

Contents

Preface xi

- 1. Real Numbers and Monotone Sequences 1**
 - 1.1 Introduction; Real numbers 1
 - 1.2 Increasing sequences 3
 - 1.3 Limit of an increasing sequence 4
 - 1.4 Example: the number e 5
 - 1.5 Example: the harmonic sum and Euler's number γ 8
 - 1.6 Decreasing sequences; Completeness property 10
- 2. Estimations and Approximations 17**
 - 2.1 Introduction; Inequalities 17
 - 2.2 Estimations 18
 - 2.3 Proving boundedness 20
 - 2.4 Absolute values; estimating size 21
 - 2.5 Approximations 24
 - 2.6 The terminology "for n large" 27
- 3. The Limit of a Sequence 35**
 - 3.1 Definition of limit 35
 - 3.2 Uniqueness of limits; the K - ϵ principle 38
 - 3.3 Infinite limits 40
 - 3.4 Limit of a^n 42
 - 3.5 Writing limit proofs 43
 - 3.6 Some limits involving integrals 44
 - 3.7 Another limit involving an integral 45
- 4. The Error Term 51**
 - 4.1 The error term 51
 - 4.2 The error in the geometric series; Applications 52
 - 4.3 A sequence converging to $\sqrt{2}$: Newton's method 53
 - 4.4 The sequence of Fibonacci fractions 56
- 5. Limit Theorems for Sequences 61**
 - 5.1 Limits of sums, products, and quotients 61
 - 5.2 Comparison theorems 64
 - 5.3 Location theorems 67
 - 5.4 Subsequences; Non-existence of limits 68
 - 5.5 Two common mistakes 71

- 6. The Completeness Property 78**
 - 6.1 Introduction; Nested intervals 78
 - 6.2 Cluster points of sequences 80
 - 6.3 The Bolzano-Weierstrass theorem 82
 - 6.4 Cauchy sequences 83
 - 6.5 Completeness property for sets 86

- 7. Infinite Series 94**
 - 7.1 Series and sequences 94
 - 7.2 Elementary convergence tests 97
 - 7.3 The convergence of series with negative terms 100
 - 7.4 Convergence tests: ratio and n -th root tests 102
 - 7.5 The integral and asymptotic comparison tests 104
 - 7.6 Series with alternating signs: Cauchy's test 106
 - 7.7 Rearranging the terms of a series 107

- 8. Power Series 114**
 - 8.1 Introduction; Radius of convergence 114
 - 8.2 Convergence at the endpoints; Abel summation 117
 - 8.3 Operations on power series: addition 119
 - 8.4 Multiplication of power series 120

- 9. Functions of One Variable 125**
 - 9.1 Functions 125
 - 9.2 Algebraic operations on functions 127
 - 9.3 Some properties of functions 128
 - 9.4 Inverse functions 131
 - 9.5 The elementary functions 133

- 10. Local and Global Behavior 137**
 - 10.1 Intervals; estimating functions 137
 - 10.2 Approximating functions 141
 - 10.3 Local behavior 143
 - 10.4 Local and global properties of functions 145

- 11. Continuity and Limits of Functions 151**
 - 11.1 Continuous functions 151
 - 11.2 Limits of functions 155
 - 11.3 Limit theorems for functions 158
 - 11.4 Limits and continuous functions 162
 - 11.5 Continuity and sequences 155

- 12. The Intermediate Value Theorem 172**
 - 12.1 The existence of zeros 172
 - 12.2 Applications of Bolzano's theorem 175
 - 12.3 Graphical continuity 178
 - 12.4 Inverse functions 179

