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The Effective Bogomolov Conjecture over Function Field

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Abstract

In this report, we introduce the height functions and semi-stable reductions first. Then we try to rewrite the Bogomolov’s conjecture to be the conjecture in the metric graph. We study various invariants associated to a given metric graph and weighted metric graph. We derive formulas relating the invariants studied in the paper Zhang [Gross-Schoen cycles and dualising sheaves][11] in terms of the tau constant of metric graphs. This enables us to use the tools developed to study the tau constant in the paper Cinkir [The tau constant of a metric graph and its behavior under graph operations][3]. We extend previous results on the tau constant to prove Zhang’s Conjecture.

Throughout the report, we use the interpretation of metric graphs as resistive electric circuits and related electrical properties such as circuit reductions. We consider metric graphs only with their combinatorial graph structure. We will use the properties of a continuous Laplacian operator on a metric graph to study the tau constant.

1 Introduction

Let $k$ be a field. Let $X$ be a smooth projective surface over $k$, let $Y$ be a smooth projective curve over $k$. Let $f : X \to Y$ be a semi-stable fibration such that the generic fiber of $f$ is smooth and of genus $g \geq 2$. Let $K$ be the function field of $Y$, with algebraic closure $\overline{K}$, and let $C$ be the generic fiber of $f$. The height pairing on the Jacobian variety $\text{Jac}(C)$ we have a canonical inclusion $j : C(\overline{K}) \to \text{Jac}(C)(\overline{K})$ defined by $j(x) = (2g - 2)x - \omega_C$.

We define $B_C(P, r) = \{ x \in C(\overline{K}) \| j(x) - P \|_{NT} \leq r \}$, where $P \in \text{Pic}^0(C)(\overline{K})$ and $r \geq 0$, and if we set $r_C(P) = -\infty$, if $(B_C(P, 0)) = \infty$. Otherwise $r_C(P) = \sup \{ r \geq 0 \| (B_C(P, r)) < \infty \}$.

Then Bogomolov’s conjecture can be stated as follows:

If $f$ is non-isotrivial, then $r_C(P)$ for all $P$.
If $f$ is non-isotrivial, then there exists an effectively calculable positive number $r_0$ such that $\inf_{P \in \text{Pic}^0(C)(\overline{K})} r_C(P) \geq r_0$. (1)

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We now describe how metric graph can be related to above conjectures.

For the semistable fibration $f : X \to Y$, let $CV(f) = \{y_1, y_2, \ldots, y_s\}$ be the set of critical values of $f$, where $s$ is the number of singular fibers. That is, $y \in CV(f)$ if and only if $f^{-1}(y)$ is singular. For any $y_i \in CV(f)$, let $\Gamma_{y_i}$ be the dual graph of the fiber $C_{y_i} := f^{-1}(y_i)$, for each $1 \leq i \leq s$. The metric graph $\Gamma_{y_i}$ is obtained as follows. The set of vertices $V_{y_i}$ of $\Gamma_{y_i}$ is indexed by reducible components of the fiber $f^{-1}(y_i)$ and the singularities of $f^{-1}(y_i)$ correspond to edges of length 1. Let $I(C_{y_i}) := \{c_{1,y_i}, c_{2,y_i}, \ldots, c_{v_i,y_i}\}$ be the set of reducible components of the fiber $C_{y_i}$, where $v_i$ is the number of irreducible components in $C_{y_i}$. Then the irreducible curve $C_{j,y_i}$ corresponds to the vertex $p_j \in V_{y_i}$ for each $1 \leq j \leq v_i$. Let $\delta_{y_i}$ be the number of singularities in $C_{y_i}$. By our construction, $\delta_{y_i} = l(\Gamma_{y_i})$, the length of $\Gamma_{y_i}$, for each $1 \leq i \leq s$. Let $\delta := \sum_{i=1}^s \delta_{y_i}$, the total number of singularities in the fibration. For any $p_j \in V_{y_i}$, let $\varphi(p_j) := g(C_{j,y_i})$, where $g(C_{j,y_i})$ is the arithmetic genus of $C_{j,y_i}$. Let $g(Y)$ be the genus of $Y$. We have $K_X, K_Y, \omega_X, \omega_Y$, and $\omega_{X/Y}$, which are the canonical divisors of $X$, the dualizing divisor of $Y$, the dualizing sheaf of $X$, the dualizing sheaf of $Y$, and the relative dualizing sheaf, respectively. For the admissible dualizing sheaf $\omega_a$, we have the following inequalities $\omega_{X/Y}^2 \geq \omega_a^2 \geq 0$.

It is showed that $\omega_a^2 > 0$ is equivalent to the Bogomolov conjecture.

Let $y \in CV(f)$, and $p \in f^{-1}(y)$ be a node. If the partial normalization of $f^{-1}(y)$ at $p$ is connected, we say that $p$ is of type 0. If it is disconnected, then it has two components, in which case $p$ will be said to be of type $i$, where $i$ is the fiber $f^{-1}(y)$ by $\delta_i(\Gamma_y)$, and we set $\delta_i(X) = \sum_{j=1}^s \delta_i(\Gamma_{y_i})$. We have $\delta_{y_i} = \sum_{i \geq 0} \delta_i(\Gamma_{y_i})$, and $\delta = \sum_{i \geq 0} \delta_i(X)$.

The following conjecture implies Bogomolov Conjecture and Effective Bogomolov Conjecture.

**Conjecture:** For any $y \in CV(f)$, there is a positive continuous function $c(\overline{y})$ of $\overline{y} \geq 2$ such that the following inequalities holds:

$$\varphi(\Gamma_y) \geq c(\overline{y})\delta_0(\Gamma_y) + \sum_{i \geq 1} \frac{2i(\overline{y} - 1)}{\overline{y}} \delta_i(\Gamma_y).$$

(2)

We prove that Conjecture holds as follows:

**Theorem:** Let $\Gamma$ be a wm-graph with genus $\overline{y}$. Then we have

$$\varphi(\Gamma) \geq t(\overline{y})\delta_0(\Gamma) + \sum_{i \geq 1} \frac{2i(\overline{y} - 1)}{\overline{y}} \delta_i(\Gamma)$$

(3)

where $t(2) = \frac{1}{27}$, $t(3) = \frac{892 - 11\sqrt{79}}{14380}$, and $t(\overline{y}) = \frac{(\overline{y} - 1)^2}{2\overline{y}(\overline{y} + 3)}$ for $\overline{y} \geq 4$.

