NOTE OF ANALYTIC AND GEOMETRIC ESTIMATES FOR COMPLEX

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This note is based on the talk given by Jian Song in Thematic Program on Kähler Geometry on June 19-23, 2017 at University of Notre Dame.

1. BACKGROUND

Let us recall the result of Calabi conjecture first proved by Yau.

Theorem 1.1 (Yau). Suppose \((X, \omega)\) is a Kähler manifold, \(\omega\) is the Kähler form, \(\Omega\) is a smooth volume form on \(X\) with \(\int_X \Omega = \int_X \omega^n\). Then there exists a unique \(\varphi \in C^\infty(X)\) such that \((\omega + \sqrt{-1} \partial \bar{\partial} \varphi)^n = \Omega\) where \(\sup_X \varphi = 0\).

This is an elliptic equation, the complex Monge-Ampere equation. Yau used the continuity method to prove this theorem, so he need some key estimate of the solution.

In history, the third order estimate was derived by Calabi. Aubin and Yau independently derived the second order estimate. The main contribution of Yau to this problem is the 0-th order estimate of the solution. To do that, he applied the Moser’s iteration technique, which now is a standard technique in elliptic equations.

In 1998, Kolodziej generalized Yau’s result as follows:

Theorem 1.2 (Kolodziej). Suppose \((X, \omega)\) is Kähler manifold, \(\Omega\) is the volume form. Suppose \(0 \leq F \in L^p(X, \Omega)\) with \(\|F\|_p \leq k\) for some \(p > 1\), and \(\int_X F \Omega = \int_X \omega^n\). Then there exists an unique plurisubharmonic function \(\varphi\) such that \((\omega + \partial \bar{\partial} \varphi)^n = F \Omega\) with \(\sup_X \varphi = 0\).

In particular, we have the following estimate

\[
\|\varphi - \sup \varphi\|_{L^\infty(X)} \leq C(X, \omega, \Omega, p, k)
\]

Date: July 24, 2018.
If $F = 1$, then the result reduces to Yau’s result, where the estimate is just the famous Yau’s 0-th order estimate. Kolodziej’s result can be viewed as a singular version of Yau’s result, since $F$ can be very singular somewhere.

The first goal of this talk is to prove Theorem 1.2. The proof is based on many other authors result, like (P Eyssidieux, V Guedj, A Zeriahi; JP Demailly, N Pali).

2. Preliminary

In this section we recall some fundamental definitions and result in complex geometry.

Plurisubharmonic Functions (PSH).

**Definition 2.1.** Let $\Omega$ be an open domain in $\mathbb{C}^n$. A function $u : \Omega \to [-\infty, \infty)$ is said to be **plurisubharmonic** if

1. $u$ is upper semi-continuous;
2. for any complex line $L$, $u|L$ is subharmonic.

We will use $PSH$ to denote the space of plurisubharmonic functions.

Just like harmonic functions on $\mathbb{R}^2$ has closed connection with holomorphic functions, plurisubharmonic functions has closed connection with holomorphic functions in higher dimensional space.

**Proposition 2.1.** Let $\Omega \subset \mathbb{C}^n$ be an open set, $u \in PSH(\Omega)$. Then $u$ satisfies the following properties:

1. For all $a \in \Omega$ and $B(a, r) \subset \Omega$ we have
   \[ u(a) \leq \frac{1}{Vol(B(a, r))} \int_{B(a, r)} u(z) dV \]
   which treat $\mathbb{C}^n$ as Euclidean space;
2. If $u \in C^2(\Omega)$, then $\omega = \sqrt{-1} \partial \bar{\partial}u$ is a non-negative real valued closed 2-form;
3. If $F(t_1, \ldots, t_k)$ is convex and increasing in each $t_i$, then $F(u_1, \ldots, u_k) \in PSH$ if $u_1, \ldots, u_k \in PSH$. For example
   - $\max(u_1, \ldots, u_k)$
   - $u_1 + \cdots + u_k$
   - $\log(e^{u_1} + \cdots + e^{u_k})$
   - if $f$ is holomorphic on $\Omega$, then $\log |f|^2$
4. Suppose $\{u_j\}$ are plurisubharrmonic functions, $u_j \searrow u$, then $u \in PSH$
5. Suppose $\{u_j\}$ are plurisubharrmonic functions, $u_j \nearrow u$, then $u$ is not necessary in $PSH$. However, if $u_j$ has a uniform upper bound $C$, then the upper envelope $u^*$ of $u$ is in $PSH$.