- 13. Continuous Functions on Compact Intervals 185**
 - 13.1 Compact intervals 185
 - 13.2 Bounded continuous functions 186
 - 13.3 Extremal points of continuous functions 187
 - 13.4 The mapping viewpoint 189
 - 13.5 Uniform continuity 190
- 14. Differentiation: Local Properties 196**
 - 14.1 The derivative 196
 - 14.2 Differentiation formulas 200
 - 14.3 Derivatives and local properties 202
- 15. Differentiation: Global Properties 210**
 - 15.1 The mean-value theorem 210
 - 15.2 Applications of the mean-value theorem 212
 - 15.3 Extension of the mean-value theorem 214
 - 15.4 L'Hospital's rule for indeterminate forms 215
- 16. Linearization and Convexity 222**
 - 16.1 Linearization 222
 - 16.2 Applications to convexity 225
- 17. Taylor Approximation 231**
 - 17.1 Taylor polynomials 231
 - 17.2 Taylor's theorem with Lagrange remainder 233
 - 17.3 Estimating error in Taylor approximation 235
 - 17.4 Taylor series 236
- 18. Integrability 241**
 - 18.1 Introduction; Partitions 241
 - 18.2 Integrability 242
 - 18.3 Integrability of monotone and continuous functions 244
 - 18.4 Basic properties of integrable functions 246
- 19. The Riemann Integral 251**
 - 19.1 Refinement of partitions 251
 - 19.2 Definition of the Riemann integral 253
 - 19.3 Riemann sums 255
 - 19.4 Basic properties of integrals 257
 - 19.5 The interval addition property 258
 - 19.6 Piecewise continuous and monotone functions 260
- 20. Derivatives and Integrals 269**
 - 20.1 The first fundamental theorem of calculus 269
 - 20.2 Existence and uniqueness of antiderivatives 270
 - 20.3 Other relations between derivatives and integrals 274
 - 20.4 The logarithm and exponential functions 276
 - 20.5 Stirling's formula 278
 - 20.6 Growth rate of functions 280

- 21. Improper Integrals 290**
 - 21.1 Basic definitions 290
 - 21.2 Comparison theorems 292
 - 21.3 The gamma function 295
 - 21.4 Absolute and conditional convergence 298
- 22. Sequences and Series of Functions 305**
 - 22.1 Pointwise and uniform convergence 305
 - 22.2 Criteria for uniform convergence 310
 - 22.3 Continuity and uniform convergence 312
 - 22.4 Integration term-by-term 314
 - 22.5 Differentiation term-by-term 316
 - 22.6 Power series and analytic functions 318
- 23. Infinite Sets and the Lebesgue Integral 329**
 - 23.1 Introduction; infinite sets 329
 - 23.2 Sets of measure zero 333
 - 23.3 Measure zero and Riemann-integrability 335
 - 23.4 Lebesgue integration 338
- 24. Continuous Functions on the Plane 347**
 - 24.1 Introduction; Norms and distances in \mathbb{R}^2 347
 - 24.2 Convergence of sequences 349
 - 24.3 Functions on \mathbb{R}^2 351
 - 24.4 Continuous functions 352
 - 24.5 Limits and continuity 354
 - 24.6 Compact sets in \mathbb{R}^2 355
 - 24.7 Continuous functions on compact sets in \mathbb{R}^2 356
- 25. Point-sets in the Plane 364**
 - 25.1 Closed sets in \mathbb{R}^2 364
 - 25.2 Compactness theorem in \mathbb{R}^2 367
 - 25.3 Open sets 368
- 26. Integrals with a Parameter 375**
 - 26.1 Integrals depending on a parameter 375
 - 26.2 Differentiating under the integral sign 377
 - 26.3 Changing the order of integration 380
- 27. Differentiating Improper Integrals 386**
 - 27.1 Introduction 386
 - 27.2 Pointwise vs. uniform convergence of integrals 387
 - 27.3 Continuity theorem for improper integrals 390
 - 27.4 Integrating and differentiating improper integrals 391
 - 27.5 Differentiating the Laplace transform 393

Appendix

- A. Sets, Numbers, and Logic 399**
 - A.0 Sets and numbers 399
 - A.1 If-then statements 403
 - A.2 Contraposition and indirect proof 406
 - A.3 Counterexamples 408
 - A.4 Mathematical induction 411
 - B. Quantifiers and Negation 418**
 - B.1 Introduction; Quantifiers 418
 - B.2 Negation 421
 - B.3 Examples involving functions 423
 - C. Picard's Method 427**
 - C.1 Introduction 427
 - C.2 The Picard iteration theorems 428
 - C.3 Fixed points 430
 - D. Applications to Differential Equations 434**
 - D.1 Introduction 434
 - D.2 Discreteness of the zeros 435
 - D.3 Alternation of zeros 437
 - D.4 Reduction to normal form 439
 - D.5 Comparison theorems for zeros 440
 - E. Existence and Uniqueness of ODE Solutions 445**
 - E.1 Picard's method of successive approximations 445
 - E.2 Local existence of solutions to $y' = f(x, y)$ 447
 - E.3 The uniqueness of solutions 450
 - E.4 Extending the existence and uniqueness theorems 452
- Index 455**