2 Height functions

2.1 Heights on Projective Space

**Definition 2.1.1:** Let $k$ be a number field, and let $P = (x_0, x_1, \ldots, x_n) \in \mathbb{P}^n(k)$ be a point whose homogeneous coordinates are chosen in $k$. Then, the height of $P$ (relative to
$k$ is the quantity

$$H_k(P) = \prod_{v \in M_k} \max\{\|x_0\|_v, \|x_1\|_v, \ldots, \|x_n\|_v\} \quad (4)$$

where $M_k$ is the set of standard absolute values on $k$, $\| \cdot \|_v$ is the normalized absolute value associated to $v$ i.e. $\|x\|_v = |x|_v^{n_v}$, $n_v$ is the local degree of $v$, $n_v = [k_v : \mathbb{Q}_v]$. We write $k_v$ for the completion of the field $k$ with respect to $v$. We define

$$h_k(P) = \log H_k(P) = \sum_{v \in M_k} \min\{v(x_0), v(x_1), \ldots, v(x_n)\} \quad (5)$$

where $v(x) = -\log |x|_v$. Usually, we call $H_k$ the multiplicative height and $h_k$ the logarithmic height.

**Remark 2.1.2:** We should prove that $H_k(P)$ is well defined i.e. it is independent of the choice of homogeneous coordinates for $P$. In fact, this follows by Product formula: Let $k$ be a number field and let $x \in k^n$, then $\prod_{v \in M_k} \|x\|_v = 1$. ensures that the definition is well defined. We can see the proof from Joseph H.Silverman, [B1, Proposition B1.2].

**Definition 2.1.3:** The absolute multiplicative height on $\mathbb{P}^n$ is the function $H : \mathbb{P}^n(\overline{\mathbb{Q}}) \to [1, \infty)$, $H(P) = H_k(P)^{1/[k : \mathbb{Q}]}$ where $k$ is any field with $P \in \mathbb{P}^n(k)$. Similarly, the absolute logarithmic height on $\mathbb{P}^n$ is the function $h : \mathbb{P}^n(\overline{\mathbb{Q}}) \to [0, +\infty)$, $h(P) = \log H(P) = \frac{h_k(P)}{[k : \mathbb{Q}]}$.

**Remark 2.1.4:** We need to prove that $H(P)$ is well defined and $H(P) \geq 1$. It is clear from the following Lemma.

**Lemma 2.1.5:** Let $k$ be a number field and let $P \in \mathbb{P}^n(k)$ be a point.

(a) $H_k(P) \geq 1$ for $P \in \mathbb{P}^n(k)$.

(b) Let $k'$ be finite extension of $k$, then $H_{k'}(P) = H_k(P)^{[k' : k]}$.

**Proof:** (a) We take homogeneous coordinates for $P$ such that some coordinates is equal to 1, then $H_k(P) \geq 1$.

(b) $H_{k'}(P) = \prod_{w \in M_{k'}} \max\{\|x_0\|_w, \|x_1\|_w, \ldots, \|x_n\|_w\}$

$$= \prod_{v \in M_k} \prod_{w \in M_{k'}, w|v} \max\{\|x_0\|_w, \|x_1\|_w, \ldots, \|x_n\|_w\}$$

$$= \prod_{v \in M_k} \prod_{w \in M_{k'}, w|v} \max\{|x_0|_v^{n_w}, |x_1|_v^{n_w}, \ldots, |x_n|_v^{n_w}\}$$

$w|v$ means $w$ lies over $v$, $n_w = [k'_w : \mathbb{Q}_w] = [k'_w : k_v]$. 


Thus we get

\[ \prod_{v \in M_k} \prod_{w \mid v} \max\{\|x_0\|_v, \|x_1\|_v, \ldots, \|x_n\|_v\}^{[k'_w : k_v]} \]

\[ = \prod_{v \in M_k} \max\{\|x_0\|_v, \|x_1\|_v, \ldots, \|x_n\|_v\}^{[k'_w : k_v]} = H_k(P)^{[k'_w : k_v]} \]

from Degree formula.

**Remark 2.1.6 (Degree formula):** Let \( k'/k \) be an extension of number fields, and let \( v \in M_k \) be an absolute value on \( k \), then

\[ \sum_{w \in M'_w, w|v} [k'_w : k_v] = [k' : k]. \]  

(6)

**Proof:** See Lang[9.2, Corollary 1 to Theorem 2][7].

**Definition 2.1.7:** Let \( \alpha \) be an element of \( k \), then define

\[ H_k(\alpha) = \prod_{v \in M_k} \max\{\|\alpha\|_v, 1\} = H_k(\alpha, 1), (\alpha, 1) \in \mathbb{P}^1(k) \]  

(7)

and similarly for \( h_k(\alpha) \), \( H(\alpha) \), and \( h(\alpha) \).

**Property 2.1.8 (Invariant with respect to the Galois action):** The action of the Galois group on \( \mathbb{P}^n(\overline{\mathbb{Q}}) \) leaves the height invariant i.e. Let \( P \in \mathbb{P}^n(\overline{\mathbb{Q}}) \) and \( \sigma \in G_\mathbb{Q} \), then \( H(\sigma(P)) = H(P) \) where \( G_\mathbb{Q} \) means the Galois group of \( \overline{\mathbb{Q}} \) over \( \mathbb{Q} \).

**Proof:** Let \( k/\mathbb{Q} \) be a number field with \( P \in \mathbb{P}^n(k) \). The automorphism \( \sigma \) of \( \overline{\mathbb{Q}} \) defines an isomorphism \( \sigma : k \to \sigma(k) \) and \( \sigma : M_k \to M_{\sigma(k)}, v \to \sigma(v) \) where for \( x \in k \) and \( v \in M_k \). The absolute value \( \sigma(v) \in M_{\sigma(k)} \) is defined by \( |\sigma(x)|_{\sigma(v)} = |x|_v \), \( \sigma \) induces an isomorphism on the completion \( k_v \cong k(\sigma(v)) \), then \( n_v = n_{\sigma(v)} \).

\[ H_{\sigma(k)}(\sigma(P)) = \prod_{w \in M_{\sigma(k)}} \max\{|\sigma(x)v|_w\} \]

\[ = \prod_{w \in M_{\sigma(k)}} \max\{|\sigma(x)v|_w\}^{n_w} = \prod_{v \in M_k} \max\{|\sigma(x)v|_{\sigma(v)}\}^{n_{\sigma(v)}} \]

\[ = \prod_{v \in M_k} \max\{|x_v|_v\}^{n_v} = \prod_{v \in M_k} \max\{|x_v|_v\} = H_k(P). \]

Also \( [k : \mathbb{Q}] = [\sigma(k) : \mathbb{Q}] \), taking \([k : \mathbb{Q}]^{|\mu|}\) roots gives the result.