Let us just make a comment about the last property, which turns out to be useful later. Suppose $u$ is a function on $\Omega$, then we define the **upper envelope of $u$** by

\[ u^*(x) = \lim_{r \to 0} \sup_{y \in B(x, r) \subset \Omega} u(y) \]

The reason (4) holds but (5) doesn’t is mainly because the increasing sequence may not be bounded or upper semi-continuous. Thus we need some more restrictions.
\textbf{Kähler Manifold.}

\textbf{Definition 2.2.} Let $X$ be a (compact) complex manifold with complex dimension $n$. Suppose $g = g_{j\bar{k}}dz^j \otimes d\bar{z}^k$ is Hermitian metric on $X$, Then $g$ is said to be \textbf{Kähler} if $\omega = \sqrt{-1}g_{j\bar{k}}dz^j \wedge d\bar{z}^k$ is closed.

We define Kähler metric locally, but it is easy to see $\omega$ defined by $g$ in the definition is intrinsically defined. We sometimes also use the Kähler form $\omega$ to indicate the metric.

\textbf{Example 2.1.} Here we list some examples of Kähler manifold.

- The simplest example is $(\mathbb{C}^n, \sum_{i=1}^{n} \sqrt{-1}dz^i \wedge d\bar{z}^i)$;
- Every Riemannian surfaces with any Hermitian metric is Kähler;
- Suppose $M \hookrightarrow \mathbb{C}P^N$ with induced metric from Fubini-Study metric $g_{FS}$, then $(M, g_{FS}|_M)$ is Kähler.

Kähler metric is very important because it links Riemannian geometry, complex geometry and symplectic geometry. We won’t discuss these influential results in this notes. Instead, we discuss some analytic property in order to do computation later.

\textbf{Proposition 2.2.} Suppose $(X, \omega)$ is a Kähler manifold, $\omega = \sqrt{-1}g_{j\bar{k}}dz^j \wedge d\bar{z}^k$, then

(1) 
$$R_{ijkl} = -\frac{\partial^2 g_{kl}}{\partial z^i \partial \bar{z}^j} + g^{pq} \frac{\partial g_{pq}}{\partial z^i} \frac{\partial g_{kl}}{\partial \bar{z}^j}$$

(2) 
$$R_{ijkl} = R_{klij} = R_{kjil}$$

(3) 
$$R_{ij} = -\frac{\partial^2 \log \det g}{\partial z^i \partial \bar{z}^j}$$

(4) Locally
$$\omega = \sqrt{-1} \partial \bar{\partial} u = \sqrt{-1} \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j} dz^i \wedge d\bar{z}^j$$

for some $u \in PSH$

\textbf{Capacity and Extremal Function.} Now we need some technical preparation. Suppose $(X, \omega)$ is a Kähler manifold. Assume $F \in C^\infty(X), F > 0$, then we have the following definitions:

\textbf{Definition 2.3.} Let $K$ be a Borel set of $X$. Then the \textbf{capacity} of $K$ with respect to $\omega$ is defined by

$$\text{Cap}_\omega(K) = \sup \{ \int_K (\omega + \sqrt{-1} \partial \bar{\partial} u)^n : u \in PSH(X, \omega), 0 \leq u \leq 1 \}$$

Intuitively, the capacity measures how large the volume of $K$ could be when we change the metric.

\textbf{Proposition 2.3.} For fixed $\omega$, we have the following properties

(1) $E_1 \subset E_2 \implies \text{Cap}_\omega(E_1) \leq \text{Cap}_\omega(E_2)$

(2) $E = \bigsqcup E_j \implies \text{Cap}_\omega(E) \leq \sum_j \text{Cap}_\omega(E_j)$

(3) $E_j \supset E_{j-1}$ and $E = \bigsqcup E_j \implies \text{Cap}_\omega(E) = \lim_{j \to \infty} \text{Cap}_\omega(E_j)$
Now we define another terminology, which turns out to have closed relation with capacity.

