3 + \cos n}$     (b)  $\frac{\sin n}{n^2 + 1}$  (make the upper bound sharp)

2. Give an upper estimate for  $\ln 3 = \int_1^3 (1/x) dx$  as in example 2.2C, by using (a) one trapezoid; (b) two trapezoids.

3. Let  $\alpha = 1/\sqrt{3}$ . Give a lower bound for  $\tan^{-1} \alpha = \int_0^\alpha \frac{dx}{1+x^2}$ , by using a single trapezoid; compare it with the exact value (use  $\sqrt{3} \approx 1.73$ ).  
(The positive point of inflection of  $1/(1+x^2)$  is  $\alpha$ ; why is this relevant?)

**2.3**

1. For each sequence, tell if it is bounded above or not (indicate a reason):  
(a)  $a_n = 1 + 1/4 + 1/7 + \dots + 1/(3n+1)$ ;  
(b)  $a_n = 1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \dots + \frac{1}{n\sqrt{n}}$ .

**2.4**

1. Prove:  $\max(a, b) = \frac{a+b+|a-b|}{2}$ ,  $\min(a, b) = \frac{a+b-|a-b|}{2}$ ,  
where  $\max$  and  $\min$  denote respectively the greater and the lesser of the two numbers  $a$  and  $b$ .  
2. If  $|a_1 \sin b + a_2 \sin 2b + \dots + a_n \sin nb| > n$ , prove that  $|a_i| > 1$  for at least one of the  $a_i$ . (Use contraposition: cf. Appendix A.2.)  
3. Give upper and lower estimates (in terms of  $n$  alone) for

$$\left| \cos na + \frac{\sin nb}{n} \right|,$$

valid for all  $a, b$ ; show these estimates are the best possible, by giving values of  $a$  and  $b$  for which the bounds are actually attained.

4. Give the best upper and lower estimates of the form  $cA^n$  ( $c$  and  $A$  are fixed numbers), valid for all  $a$  and  $b$ , that you can for

$$\left| \frac{\sin^3 na - 2}{(1 + (\cos b)/2)^n} \right|;$$

then tell by using your estimates for what values of  $n$  you can be sure that the expression is  $< 100$ , and for what values you can be sure it is  $> 1/3$ , regardless of what  $a$  and  $b$  are.

5. Consider the sum  $S_n = \sum_1^n c_k \cos kt$ , where  $c_k = 1/2^k$  (cf. Ex. 2.4A).  
Prove the lower estimate  $S_n \geq .7$ , for  $0 \leq t \leq .1$ .

(Find a lower bound for the first three terms — use  $\cos u > .95$ ,  $0 \leq u \leq .3$ ; estimate the remaining terms, and then use the difference form of the triangle inequality.)

6. Give an alternate proof of the triangle inequality (4), based on showing that if both sides of (4) are squared, an inequality holds between them. Write up the proof carefully, citing a reference for any non-trivial step.

7. Prove the extended triangle inequality (4a) by induction.

**2.5**

1. If  $a \underset{\epsilon}{\approx} b$ , and  $|a| \leq K$ ,  $|b| \leq K$ , give an estimate for the accuracy of the approximation  $a^n \approx b^n$ , where  $n$  is a positive integer.

2. If  $a \underset{\epsilon}{\approx} 1$  and  $a \underset{\epsilon}{\approx} 2$ , then  $\epsilon > \frac{1}{2}$ . This is obvious if you draw a picture, but prove it without a picture.

3. Suppose that  $a$  and  $b$  are two numbers with this property: for any  $\epsilon > 0$ , it is true that  $a \underset{\epsilon}{\approx} b$ . Prove  $a = b$ . ( Use contraposition, cf. Appendix A.2.)

4. Prove Theorem 2.5(i) a different way using this suggestion.

We are looking for integers  $m$  and  $n$  such that  $an < m < bn$  (why?) What property of two real numbers guarantees there is an integer between them?

5. If  $a \underset{\epsilon}{\approx} b$ ,  $\epsilon = \frac{1}{10^n}$ , state and prove the best estimate you can give for  $a^{(n)} \approx b^{(n)}$ . Show by example that it is the best possible. (Assume  $a, b > 0$ .)