**Property 2.1.9:** For any numbers \( B, D \geq 0 \), the set \( \{P \in \mathbb{P}^n(\overline{\mathbb{Q}}) | H(P) \leq B, [\mathbb{Q}(P) : \mathbb{Q}] \leq D\} \) is finite, then for any fixed number field \( k \), the set \( \{P \in \mathbb{P}^n(k) | H_k(P) \leq B\} \) is finite where \( \mathbb{Q}(P) \) is the field of definition of a point \( P = (x_0, x_1, \ldots, x_n) \in \mathbb{P}^n(\overline{\mathbb{Q}}), \mathbb{Q}(P) = \mathbb{Q}\left(\frac{x_0}{x_j}, \frac{x_1}{x_j}, \ldots, \frac{x_n}{x_j}\right) \) for any \( j \) with \( x_j \neq 0 \).
**Proof:** Without loss of generality, we assume that some coordinate of \( P \) equals 1, then for any absolute value \( v \) and any index \( i \), we have \( \max\{\|x_0\|_v, \|x_1\|_v, \ldots, \|x_n\|_v\} \geq \max\{\|x_i\|_v, 1\} \). Now we multiply over all \( v \) and take a root, then \( H(P) \geq H(x_i) \) for \( 0 \leq i \leq n \). Moreover, it is easy to see that \( \mathbb{Q}(x_i) \subset \mathbb{Q}(P) \). Thus it suffices to prove that for any \( 1 \leq d \leq D \), the set \( \{x \in \overline{\mathbb{Q}}|H(x) \leq B, |\mathbb{Q}(x) : \mathbb{Q}| = d\} \) is finite. Choose \( x \) such that \( x \in \overline{\mathbb{Q}} \) with degree \( d \) and \( k = \mathbb{Q}(x) \), \( x_1, \ldots, x_d \) are the conjugates of \( x \) over \( \mathbb{Q} \). Define \( F_x(T) = \prod_{j=1}^{d} (T - x_j) = \sum_{r=0}^{d} (-1)^r S_r(x) T^{d-r} \). It is the minimal polynomial of \( x \) over \( \mathbb{Q} \). ∀\( v \in M_k \), we have

\[
|S_r(x)|_v = \left| \sum_{1 \leq i_1 \leq \cdots \leq i_r \leq d} x_{i_1} x_{i_2} \ldots x_{i_r} \right|_v
\]

\[
\leq c(v, r, d) \max_{1 \leq i_1 \leq \cdots \leq i_r \leq d} |x_{i_1} x_{i_2} \ldots x_{i_r}|_v
\]

\[
\leq c(v, r, d) \max_{1 \leq i \leq d} |x_i|^v
\]

where \( c(v, r, d) = \binom{d}{r} \leq 2^d \) if \( v \) is archimedean, we can take \( c(v, r, d) = 1 \), if \( v \) is nonarchimedean. Then

\[
\max\{|s_0(x)|_v, \ldots, |s_d(x)|_v\} \leq c(v, d) \prod_{i=1}^{d} \max\{|x_i|_v, 1\}^d,
\]

(8)

here \( c(v, d) = 2^d \) if \( v \) is archimedean and \( c(v, d) = 1 \) otherwise. Multiplying this inequality over all \( v \in M_k \) and taking the \([k : \mathbb{Q}]^{th}\) root, we get the estimate \( H(s_0(x), s_1(x), \ldots, s_d(x)) \leq 2^d \prod_{i=1}^{d} H(x_i)^d \). However, the \( x_i \)'s are conjugates, so Property 1.8 tells us that all of the \( H(x_i) \)'s are equal, then \( H(s_0(x), s_1(x), \ldots, s_d(x)) \leq 2^d H(x)^d \). Suppose that \( x \) belongs to \( \{x \in \overline{\mathbb{Q}}|H(x) \leq B, |\mathbb{Q}(x) : \mathbb{Q}| = d\} \). Thus we provided that \( x \) is the root of a polynomial \( F_x(T) \in \mathbb{Q}[T] \) whose coefficients \( s_0, \ldots, s_d \) satisfy \( H(s_0, \ldots, s_d) \leq 2^d B^{d^2} \). But \( \mathbb{P}^d(\mathbb{Q}) \) has only finitely many points of bounded height, then there are only finitely many possibilities for the polynomial \( F_x(T) \) and only finitely many possibilities for \( x \). This gives the desired result.

**Remark 2.1.10:** For any \( B > 0 \), the set \( \{P \in \mathbb{P}^n(\mathbb{Q})|H(P) \leq B\} \) is finite, since there are only finitely many integers \( x \in \mathbb{Z} \) satisfying \( |x| \leq B \).

2.2. Heights on Varieties

**Definition 2.2.1:** Let \( \Phi : V \rightarrow \mathbb{P}^n \) be a morphism, here \( V \) is a projective variety defined over \( \overline{\mathbb{Q}} \). The absolute logarithmic height on \( V \) relative to \( \Phi \) is the function \( h_{\Phi} : V(\overline{\mathbb{Q}}) \rightarrow [0, \infty) \), \( h_{\Phi}(P) = h(\Phi(P)) \) where \( h : \mathbb{P}^n(\overline{\mathbb{Q}}) \rightarrow [0, \infty) \) is the height function on projective space.

Now, we need to give Weil's construction that associates a height function to every divisor.

**Theorem 2.2.2 (Weil’s Height Machine):** Let \( k \) be a number field. ∀ smooth projective variety \( V/k \), there exists a map

\[
h_v : \text{Div}(V) \rightarrow \{\text{functions } V(\overline{k}) \rightarrow \mathbb{R}\}
\]