**Definition 2.4.** Let $K$ be a Borel set of $X$, the extremal function with respect to $K$ and $\omega$ is defined by

$$\psi_{K,\omega}(x) = \sup\{u(x) : u(x) \in PSH(X, \omega), u \leq 0 \text{ on } K\}$$

We will use $\psi_{K,\omega}^*$ to denote the upper envelope of $\psi_{K,\omega}$.

**Proposition 2.4.** $\psi_{K,\omega}^*$ has the following properties

1. $\psi_{K,\omega}^* \in PSH(X, \omega)$;
2. $\psi_{K,\omega}^* \geq 0$;
3. If $\emptyset \neq K \subset X$ is open, then $\psi_{K,\omega}^* = 0$ on $K$, and $(\omega + i\partial \bar{\partial} \psi_{K,\omega}^*)^n = 0$ on $X \setminus K$.

### 3. Proof of Kolodziej Theorem

In this section we will prove Theorem 1.2 assuming some propositions. We will prove the propositions in the following sections.

**Proposition 3.1.** Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a right-continuous decreasing function satisfying

1. $\lim_{s \to \infty} f(s) = 0$
2. $rf(s + r) \leq Af(s)^{1+\alpha}$ for all $s > 0$, $0 \leq r \leq 1$ with fixed $\alpha, A > 0$

Then there exists $S > 0$ such that $f(s) = 0$ for all $s > S$.

**Proposition 3.2.** There exists $\delta, C > 0$ such that for all open $K \subset X$ we have

$$\int_K \omega^n \leq Ce^{-\delta(Cap_{\omega,K})^{-1/n}}$$

**Proposition 3.3.** For any $\phi \in PSH(X, \omega) \cap L^\infty(X)$, for all $s > 0$ and $0 \leq r \leq 1$ we have

$$r^n Cap_{\omega}(\{\phi < -s - r\}) \leq \int_{\phi \leq -s} (\omega + i\partial \bar{\partial} \phi)^n$$

The Proposition 3.1 is about how capacity can control the measure, and the Proposition 3.2 is about how measure can control the capacity.

**Proposition 3.4** (Comparison Principle). Suppose $\varphi, \psi \in PSH(X, \omega) \cap L^\infty(X)$, then

$$\int_{\{\varphi < \psi\}} (\omega + i\partial \bar{\partial} \varphi)^n \leq \int_{\{\varphi < \psi\}} (\omega + i\partial \bar{\partial} \psi)^n$$

This is somewhat the generalization of maximal principle.

Now we can prove the Theorem 1.2.

**Proof.** We only need to consider the case $F$ is smooth, then we can use smooth functions to approximate the original $L^p$ function.

Let us show the apriori estimate. Suppose $\varphi$ is a plurisubharmonic function such that $(\omega + i\partial \bar{\partial} \varphi)^n = F\Omega$. Let us define a function

$$f(s) = (Cap_{\omega}(\{\varphi < -s\}))^{1/n}$$
It is not hard to check $f$ is deceasing to 0 and right-continuous. Now we want to verify the condition 2 in the Proposition 3.1. Suppose $s > 0, 0 \leq r \leq 1,$

$$
(rf(s + r))^n = r^n \text{Cap}_\omega(\varphi < -s - r)
$$

$$
\leq \int_{\{\varphi < -s\}} (\omega + i\partial \bar{\partial} \varphi)^n \quad \text{by Proposition 3.3}
$$

$$
= \int_{\{\varphi < -s\}} F \Omega
$$

$$
\leq \|F\|_{L^p(\{\varphi < -s\})}(\int_{\{\varphi < -s\}} \Omega)^{1/q} \quad \text{by Cauchy-Schwartz inequality, here } 1/p + 1/q = 1
$$

$$
\leq CK(\int_{\{\varphi < -s\}} \omega^n)^{1/q}
$$

$$
\leq Ce^{-\delta'(\text{Cap}_\omega(\{\varphi < -s\}))^{-1/n}} \quad \text{by Proposition 3.2}
$$

