(All Exercises are references to *Introduction to Commutative Algebra* by M. Atiyah and I. Macdonald.)

**Problem 1.** Chapter 8, Exercise 3, implication (i) $\implies$ (ii). (Artinian f.g. $k$-algebras are finite $k$-algebras - you’ve already done the other implication.)

**Problem 2.** Let $A$ be a finite ring (i.e. a ring with finitely many elements). Show that $A$ is isomorphic to a product of rings, each of which has a prime power number of elements (i.e. $p^n$ for prime $p$ and $n \geq 1$).

**Problem 3.** Let $A = \mathbb{C}[x, y]$ and $X = \mathbb{C}^2$. For each $n \geq 0$, define
$$X^{[n]} := \{I \subseteq A \mid I \text{ is an ideal and } \dim_{\mathbb{C}}(A/I) = n\}.$$ (Here $\dim_{\mathbb{C}}$ means dimension as a $\mathbb{C}$-vector space. We say that such ideals $I$ have *colength* $n$. The set $X^{[n]}$ is sometimes called the *Hilbert scheme of points* of $X$.)

Also let
$$S^n(X) := X^n/S_n = \{\text{multisets of size } n \text{ consisting of elements of } X\}.$$ (Here $S_n$ is the symmetric group on $n$ elements, acting on the $n$ factors of the cartesian product $X^n$.)

Show that there exist functions
$$\phi_n : X^{[n]} \to S^n(X)$$
for each $n \geq 0$ such that
(a) $\phi_n$ is surjective for each $n \geq 0$.
(b) If $I$ and $J$ are ideals of finite colengths $m$ and $n$ respectively and $I \subseteq J$, then $\phi_n(J)$ is a sub-multiset of $\phi_m(I)$.
(c) If $I$ and $J$ are ideals of finite colengths $m$ and $n$ respectively, then $I \cap J$ has colength $m + n$ if and only if $\phi_m(I)$ and $\phi_n(J)$ are disjoint, and in this case $\phi_{m+n}(I \cap J) = \phi_m(I) \cup \phi_n(J)$.

**Problem 4.** Let $k$ be a field. Describe all discrete valuations on the field $k(x)$ satisfying $v(f) \geq 0$ for all polynomials $f \in k[x]$ and show that the corresponding DVRs are localizations of $k[x]$ at maximal ideals.

**Problem 5.** Give an example of a discrete valuation on the field $\mathbb{C}(x,y)$ such that
(a) $v(f) \geq 0$ for all polynomials $f \in \mathbb{C}[x,y]$;
(b) the corresponding DVR is not equal to a localization of $\mathbb{C}[x,y]$ (both viewed as subrings of $\mathbb{C}(x,y)$).

(Hint: look for a discrete valuation such that $v(x) = v(y) = 1$.)