(9)
with the following properties. (\(Div(V)\) is the group of Weil divisors on \(V\))
(a) (Normalization) Let \(H \subset \mathbb{P}^n\) be a hyperplane and let \(h(P)\) be the absolute logarithmic height on \(\mathbb{P}^n\) defined earlier. Then
\[
h_{\mathbb{P}^n,H}(P) = h(P) + O(1), \forall P \in \mathbb{P}^n(\bar{k}). \tag{10}\]
(b) (Functoriality) Let \(\Phi : V \to W\) be a morphism and \(D \in Div(W)\). We have
\[
h_{V,\Phi^*D}(P) = h_{W,D}(\Phi(P)) + O(1), \forall P \in V(\bar{k}). \tag{11}\]
(c) (Additivity) Let \(D, E \in Div(V)\), we obtain that
\[
h_{V,D+\mathcal{E}}(P) = h_{V,D}(P) + h_{V,E}(P) + O(1), \forall P \in V(\bar{k}). \tag{12}\]
(d) (Linear Equivalence) Let \(D, E \in Div(V)\) with \(D\) linearly equivalent to \(E\). Then,
\[
h_{V,D}(P) = h_{V,E}(P) + O(1) \text{ for all } P \in V(\bar{k}). \tag{13}\]
(e) (Positivity) Let \(D \in Div(V)\) be an effective divisor and let \(B\) be the base locus of the linear system \(|D|\), we have
\[
h_{V,D}(P) \geq O(1), \forall P \in (V - B)(\bar{k}). \tag{14}\]
(f) (Algebraic Equivalence) Let \(D, E \in Div(V)\) with \(D\) ample and \(E\) algebraically equivalent to 0. Then
\[
\lim_{P \in V(\bar{k}), h_{V,D}(P) \to \infty} \frac{h_{V,E}(P)}{h_{V,D}(P)} = 0. \tag{15}\]
(g) (Finiteness) Let \(D \in Div(V)\) be ample. Then for every finite extension \(k'/k\) and every constant \(B\), the set \(\{P \in V(k')|h_{V,D}(P) \leq B\}\) is finite.
(h) (Uniqueness) The height functions \(h_{V,D}\) are determined up to \(O(1)\) by normalization (a), functoriality (b) just for embedding \(\Phi : V \hookrightarrow \mathbb{P}^n\) and additivity (c).

**Remark 2.2.3:** The "\(O(1)\)" constants depend on the varieties, divisors, and morphisms but they are independent of the points on the varieties.

**Proof:** First, \(\forall D \in Div(V)\) whose linear system has no base points. Now, we choose a morphism \(\Phi_D : V \to \mathbb{P}^n\) associated to \(D\) i.e. \(\Phi_D\) is a morphism such that \(\Phi_D^*H \sim D\) for any hyperplane \(H\) in \(\mathbb{P}^n\). Define
\[
h_{V,D}(P) = h(\Phi_D(P)), \forall P \in V(\bar{k}). \tag{16}\]
We know that every divisor can be written as the difference of two very ample divisors. Say \(D = D_1 - D_2\), then for any \(D \in Div(V)\), we define
\[
h_{V,D}(P) = h_{V,D_1}(P) - h_{V,D_2}(P), \forall P \in V(\bar{k}). \tag{17}\]
We need to prove that \(h_{V,D}\) is independent of the morphism \(\Phi_D\) up to \(O(1)\). Let \(\psi_D : \)
$V \rightarrow \mathbb{P}^m$ be another morphism associated to $D$, this means that $\Phi_D^*H \sim D \sim \psi_D^*H'$, here $H$ and $H'$ are hyperplanes in $\mathbb{P}^n$ and $\mathbb{P}^m$, respectively, we claim that

$$h(\Phi_D(P)) = h(\psi_D(P)) + O(1), \forall P \in V(\overline{k}). \quad (18)$$

It comes from the following lemma.

**Lemma 2.2.4:** Let $V$ be a projective variety defined over $\overline{k}$, $\Phi : V \rightarrow \mathbb{P}^n$ and $\psi : V \rightarrow \mathbb{P}^m$ are morphisms. $H$ and $H'$ are hyperplanes in $\mathbb{P}^n$ and $\mathbb{P}^m$, respectively. Suppose that, $\Phi^*H \sim \psi^*H'$. Then $h_\Phi(P) = h_\psi(P) + O(1)$ for all $P \in V(\overline{\mathbb{Q}})$. The $O(1)$ constant will depend on $V$, $\Phi$ and $\psi$, but it is independent of $P$.

**Proof:** $D \in \text{Div}(V)$ is any positive divisor in the linear equivalence class of $\Phi^*H$ and $\psi^*(H')$. $\Phi$ and $\psi$ are determined respectively by certain $V_0$ and $V_0'$ in the vector space $L(D)$ and choices of bases for $V_0$ and $V_0'$. (See Joseph Hindry-Silverman [Diophantine Geometry Section A3.1][6])

We choose a basis $h_0, h_1, \ldots, h_N$ for $L(D)$, then $\exists f_i = \sum_{j=0}^N a_{ij}h_j, 0 \leq i \leq n, g_i = \sum_{j=0}^N b_{ij}h_j, 0 \leq i \leq m$ such that $\Phi$ and $\psi$ are given by $\Phi = (f_0, f_1, \ldots, f_n)$, $\psi = (g_0, g_1, \ldots, g_n)$ where $a_{ij}$'s and $b_{ij}$'s are constants. Now, $\lambda = (h_0, h_1, \ldots, h_N) : V \rightarrow \mathbb{P}^n$ is the morphism associated to $|D|$. Let $A$ be the linear map $A : \mathbb{P}^N \rightarrow \mathbb{P}^n$ defined by the matrix$(a_{ij})$, $B : \mathbb{P}^N \rightarrow \mathbb{P}^m$ is the linear map defined by $(b_{ij})$. Then we have following two equalities $\Phi = A \circ \lambda$ and $\psi = B \circ \lambda$.

Notation: $A$, $B$ are not morphisms on all of $\mathbb{P}^N$.

However, $\Phi$ and $\psi$ are morphisms associated to the linear system $L(D)$ implies that $A$, $B$ are both defined at every point of the image $\lambda(V(\overline{k}))$. Thus we have $h(A(Q)) = h(Q) + O(1)$ and $h(B(Q)) = h(Q) + O(1)$ for all $Q \in \lambda(V(\overline{\mathbb{Q}}))$. This claims comes from the following lemma.

**Lemma 2.2.5:** Let $A : \mathbb{P}^n \rightarrow \mathbb{P}^m$ be a linear map defined over $\overline{k}$ i.e. $A$ is given by $m+1$ linear forms $(L_0, \ldots, L_m)$, $Z \subset \mathbb{P}^n$ is the linear subspace where $L_0, L_1, \ldots, L_m$ vanish at the same time and $X \subset \mathbb{P}^n$ is a closed subvariety with $X \cap Z = \emptyset$, then $h(A(P)) = h(P) + O(1)$ for all $P \in X(\overline{\mathbb{Q}})$.

**Proof:** Joseph Hindry-Silverman [Diophantine Geometry B.2 Corollary B2.6][6].

Now, we are back to Lemma 2.4. We write $Q = \lambda(P)$ with $P \in V(\overline{k})$ and use the commutative diagrams, thus

$$h(\Phi(P)) = h(A(\lambda)) = h(\lambda(P)) + O(1) = h(B(\lambda(P))) + O(1) = h(\psi(P)) + O(1).$$

The proof of Lemma 2.4 is done.