$$
= Ce^{-\delta(f(s))^{-1}}
$$

$$
\leq A^n f(s)^{n(1-\alpha)} \quad \text{for some } A, \alpha > 0
$$

Then the $f$ we defined satisfies the conditions in Proposition 3.1, hence $f(s) = 0$ for $s >> 1.$ So $\text{Cap}_\omega(\{\varphi < -s\}) = 0$ for all $s >> 1.$ By definition

$$
\text{Cap}_\omega(\{\varphi < -s\}) \geq \int_{\{\varphi < -s\}} \omega^n
$$

, which means $\int_{\{\varphi < -s\}} \omega^n = 0$ for $s$ large, i.e. $\{\varphi < -s\} = \emptyset$ if $s$ is large. \qed

Now we can go back to prove those propositions.

**proof of proposition 3.1.** Let us pick

$$
\left\{ \begin{array}{c}
    s_0 = \inf \{s | f^\alpha(s) \leq \frac{1}{2A} \} \\
    s_{j+1} = \sup \{s > s_j | f(s) \geq \frac{1}{2} f(s_j) \}
\end{array} \right.
$$

Then we have

$$
(s_{j+1} - s_j)f(s) \leq A[f(s_j)]^{1+\alpha} \leq 2Af(s_{j+1})f(s_j)^\alpha
$$

So we have

$$
s_{j+1} - s_j \leq 2Af(s_j)^\alpha \leq 2A(\frac{1}{2})^{\alpha j} f(s_0)^\alpha
$$

$$
\leq (\frac{1}{2})^{\alpha j}
$$

Sum up we get $s_j - s_0 \leq 2,$ which means $f$ must be 0 after some $S.$ \qed

Before prove proposition 3.2, we introduce an estimate by Hörmander (locally) and Tian (globally) without proof:

**Lemma 3.1.** There exists $\alpha, C_\alpha > 0$ such that for all $\varphi \in PSH(X, \omega),$

$$
\int_X e^{-\alpha(\varphi - \sup_X \varphi)} \omega^n \leq C_\alpha
$$
proof of proposition 3.2. Let us consider any $K' \subset K$, $K'$ open. Then we have

$$\int_{K'} \omega^n = e^{-\delta \sup X \psi^*_{K', \omega}}$$

Note the equality holds because $\psi^*_{K', \omega} = 0$ on $K'$ and $(\omega + i\partial\bar{\partial} \psi^*_{K', \omega})^n = 0$ on $X \setminus K'$. Now use the lemma we have

$$\int_{K'} \omega^n \leq C e^{-\delta \sup X \psi^*_{K', \omega}}$$

We consider two different cases.

**case 1:** $\sup X \psi^*_{K', \omega} > 1$. Then we have

$$(\sup_X \psi^*_{K', \omega})^{-n} = \left(\sup_X \psi^*_{K', \omega}\right)^{-n} \int_X (\omega + i\partial\bar{\partial} \psi^*_{K', \omega})^n = \left(\sup_X \psi^*_{K', \omega}\right)^{-n} \int_X \omega^n$$

by the property of $\psi^*_{K', \omega}$

$$\leq \frac{\int_{K'} (\omega + i\partial\bar{\partial} \psi^*_{K', \omega})}{\int_X \omega^n}$$

since $\sup_X \psi^*_{K', \omega} > 1$

Then we proved the inequality for proposition 3.2.

**case 2:** $\sup_X \psi^*_{K', \omega} \leq 1$. Then

$$\int_X \omega^n = \int_{K'} (\omega + i\partial\bar{\partial} \psi^*_{K', \omega})^n \leq \text{Cap}_\omega(K') \leq \text{Cap}_\omega(X) = \int_X \omega^n$$

Hence all the inequalities are equalities. Thus we have $\text{Cap}_\omega(K) = \int_X \omega^n$ is a constant depending on fixed metric. Then we also have proposition 3.2. \hfill \Box

It should be noted that case 2 in the proof is not so surprising: it just indicate that if the case happens, then $K$ somewhat is just the whole $X$.