## 2.6

1. Prove that if  $a > 0$ , the sequence  $a^n/n!$  is monotone for large  $n$ .

2. Prove: if  $a > 1$ , the sequence  $n/a^n$  is monotone for large  $n$ .

3. Prove that if  $\epsilon > 0$  is given, then  $\frac{n}{n+2} \underset{\epsilon}{\approx} 1$ , for  $n \gg 1$ .

4. Prove  $\{a_n\}$  is decreasing for  $n \gg 1$ , if  $a_0 = 1$  and

$$(a) \quad a_{n+1} = \frac{n-5}{(n+1)(n+2)} a_n \quad (b) \quad a_{n+1} = \frac{n^2+15}{(n+1)(n+2)} a_n$$

## Problems

**2-1** Let  $\{a_n\}$  be a sequence. We construct from it another sequence  $\{b_n\}$ , its *sequence of averages*, defined by

$$b_n = \frac{a_1 + \dots + a_n}{n} = \text{average of the first } n \text{ terms.}$$

(a) Prove that if  $\{a_n\}$  is increasing, then  $\{b_n\}$  is also increasing.

(b) Prove that if  $\{a_n\}$  is bounded above, then  $\{b_n\}$  is bounded above.

**2-2** Prove that a bounded increasing sequence  $\{a_n\}$  of integers is constant for  $n \gg 1$ .

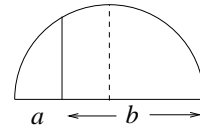
(Do not use an indirect argument. Use a direct argument which, starting from the hypotheses, produces the constant value  $L$ ; then prove that  $L$  has the desired property.)

**2-3** *The arithmetic-geometric mean inequality.*

(a) Prove: for any  $a, b \geq 0$ ,  $\sqrt{ab} \leq \frac{a+b}{2}$ ,

with equality holding if and only if  $a = b$ .

(b) How does the figure illustrate the inequality?



**2-4** A positive sequence is defined by  $a_{n+1} = \sqrt{1 + a_n^2/4}$ ,  $0 \leq a_0 < 2/\sqrt{3}$ .

(a) Prove the sequence is strictly increasing.

(b) Prove the sequence is bounded above.



**Answers to Questions**

**2.1 1.**  $1 < 2$  and  $-3 < 1$ , for example.

**2.**

$$\begin{aligned} -a < -b &\Rightarrow a > b && \text{(sign-change law)} \\ &\Rightarrow 1/a < 1/b && \text{(reciprocal law; since } a, b > 0) \\ &\Rightarrow 1/(-a) > 1/(-b) && \text{(sign-change law)} \end{aligned}$$

**3.** (i) true. Note that  $a, b \geq 0$  since their square roots exist.

By contraposition:  $\sqrt{a} \geq \sqrt{b} \Rightarrow a \geq b$  (multiplication law).

(ii) false:  $1 < 4$ , but  $-1 > -2$ . If  $a, b \geq 0$ , it follows from (i).

(iii) false:  $\{-1, 1, 2, 3, \dots\}$  is increasing, but  $\{-1, 1, 1/2, 1/3, \dots\}$  is not decreasing. True if all  $a_n > 0$  or all  $a_n < 0$ , by the reciprocal law and 2.1/2 .

**4.**  $a_n > c > 0$  for all  $n \Rightarrow 1/a_n < 1/c$  for all  $n$ .

(The hypothesis is usually phrased: “ $a_n$  is bounded away from 0”.)

**2.2 1.** We have

$$\begin{aligned} 1 \leq B \leq 2 &\Rightarrow 1/2 \leq 1/B \leq 1 && \text{(reciprocal law);} \\ &1 \leq A \leq 2; \end{aligned}$$

therefore, by the multiplication law,

$$1/2 \leq A/B \leq 2, \quad \text{since everything is } > 0.$$

**2.** Call it  $A/B$ .

$$2 \leq B \leq 4 \Rightarrow 1/4 \leq 1/B \leq 1/2 \quad \text{by the reciprocal law;}$$

$$3 \leq A \leq 5 \Rightarrow 3/4 \leq A/B \leq 5/2 \quad \text{by the multiplication law.}$$

**3.** Area of left rectangle =  $1/3$ , area of left trapezoid =  $3/8$ . Since  $1/3 < 3/8$ , the rectangles give a weaker lower estimate of the area under the curve.