Hence for base point free divisors, we can use any associated morphism to compute the height. We check (c) for base point free divisors, which we use to show that $h_{V,D}$ is independent of the decomposition $D = D_1 - D_2$.

$D$ and $E$ are base point free divisors, $\Phi_D : V \rightarrow \mathbb{P}^n$, $\Phi_E : \mathbb{P}^m$ are associated morphisms. We compose the product $\Phi_D \times \Phi_D : V \rightarrow \mathbb{P}^n \times \mathbb{P}^m$ with the Segre embedding
$S_{n,m}$ gives a morphism $\Phi_D \otimes \Phi_E : V \to \mathbb{P}^N$, $\Phi_D \otimes \Phi_E(P) = S_{n,m}(\Phi_D(P), \Phi_E(P))$.

**Notation 2.2.6 (Segre embedding):** $m, n \geq 1$ are integers, $N = (n + 1)(m + 1) - 1$, $S_{n,m} : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$, $(x, y) \to (x_i y_j)_{0 \leq i \leq n, 0 \leq j \leq m}$, here we have written $x = (x_0, x_1, \ldots, x_n) \in \mathbb{P}^n$ and $y = (y_0, y_1, \ldots, y_m) \in \mathbb{P}^m$, $S_{n,m}$ is the morphism and gives embedding of the product $\mathbb{P}^n \times \mathbb{P}^m$ into $\mathbb{P}^N$.

The morphism $\Phi_D \otimes \Phi_E$ is associated to the divisor $D + E$ means that $(\Phi_D \otimes \Phi_E)^* H \sim D + E$. This is a result of the following proposition.

**Proposition 2.2.7:** Let $S_{n,m}$ be the Segre embedding described in the following statement, $S_{n,m} : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$, $(x, y) \to (x_0 y_0, x_0 y_1, \ldots, x_i y_j, \ldots, x_n y_m)$. Let $H_n, H_m$ and $H_N$ be hyperplanes in $\mathbb{P}^n, \mathbb{P}^m$, and $\mathbb{P}^N$, respectively.

(a) $S_{n,m}^*(H_N) \sim H_n \times \mathbb{P}^m + \mathbb{P}^n \times H_m \in \text{Div}(\mathbb{P}^n \times \mathbb{P}^m)$.

(b) $h(S_{n,m}(x, y)) = h(x) + h(y)$ for all $x \in \mathbb{P}^n(\overline{k})$ and $y \in \mathbb{P}^m(\overline{k})$.

**Proof:** See Joseph Hindry-Silverman [Diophantine Geometry B.2 Proposition B2.4][6].

Thus $h_{V,D+E}(P) = h(\Phi(D) \otimes \Phi_E(P)) + O(1)$ for all $P \in V(\overline{k})$. Then we use Proposition 2.7(b), we can obtain that

$$h_{V,D+E}(P) = h(\Phi(D) \otimes \Phi_E(P)) + O(1) = h(S_{n,m}(\Phi_D(P), \Phi_E(P))) + O(1)$$

$$= h(\Phi_D(P)) + h(\Phi_E(P)) + O(1) = h_{V,D}(P) + h_{V,E}(P) + O(1)$$

This gives us additivity for base point free divisors.

If we have two decompositions $D = D_1 - D_2 = E_1 - E_2$ of $D$ as the difference of the base point divisors, $D_1 + E_2 = D_2 + E_1$, thus

$$h_{V,D_1} + h_{V,E_2} = h_{V,D_1+E_2} + O_1$$

$$= h_{V,D_2+E_1} + O_1 = h_{V,D_2} + h_{V,E_1} + O_1$$

Now, we check properties (a) and (b).

If $H$ is a hyperplane in $\mathbb{P}^n$, then the identity map $\mathbb{P}_n \to \mathbb{P}_n, P \to P$ is associated to $H$, this tells us that (a) is correct.

In order to verify (b), $D \in \text{Div}(W)$, we write it as a difference of base point free divisors $D = D_1 - D_2$. $\Phi^*(D_1)$ and $\Phi^*(D_2)$ are base point free (Since $D_1, D_2$ are base point free and $\Phi$ is a morphism between two projective varieties) with associated morphisms $\Phi_D, \Phi_E$ and $\Phi_{D_1} \circ \Phi$ and $\Phi_{D_2} \circ \Phi$, respectively.

Then:

$$h_{W,\Phi^*D} = h_{W,\Phi^*D_1} - h_{W,\Phi^*D_2} + O(1) = h \circ \Phi_D \circ \Phi - h \circ \Phi_D \circ \Phi + O(1)$$

$$= h_{W,D_1} \circ \Phi - h_{W,D_2} \circ \Phi + O(1) = h_{W,D} \circ \Phi + O(1).$$

It is done.

As to property (c), which we have proved for base point free divisors. $D, E$ are arbitrary divisors, write them as differences $D = D_1 - D_2$ and $E = E_1 - E_2$ of base point
free divisors. Then \( D_1 + E_1 \) and \( D_2 + E_2 \) are base point free. Then
\[
h_{V,D+E} = h_{V,D_1+E_1} - h_{V,D_2+E_2} + O(1) = h_{V,D_1} + h_{V,E_2} - h_{V,D_2} - h_{V,E_2} + O(1) = h_{V,D} + h_{V,E} + O(1).\]

(c) is done!
We note that properties (a), (b) and (c) determine the height function up to \( O(1) \).
If \( D \) is very ample with associated embedding \( \Phi_D : V \to \mathbb{P}^n \), (a), (b) imply that
\( h_{V,D} = h \circ \Phi_D + O(1) \). This determines the height function for very ample divisors. On
the other hand, any divisor \( D \) can be written as the differences \( D_1 - D_2 \) of very ample
divisors, then (c) means that \( h_{V,D} = h_{V,D_1} - h_{V,D_2} + O(1) \). The property (h) is proved.

We want to verify property (d), so suppose that \( D, E \) are linearly equivalent.

As usual, we write \( D = D_1 - D_2 \) and \( E = E_1 - E_2 \) as the difference of base point free
divisors, then \( D_1 + E_2 \sim D_2 + E_1 \) which means that the morphisms \( \Phi_{D_1+E_2} \) and \( \Phi_{D_2+E_1} \)
are associated to the same linear system. Lemma 2.4 tells us that \( \Phi_{D_1+E_2} \sim \Phi_{D_2+E_1} \).

We note that properties (a), (b) and (c) determine the height function up to
\( O(1) \).

Then:
\[
h_{V,D} - h_{V,D_1} + h_{V,D_2} - h_{V,E_2} + O(1) = h_{V,E_1} - h_{V,E_2} + O(1) = h_{V,E} + O(1).\]

(d) is done!
In order to prove (e), we take \( D > 0 \) and write \( D = D_1 - D_2 \) as usual.