**proof of proposition 3.3.** Pick any $u \in PSH(X, \omega), 0 \leq u \leq 1$, we have

$$r^n \int_{\{\phi < -s-r\}} (\omega + i\partial\bar{\partial} u)^n \leq \int_{\{\phi < -s-r\}} (r\omega + i\partial\bar{\partial} (ru))^n$$

$$\leq \int_{\{\phi < -s-r\}} (\omega + i\partial\bar{\partial} (ru))^n$$

$$\leq \int_{\{\phi < -s-r+ru\}} (\omega + i\partial\bar{\partial} (ru))^n$$

$$\leq \int_{\{\phi < -s-r+ru\}} (\omega + i\partial\bar{\partial} (ru - s - r))^n$$

$$\leq \int_{\{\phi < -s-r+ru\}} (\omega + i\partial\bar{\partial} \phi)^n$$

by comparison principle

$$\leq \int_{\{\phi < -s\}} (\omega + i\partial\bar{\partial} \phi)^n$$

since $1 \leq u \leq 0$

Then we showed proposition 3.3. \hfill \Box
Remark: With the similar techniques, one could show some other generalization of complex Monge-Ampere equation. For instance

\begin{itemize}
  \item \((\omega + i\partial \bar{\partial} \phi)^n = e^{\lambda \phi} F \Omega\)
  \item \((x + \delta \omega + i\partial \bar{\partial} \phi)^n = e^{\lambda \phi + \alpha(s)} F \Omega\)
\end{itemize}

4. Local Schwartz Lemma

Next tool we will going to use is a generalization of Yau’s Schwartz lemma. Let us first recall Yau’s result:

**Theorem 4.1 (Yau 76’)**. Suppose \(F : (X, g) \to (Y, h)\) is holomorphic, where \(X, Y\) are Kähler and \(X\) is complete. Suppose

\begin{itemize}
  \item \(\text{Ric}(g) \geq -K_1 g\)
  \item \(R_{ijkl}(h) \leq -K_2 (h_{ij} h_{kl} + h_{ik} h_{jl})\)
\end{itemize}

Then \(F^* h \leq K_1 K_2 g\)

In this section we will prove the following stronger version of Yau’s Schwartz lemma.

**Theorem 4.2.** Suppose \(F : (X, g) \to (Y, h)\) is holomorphic, where \(X, Y\) are Kähler and \(X\) is complete. Suppose

\begin{itemize}
  \item \(\text{Ric}(g) \geq -K_1 g\) on \(B_{g}(p, 2R) \subset X\);
  \item \(R_{ijkl}(h) \leq -K_2 (h_{ij} h_{kl} + h_{ik} h_{jl})\) on \(Y\)
\end{itemize}

Then \(F^* h|_{B_{g}(p, R)} \leq K_1 K_2 g|_{B_{g}(p, R)} + C(K_1, K_2) R^{-1}\) for \(R\) large.

In order to prove this theorem, we introduce the following comparison theorem:

**Theorem 4.3 (Laplace Comparison).** Suppose \((M, g)\) is a Riemannian complete with \(\text{Ric}(g) \geq -k, k > 0\). Fix \(p \in M\) and define the distance function \(r(x) = d_g(x, p)\). Then

\[\Delta r \leq \sqrt{k} \cot(\sqrt{k} r)\]

In particular, we have

\[\Delta r \leq r^{-1} + \sqrt{k}\]

**proof of theorem 4.2.** We will use the standard elliptic maximum principle method.

**Step 1:** Let us define the cut-off function \(\rho(y) : \mathbb{R} \to \mathbb{R}\), which is 1 when \(y \leq 1\) and 0 when \(y \geq 2\) such that \(0 \leq \rho \leq 1, 0 \leq \rho^{-1}(\rho')^2 \leq C, \|\rho''\| \leq C\). Then define \(\psi : M \to \mathbb{R}\) by \(\psi(x) = \rho(\frac{r(x)}{R})\). Then we have

\[|\psi|^2 \leq \frac{C \psi}{R^2}\]

\[\Delta \psi = \frac{\rho'' |\nabla r|^2}{R^2} + \frac{\rho' \Delta r}{R} \geq -C^2 (1 + \sqrt{k} r)\]

By Laplacian comparison.