**2.3 1.** Get an upper estimate by comparing the terms to the area under the curve  $1/x^2$  and lying over the interval  $1 \leq x < \infty$ .

**2.4 1.** Both false. (i)  $-3 < -2$  (ii)  $|-2| < |-3|$

**2.** (b) Using the triangle inequality (4) twice:

$$\begin{aligned} |a + b + c| &= |(a + b) + c| \leq |a + b| + |c| \\ &\leq |a| + |b| + |c|. \end{aligned}$$

**3.** We have

$$\begin{aligned} |S_n| &\leq |c_1| |\cos t| + |c_2| |\cos 2t| + \dots + |c_n| |\cos nt| \\ &\leq (1/1!) \cdot 1 + (1/2!) \cdot 1 + \dots + (1/n!) \cdot 1 < e - 1, \text{ by Ques. 1.4/2 .} \end{aligned}$$

**4.**  $|a| \geq 2$ ;  $-|b| \geq -1/2$  (sign-change law); add the inequalities.

**5.** (a)  $-1 \leq a_n \leq 1/2$  (b)  $|a_n| \leq 1$

## 2.5

1. (a) any  $\epsilon > .002$ , based on the given information. (If we knew that  $e > 2.718$ , then we could also take  $\epsilon = .002$ .)

$$(b) \quad a - a^{(n)} = .0 \dots 0 a_{n+1} a_{n+2} \dots \geq 0 \quad (n \text{ zeros});$$

$$\text{Therefore,} \quad 0 \leq a - a^{(n)} < .00 \dots 01 \quad (n \text{ places}) \\ = 1/10^n .$$

2. (a) To save space we write horizontally; it would be clearer vertically.

$$\text{First part: } n > -\ln(b-a)/\ln 10 \Rightarrow \frac{1}{10^n} < b-a \Rightarrow a < b - \frac{1}{10^n} < b .$$

$$\text{Second part: by (7), } b - b^{(n)} < \frac{1}{10^n} \Rightarrow b - \frac{1}{10^n} < b^{(n)} \Rightarrow a < b - \frac{1}{10^n} < b^{(n)} < b .$$

$$(b) \text{ We have } r + \sqrt{2}/n < b \Leftrightarrow n > \frac{\sqrt{2}}{b-r} .$$

Indirect proof of irrationality (cf. appendix A.2):

If  $r + \sqrt{2}/n = s$ , a rational number, then  $\sqrt{2} = n(s - r)$ , contradicting the fact that  $\sqrt{2}$  is irrational.

$$3. (a) |a - c| = |(a - b) + (b - c)| \leq |a - b| + |b - c| < \epsilon + \epsilon'$$

(b)

$$|a_1 - a_n| = |(a_1 - a_2) + (a_2 - a_3) + \dots + (a_{n-1} - a_n)| \\ \leq |a_1 - a_2| + |a_2 - a_3| + \dots + |a_{n-1} - a_n| \\ < (n-1)\epsilon .$$

$$4. (i) |a^2 - b^2| = |a - b||a + b| < \epsilon \cdot |a + b|, \text{ i.e., } a^2 \underset{|a+b|\epsilon}{\approx} b^2 .$$

$$(ii) \left| \frac{1}{a} - \frac{1}{b} \right| = \frac{|b - a|}{|ab|} < \frac{\epsilon}{|ab|}, \text{ thus } \frac{1}{a} \underset{\epsilon'}{\approx} \frac{1}{b}, \text{ where } \epsilon' = \epsilon/|ab| .$$

## 2.6

1. (a)  $\cos x$  is decreasing for  $0 \leq x \leq \pi$ . So the sequence is increasing if  $20\pi/n \leq \pi$ , i.e., if  $n \geq 20$ .

$$(b) \quad n^4 - 100n^3 - 1 = n^3(n - 100) - 1, \text{ which is } > 0 \text{ if } n > 100 .$$

2. Counterexample:  $a_n = -2$  for all  $n$ ,  $k = -1$ .

$$3. \left| \frac{2n}{n-1} - 2 \right| = \left| \frac{2}{n-1} \right| < .1 \text{ if } n-1 > 20, \text{ or } n > 21 .$$

4. Some  $a_N > 0$ , since otherwise all  $a_n \leq 0$ , and the sequence would be bounded above. Then  $a_n > 0$  for all  $n \geq N$ , since  $\{a_n\}$  is increasing.