Choose a basis \( f_0, f_1, \ldots, f_n \) for \( L(D_2) \). Then \( D \) is positive implies that \( D + \text{div}(f_i) = D + f_0, f_1, \ldots, f_n \) are also in \( L(D_1) \). Extending the set to form a basis
\( f_0, f_1, \ldots, f_n, f_{n+1}, \ldots, f_m \in L(D_1) \), these basis give us morphisms \( \Phi_{D_1} = (f_0, f_1, \ldots, f_m) : V \to \mathbb{P}^n \), \( \Phi_{D_2} = (f_0, f_1, \ldots, f_n) : V \to \mathbb{P}^n \) associated to \( D_1 \) and \( D_2 \).

The functions \( f_0, f_1, \ldots, f_m \) are regular at all points not in the support of \( D_1 \), so for any \( P \in V \) with \( P \not\in \text{supp}(D_1) \),
\[
h_{V,D}(P) = h_{V,D_1}(P) - h_{V,D_2}(P) + O(1) = \mu(\Phi_{D_1}(P)) - \mu(\Phi_{D_2}(P)) + O(1)\]
\[
= h(f_0(P), f_1(P), \ldots, f_m(P)) - h(f_0(P), f_1(P), \ldots, f_n(P)) + O(1) \geq O(1)\]

This last inequality follows from the definition of the height, since \( m \geq n \) implies
\[
\prod_{v \in M_k} \max_{0 \leq i \leq m} \{ \| f_i(P) \|_v \} \geq \prod_{v \in M_k} \max_{0 \leq i \leq n} \{ \| f_i(P) \|_v \}.\]

This gives us the estimate for points not in the support of \( D_1 \). We choose very ample
divisors \( H_0, H_1, \ldots, H_r \) on \( V \) with property that \( H_0 \cap H_1 \cap \cdots \cap H_r = \emptyset \) and \( H_i + D \) is
very ample.

To explain the reason we can do this, we use the following lemma.

**Lemma 2.2.8:** Suppose that \( D \) is an arbitrary divisor and \( H \) is a very ample divisor,
then \( \exists m \geq 0 \) s.t. \( mH + D \) is base point free. Moreover, if \( D \) is base point free, then \( D + H \)
is very ample.
Proof: See Joseph Hindry-Silverman [Diophantine Geometry A.3.2 Theorem A.3.2.3][6].

We can find a very ample divisor $H$ such that $D + H$ is also very ample. Take an embedding $V \hookrightarrow \mathbb{P}^n$ corresponding to $H$, and take the $H_i$'s to be the pullbacks of the coordinate hyperplanes in $\mathbb{P}^n$. Then we apply our result to each of the decompositions $D = (D + H_i) - H_i$ to deduce the inequality $h_{V,D} \geq O(1)$ for all points not in the support of $D$.

In the end, varying $D$ in its linear system $|D|$, we get the (e) for all points not lying in the base locus of $|D|$.

Now, we use the fact that if $D$ is ample and $E$ is algebraically equivalent to 0, then $\exists m > 0 (m \in \mathbb{Z})$ s.t. $mD + nE$ is base point free for all $n \in \mathbb{Z}$ (See Lang [Fundamentals of Diophantine Geometry, Chapter 4, Lemma 3.2][7]).

The height associated to a base point free divisor is nonnegative by construction, then $h_{V,mD+nE}(P) \geq O(1)$ for all $P \in V(\mathbb{K})$. Using (c), we obtain $mh_{V,D}(P) + nh_{V,E}(P) \geq -c$ for all $P \in V(\mathbb{K})$. Here the constant $-c$ will depend on $D, E, m$ and $n$, but it is independent of $P$. This holds for all $n \in \mathbb{Z}$, we can rewrite using positive and negative values for $n$.

Then for any $n \geq 1$, we have

$$
\frac{m}{n} + \frac{c}{nh_{V,D}(P)} \geq \frac{h_{V,E}(P)}{h_{V,D}(P)} \geq -\frac{m}{n} - \frac{c}{nh_{V,D}(P)}, \forall P \in V(\mathbb{K}). \tag{22}
$$

Note that $c$ depends on $n$.

Now, let $h_{V,D}(P) \to \infty$, then

$$
\frac{m}{n} \geq \limsup_{h_{V,D}(P) \to \infty} \frac{h_{V,E}(P)}{h_{V,D}(P)} \geq \liminf_{h_{V,D}(P) \to \infty} \frac{h_{V,E}(P)}{h_{V,D}(P)} \geq -\frac{m}{n}. \tag{23}
$$

These inequalities hold for all $n \geq 1$. Let $n \to \infty$, we have

$$
\lim_{h_{V,D}(P) \to \infty} \frac{h_{V,E}(P)}{h_{V,D}(P)} = 0. \tag{24}
$$

(f) is done!

Finally, we have to prove the property (g). We replace the ample divisor $D$ by a very ample $mD$, then (c) implies that $h_{V,mD} = mh_{V,D} + O(1)$. Thus, without loss of generality, we can suppose that $D$ is very ample. $\Phi : V \hookrightarrow \mathbb{P}^n$ is an embedding associated to $D$, so $\Phi^*H = D$. Property (a) and (b) imply that

$$
h_{V,D} \circ \Phi = h_{\mathbb{P}^n, \Phi^*D} + O(1) = h_{\mathbb{P}^n,H} + O(1) = h + O(1). \tag{25}
$$

We need only to show that $\mathbb{P}^n(k)$ has finitely many points of bounded height, it is the result we have proved earlier.

2.3. Canonical Heights

We have defined the height machine which associates to each divisor $D \in \text{Div}(V)$, a height function $h_D : V(\mathbb{K}) \to \mathbb{R}$. In some case, it is possible to find a particular height
function within its O(1) equivalence class that has particularly good properties.

**Theorem 2.3.1:** \( V/k \) is a smooth variety defined over a number field. Let \( D \in \text{Div}(V) \) and \( \Phi : V \to V \) be a morphism. Suppose that \( \Phi^* D \sim \alpha D \) for some some number \( \alpha > 1 \). Then \( \exists \) a unique function, called the canonical height on \( V \) relative to \( \Phi \) and \( D \), \( \hat{h}_{V, \Phi, D} : V(\overline{k}) \to \mathbb{R} \) with the following two properties.

\[
\hat{h}_{V, \Phi, D}(P) = h_{V, D}(P) + O(1) \quad \text{for all } P \in V(\overline{k})
\]

\[
\hat{h}_{V, \Phi, D}(\Phi(P)) = \alpha \hat{h}_{V, \Phi, D}(P) \quad \text{for all } P \in V(\overline{k})
\]

This canonical height depends only on the linear equivalence class of \( D \). Moreover, it can be computed as the limit

\[
\hat{h}_{V, \Phi, D}(P) = \lim_{n \to \infty} \frac{h_{V, D}(\Phi^n(P))}{\alpha^n} \quad \text{where } \Phi^n = \Phi \circ \Phi \circ \ldots \circ \Phi
\]

\( n \)-fold iterate of \( \Phi \).