**Step 2:** Now we prove Chern-Lu inequality, which is an elliptic inequality for the evaluate of the pull back metric. Let \(u = \text{tr}_g(F^* h)\), in local coordinate it is \(g^{ij} F^i_\alpha F^\beta_j h_{\alpha \beta}\). For \(p \in M\), let us use the normal coordinate for \(g, h\), i.e. \(g_{i\bar{j}}(p) =\)
\[ \delta_{ij}, h_{\alpha \beta}(p) = \delta_{\alpha \beta} \text{ and the first order derivatives all vanish at } p. \text{ Then} \]

\[ \Delta u = g^{kl} u_{kl} \]

\[ = g^{kl} (g^{ij} F_i^\alpha F_j^\beta h_{\alpha \beta})_{kl} \]

\[ = g^{kl} (-g^{ij} q_{pq,k} F_i^\alpha F_j^\beta h_{\alpha \beta} + g^{ij} F_i^\alpha F_j^\beta h_{\alpha \beta}) \]

\[ = -g^{ij} k_{kl} F_i^\alpha F_j^\beta + F_i^\alpha F_j^\beta h_{\alpha \beta, \gamma} \]

\[ = R(p)_{ijkl} F_i^\alpha F_j^\beta + |\nabla^2 F|^2 - F_i^\alpha F_j^\beta F_k^\gamma F_l^\delta R(h)_{\alpha \beta, \gamma \delta} \]

\[ \geq -K_1 u + K_2 u^2 \]

**Step 3:** Chern-Lu inequality is an elliptic inequality, then we can apply maximal principle. Let \( H = \psi u = \psi \text{tr}_q (F^* h) \), then by Chern-Lu inequality we have

\[ \Delta H = \psi \Delta u + 2 \Re (\nabla \psi, \nabla u)_g + u \Delta \psi \]

\[ \geq \psi(-K_1 u + K_2 u^2) + 2 \Re (\nabla \psi, \nabla (\frac{H}{\psi}))_g - Cu R^{-2} (1 + R \sqrt{K_1}) \]

\[ = -K_1 H + K_2 \psi^{-1} H^2 + 2 \psi^{-1} \Re (\nabla \psi, \nabla H)_g - 2 H \psi^{-2} |\nabla \psi|^2 - C \psi^{-1} R^{-2} H (1 + R \sqrt{K_1}) \]

\[ \geq \psi^{-1} H (K_2 H - CR^{-2} (1 + R \sqrt{K_1}) - K_1 \psi) + 2 \psi^{-1} \Re (\nabla \psi, \nabla H)_g \]

Now we consider the point where \( H \) takes the maximum. If the point does not lie on the cut locus, then \( \Delta H(q) \leq 0, \nabla H(q) = 0 \), so we get

\[ K_2 H - CR^{-2} (1 + R \sqrt{K_1}) - \psi K_1 \leq 0 \]

i.e. \( H \leq H(q) \leq \frac{K_1}{K_2} + CR^{-2} (\frac{1}{K_2} + \frac{R \sqrt{K_1}}{K_2}) \).

**Step 4:** The only thing left is to deal with the case that maximum of \( H \) is achieved as a cut point of \( p \), let us call it \( q \). We apply a trick of Calabi.

Let \( \gamma \) be a minimizing geodesic joining \( p \) and \( q \) with \( \gamma(0) = p, \gamma(r(q)) = q \). Then let \( p_\epsilon = \gamma_\epsilon \) for some small \( \epsilon < q \) and new distance function \( r_\epsilon(x) = d(p_\epsilon, x) \). Note now \( q \) is not a cut point of \( p_\epsilon \).