**Proof:** We apply the height machine to the relation \( \Phi^* D \sim \alpha D \), there exists a constant \( C \) such that

\[
|h_{V, D}(\Phi(Q)) - \alpha h_{V, D}(Q)| \leq C \quad \text{for all } Q \in V(\overline{k}).
\]

Notation: \( C \) depends on \( V, D, \Phi \), and the choice of the height function \( h_{V, D} \).

\( \forall P \in V(\overline{k}) \), we want to prove that \( \alpha^{-n} h_{V, D}(\Phi^n(P)) \) converges. Taking \( n \geq m \), computing

\[
|\alpha^{-n} h_{V, D}(\Phi^n(P)) - \alpha^{-m} h_{V, D}(\Phi^m(P))| \\
= \left| \sum_{i=m+1}^{n} \alpha^{-i}(h_{V, D}(\Phi^i(P)) - \alpha h_{V, D}(\Phi^{i-1}(P))) \right| \\
\leq \sum_{i=m+1}^{n} \alpha^{-i}|(h_{V, D}(\Phi^i(P)) - \alpha h_{V, D}(\Phi^{i-1}(P)))| \\
\leq \sum_{i=m+1}^{n} \alpha^{-i}C = \frac{\alpha^{-m} - \alpha^{-n}}{\alpha - 1}C
\]

which means this sequence is Cauchy, hence converges. Then we can define \( \hat{h}_{V, \Phi, D}(P) \) to be the limit.

\[
\hat{h}_{V, \Phi, D}(P) = \lim_{n \to \infty} \frac{h_{V, D}(\Phi^n(P))}{\alpha^n} \quad \text{(26)}
\]

In order to verify property (1), we take \( m = 0 \) and let \( n \to \infty \), then

\[
|\hat{h}_{V, \Phi, D}(P) - h_{V, D}(P)| \leq \frac{c}{\alpha - 1}. \quad \text{(27)}
\]

As to the Property (2), we know that:

\[
\hat{h}_{V, \Phi, D}(\Phi(P)) = \lim_{n \to \infty} \frac{h_{V, D}(\Phi^n(\Phi(P)))}{\alpha^n} \\
= \lim_{n \to \infty} \frac{\alpha h_{V, D}(\Phi^{n+1}(P))}{\alpha^{n+1}} = \alpha \hat{h}_{V, \Phi, D}(P).
\]


In the end, we want to prove the uniqueness, suppose that \( \hat{h} \) and \( \hat{h}' \) are two functions with properties (1) and (2). \( g = \hat{h} - \hat{h}' \). Then (1) implies that \( g \) is bounded, say \( |g(P)| \leq c' \) for all \( P \in V(\overline{k}) \). Moreover, (2) says that \( g \circ \Phi = \alpha g \), \( g \circ \Phi^n = \alpha^n g \) for all \( n \geq 1 \). Thus

\[
|g(P)| = \frac{|g(\Phi^n(P))|}{\alpha^n} \leq \frac{c'}{\alpha^n}
\]

and \( \frac{c'}{\alpha^n} \to 0 \) as \( n \to \infty \) which means that \( g(P) = 0 \) for all \( P \), then \( \hat{h} = \hat{h}' \).

Theorem 2.3.1 associates a canonical height to any morphism \( \Phi : V \to V \) with an eigendivisor \( \Phi^*D \sim \alpha D \) having eigenvalue \( \alpha > 1 \). Now, we get an important example which is the case of an abelian variety \( A \), a symmetric divisor \( D \), and a multiplication by \( m \) map \([m] : A \to A\).

**Theorem 2.3.2:** Suppose that \( A/k \) is an abelian variety defined over a number field and \( D \in \text{Div}(A) \) is a divisor whose divisor class is symmetric (which means that \([−1]^*D \sim D\) ). Then, for any height function \( \hat{h}_{A,D} : A(\overline{k}) \to \mathbb{R} \), named the canonical height on \( A \) relative to \( D \) with the following properties:

(a) \[\hat{h}_{A,D}(P) = h_{A,D}(P) + O(1), \forall P \in A(\overline{k}).\]  

(b) \[\forall m \in \mathbb{Z}, \hat{h}_{A,D}([m]P) = m^2 \hat{h}_{A,D}(P), \forall P \in A(\overline{k}).\]  

(c)(Parallelogram Law) \[\hat{h}_{A,D}(P + Q) + \hat{h}_{A,D}(P - Q) = 2\hat{h}_{A,D}(P) + 2\hat{h}_{A,D}(Q), \forall P, Q \in A(\overline{k}).\]  

(d)\(\hat{h}_{A,D} : A(\overline{k}) \to \mathbb{R} \) is a quadratic form. The associated pairing \( \langle \cdot, \cdot \rangle_D : A(\overline{k}) \times A(\overline{k}) \to \mathbb{R} \) defined by \[\langle P, Q \rangle_D = \frac{\hat{h}_{A,D}(P + Q) - \hat{h}_{A,D}(P) - \hat{h}_{A,D}(Q)}{2}\]  

is bilinear and satisfies \( \langle P, P \rangle_D \geq \hat{h}_{A,D}(P) \).

(e)(Uniqueness) The canonical height \( \hat{h}_{A,D} \) depends only on the divisor class of the divisor \( D \). It is uniquely determined by (a) and (b) for any \( m \geq 2(m \in \mathbb{Z}) \).