Now choose a new cut-off function \( \psi_\epsilon(x) = \rho(\frac{r_\epsilon(x) + \epsilon}{R}) \). Then it is easy to check \( \psi_\epsilon \) also satisfies

\[ |\psi_\epsilon|^2 \leq \frac{C \psi_\epsilon}{R^2}, \Delta \psi_\epsilon \geq -CR^{-2} (1 + R \sqrt{K_1}) \]

Let \( H_\epsilon = \psi_\epsilon u \), then we have

\[ H_\epsilon \leq H, H_\epsilon(q) = H(q) \]

As a result, \( q \) is again the point where \( H_\epsilon \) achieves maximum, but now it is not on the cut locus of \( p_\epsilon \), i.e. \( H_\epsilon \) is smooth locally near \( q \). Then just as Step 3 we finish the proof. \( \square \)

5. Geometric Estimate

In this section, we will use the ideas and techniques in previous sections to study the geometry of Kähler manifold. The main result of this section follows:

**Theorem 5.1** (Fu-Guo-Song). Let \((X, \theta)\) be a Kähler manifold of complex dimension \( n \). Consider \( \omega = \theta + i \partial \bar{\partial} \phi \), \( g \) is the associated metric, such that \((\omega)^n = \Omega = e^{-f} \theta^n \) with \( \int_X \Omega = \int_X \theta^n \). If

- \( \|e^{-f}\|_{L^p(X)} \leq k \), for some \( p > 1 \)
• \( i\partial\bar{\partial}f + A\theta + \operatorname{Ric}(\theta) \geq 0 \) (Equivalently, \( \operatorname{Ric}(\Omega) = -i\partial\bar{\partial}\log \Omega \leq -A\theta \))

Then there exists \( C = C(X, \theta, p, k, A) > 0 \) such that

1. \( \|\varphi - \sup_X \varphi\|_{L^\infty(X, g)} + \|\nabla \varphi\|_{L^\infty(X, g)} \leq C \)
2. \( \operatorname{Ric}(g) \geq -Cg \)
3. \( \text{Diam}(X, g) \leq C \)

**Proof.** Let us first show the following lemma, which is useful for the proof of (1) and (2):

**Lemma 5.1.** There exists \( C = C(X, \theta, p, k, A) > 0 \) such that \( \omega \geq C\theta \).

**proof of the lemma.** We can not directly apply Yau’s Schwartz lemma because we have no Ricci curvature bound for \((X, \omega)\) yet. Just like the Chern-Lu inequality, we get

\[
\Delta_g \log \operatorname{tr}_\omega(\theta) \geq -C\operatorname{tr}_\omega(\theta) - C
\]

Here \( C = C(X, \theta, A) \). Then note \( \Delta_g \varphi = \operatorname{tr}_\omega \theta - n \). Hence for some large \( C' \), let \( H' = H - C'\varphi - \sup_X \varphi \) we have

\[
\Delta_g H' \geq C\operatorname{tr}_\omega(\theta) - C
\]

Again by maximal principle, the place \( H' \) achieves maximum (suppose point \( p \)) we have

\[
\operatorname{tr}_\omega(\theta)(q)e^{-C'\varphi - \sup_X \varphi}(q) \leq \operatorname{tr}_\omega(\theta)(p)e^{-C'\varphi - \sup_X \varphi}(p)
\]

Then by Kolodziej theorem we have

\[
\operatorname{tr}_\omega(\theta) \leq C = C(X, \theta, A, p, k)
\]

□

Now we go back to the proof.

**proof of (1):** the \( L^\infty \) bound for \( \varphi - \sup_X \varphi \) is just the result of Kolodziej. Now we consider the gradient estimate.

We still want to use the technique of Yau’s Schwartz lemma. Let \( G = \frac{\|\nabla \varphi\|^2}{A + (\varphi - \sup_X \varphi)} \). Then we have the inequality

\[
\Delta Q \geq Q\left(\frac{\|\nabla \varphi\|^2}{A + (\varphi - \sup_X \varphi)} - C\operatorname{Ric}(g) + C + \operatorname{tr}_\omega(\theta)\right) + \nabla Q \text{ terms}
\]

Then by maximal principle we get

\[
Q \leq C = C(X, \theta, A, p, k)
\]

**proof of (2):** This is straightforward by

\[
\operatorname{Ric}(g) = -i\partial\bar{\partial}\log \omega^n = -i\partial\bar{\partial}\log \Omega \geq -A\theta \geq -ACg
\]

The first inequality comes from the lemma and the second inequality comes from (1).

**proof of (3):** In order to prove (3) we need to analyze the relations between analysis and geometry. We prove (3) in several steps:
step 1: Suppose $\text{Diam}(X) = D \geq 4$, otherwise we have already down. Let $\gamma: [0, D] \to X$ be a minimizing geodesic, and pick $\{x_i = \gamma(6i)\}_{i=\lfloor 6D/6 \rfloor}$ Then the balls $\{B_x(x_i, 3)\}_{i=\lfloor 6D/6 \rfloor}$ are disjoint, hence we get

$$\sum_{i=0}^{\lfloor D/6 \rfloor} \text{Vol}_g(B_x(x_i, 3)) \leq \int_X \omega^n = \int_X \theta^n =: V$$

So we can choose $p_0 = x_i$ such that $\text{Vol}(B_x(p_0, 3)) \leq 6V/D$.

step 2: Just like the proof of Yau’s Schwartz lemma, we choose a cut-off function $\rho$ such that

$$\begin{cases} 
0 \leq \rho \leq 1 \\
0 \leq \rho^\prime \leq C \\
|\rho^\prime\prime| + |\rho^\prime| \leq C
\end{cases}$$

Again let $\psi(x) = \rho(r(x))$, $r(x) = d(p_0, x)$, then we have

$$|\nabla \psi|^2 \leq C\psi, \quad \Delta \psi \geq -C$$

step 3: Define a new function $0 < F_\epsilon \in C^\infty(X)$ satisfying the following conditions:

(1) $F_\epsilon = \begin{cases} 
D \rho(p_0, 2) \\
1 
\end{cases}$ on $B_x(p_0, 2)$ outside $B_x(p_0, 3)$

Here we fix $0 < \epsilon < p - 1$;

(2) $\int X F_\epsilon \Omega = \int X \theta^n =: V$

Then by Cauchy-Schwartz inequality we have

$$\int_X (F_\epsilon \Omega \theta^n)^{p-1} \theta^n \leq C(\epsilon)$$

Intuitively, if $D$ is large, $F_\epsilon$ is large in a small ball, and constant 1 out side a larger ball. In between we can arrange the value such that it satisfies the other condition.

Then we can replace $F$ by $F_\epsilon$ in Kolodziej’s theorem, then there is an unique $\phi_\epsilon$ such that

$$\begin{cases} 
(\theta + i\bar{\theta}\partial \bar{\partial} \phi_\epsilon)^n = F_\epsilon \Omega \\
\|F_\epsilon \Omega \bar{\theta}\|_{L^{p-1}(X)} \leq C, p - \epsilon > 1
\end{cases}$$

Moreover we have

$$\|\phi_\epsilon - \sup_X \phi_\epsilon\|_{L^\infty(X)} \leq C$$

step 4: Now let $\omega_\epsilon = (\theta + i\bar{\theta}\partial \bar{\partial} \phi_\epsilon)$. Then on $B_x(p_0, 2)$, $\omega_\epsilon^n = D \rho(p_0, 2) \omega^n$.

step 5: Finally, put

$$H = \psi \frac{\omega^n}{\omega_\epsilon^n} e^{-(\phi - \sup_X \phi_\epsilon) + (\phi_\epsilon - \sup_X \phi_\epsilon)}$$

We get the elliptic inequality

$$\Delta \psi H \geq -H + \psi^{-1} e^{-(\phi - \sup_X \phi_\epsilon) + (\phi_\epsilon - \sup_X \phi_\epsilon)} H^2 + 2 \frac{\psi}{\psi^2} R\text{ic}(\nabla H, \nabla H) - \frac{2H}{\psi^2} |\nabla \psi|^2$$

Then by maximal principle, we have

$$H \leq C(X, \theta, p, k, A)$$

i.e.

$$\psi \frac{\omega^n}{\omega_\epsilon^n} e^{-(\phi - \sup_X \phi_\epsilon) + (\phi_\epsilon - \sup_X \phi_\epsilon)} \leq C$$
Then on $B_g(p_0, 1)$, we have

$$F_\epsilon = \frac{\omega^n}{\omega^n} \leq C$$

as a result, we have $D \leq C$. Then we proved (3). □

REFERENCES