**Proof:** See Joseph Hindry-Silverman [Diophantine Geometry, B.5 Theorem B5.1][6].

### 2.4 Heights in function fields

**Definition 2.4.1** Let \( W \) be a projective variety in \( \mathbb{P}^r \), non-singular in codimension 1, and define over a field \( k \). Let \( c \) be a number, \( 0 < c < 1 \). Let \( M_k = M_k(W) \) be the set of discrete absolute values of the function field \( k(W) \) obtained from the prime rational divisors of \( W \) over \( k \). We then have by definition \(|x|_p = c^{(\text{ord}_p(x))\deg(p)}\) for each such prime
divisor $p$, and our set $M_k$ satisfies the product formula. If $P$ is a point in $\mathbb{P}^n$ rational over $k(W)$ with coordinates $(y_0, \ldots, y_n)$ in $k(W)$, then

$$H_{k(W)}(P) = H_W(P) = \prod_P \sup_i |y_i|_p.$$ (33)

We have the following fact: Let $d = \deg \sup_i(y_i)_\infty$ be the projective degree in $\mathbb{P}^r$ of the sup of the polar divisors of the $y_i$. Then

$$H_W(P) = \left(\frac{1}{c}\right)^d.$$ (34)

We define the logarithmic height, or simply height to be Then

$$h_W(P) = \deg \sup_i(y_i)_\infty.$$ (35)

**Proposition 2.4.2:** Let $W$ be a projective variety in $\mathbb{P}^m$, non-singular in codimension 1, defined over the field $k(W)$. Let $f : W \in \mathbb{P}^n$ be the rational map defined over $k$, determined by $P$. Then

$$h_W(P) = \deg f^{-1}(L).$$ (36)

for any hyperplane $L$ of $\mathbb{P}^n$, such that $f^{-1}(L)$ is defined, the degree being that in the given projective embedding of $W$ in $\mathbb{P}^m$.

**Proof:** S.Lang, [Fundamentals of diophantine geometry, Proposition 3.2][7].

**Theorem 2.4.3:** Let $W^r$ be a projective variety, non-singular in codimension 1 and defined over a field $k$. Let $P$ be a point in $\mathbb{P}^n$ rational over $k(W)$, and let $T$ be the locus of $P$ over $k$, so that we have a generically surjective rational map $g : W \to T$. Then

$$\deg_{\mathbb{P}^n} T \leq h_W(P).$$ (37)

### 3 Semi-stable reduction theorem

1.Models of algebraic curves

**Definition 3.1.1:** The scheme $S$ is called an integral scheme if $S$ is reduced and irreducible. $S$ is said to be Noetherian if it is a finite union of affine open $S_i$ such that $O_S(S_i)$ is a Noetherian ring for every $i$. The scheme is locally Noetherian if every point has a Noetherian open neighborhood.

We call $S$ is normal at $s \in S$ if $O_{S,s}$ is normal. Then we say that $S$ is normal if it is irreducible and normal at all of its points.

Finally, we call a normal locally Noetherian scheme of dimension 0 or 1 a Dedekind scheme.
Definition 3.1.2: Let $k$ be a field. An **affine variety** over $k$ is the affine scheme associated to a finitely generated algebra over $k$. An algebraic variety over $k$ is a $k$-scheme $X$ such that there exists a covering by a finite number of affine open subschemes $X_i$ which are affine varieties over $k$. A **projective variety** over $k$ is a projective scheme over $k$.

An algebraic (resp. projective) variety over $k$ whose irreducible components are of dimension 1 is called an **algebraic curve** (projective curve) over $k$.

Definition 3.1.3: Let $S$ be a scheme and let $X, Y$ be two $S$-schemes. We define the **fibered product** of $X, Y$ over $S$ to be an $S$-scheme $X \times_S Y$, together with two morphisms of $S$-schemes $p : X \times_S Y \to X, q : X \times_S Y \to Y$, verifying the following universal property:

Let $f : Z \to X, g : Z \to Y$ be two morphisms of $S$-schemes. Then there exists a unique morphism of $S$-schemes $(f, g) : Z \to X \times_S Y$ making the following diagram commutative:

\[
\begin{array}{ccc}
Z & \xrightarrow{(f,g)} & X \\
\downarrow{g} & & \downarrow{p} \\
Y & \leftarrow & X \times_S Y
\end{array}
\]

Let $f : X \to Y$ be a morphism of schemes. For any $y \in Y$, we set $X_y = X \times_Y \text{Spec } k(y)$ where $k(y)$ means that the residue field of $Y$ at $y$. This is the fiber of $f$ over $y$. Let $S$ be a Dedekind scheme. We call an integral, projective, flat $S$-scheme $\pi : X \to S$ of dimension 2 a **fibered surface** over $S$. The generic fiber of $S$ will be denoted by $\eta$. We call $X_{\eta}$ the **generic fiber** of $X$. We will say that $X$ is a normal (resp. regular) fibered surface if $X$ is normal (resp. regular).

Definition 3.1.4: Let $X$ be an integral scheme, with generic point $\xi$, we have that $\text{Frac}(\mathcal{O}_X(V)) \simeq \mathcal{O}_{X, \xi}$, called the **function field** of $X$, denoted as $k(X)$. See Qingliu [Algebraic Geometry and Arithmetic Curves, 2.4.3 Proposition 4.18].

Definition 3.1.5: Let $S$ be a Dedekind scheme of dimension 1, with function field $k$. Let $C$ be a normal, connected, projective curve over $k$. We call a normal fibered surface $\pi : C \to S$ together with an isomorphism $f : C_\eta \simeq C$ a model of $C$ over $S$. We will say that a model $(C, f)$ verifies a property $(P)$ if $C \to S$ verifies $(P)$. A morphism $C \to C'$ of two models of $C$ is a morphism of $S$-schemes that is compatible with the isomorphisms $C_\eta \simeq C, C'_\eta \simeq C$.

Definition 3.1.6: We call a regular fibered surface $X \to S$ is **relatively minimal** if it does not contain any exceptional divisor (means that an integral curve that can be contracted to a regular point). This is equivalent to say that every birational morphism of regular fibered surfaces $X \to Y$ is an morphism(See Qingliu [Algebraic Geometry and Arithmetic Curves, 9.2, Theorem 2.2][8]). We say that $X \to S$ is **minimal** if every birational map of regular fibered $S$-surfaces $Y \dashrightarrow X$ is a birational morphism.
Definition 3.1.7: A regular fibered surface \(X \to S\) over a Dedekind scheme \(S\) of dimension 1 is called an arithmetic surface.

Definition 3.1.8: Let \(X \to S\) be an arithmetic surface. We say that \(X \to S\) has normal crossings if for every closed point \(s \in S\), the divisor \(X_s\) on \(X\) has normal crossings.

Proposition 3.1.9: Let \(S\) be an affine Dedekind scheme of dimension 1 with function field \(k\). Let \(C\) be a smooth projective curve of genus \(g\) over \(k\). Then \(C\) admits a relatively minimal regular model (resp. a regular model with normal crossings) over \(S\). If moreover, \(g \geq 1\), then \(C\) admits a unique minimal regular model \(C_{\text{min}}\) and a unique minimal regular model with normal crossings.

Proof: See Qingliu [Algebraic Geometry and Arithmetic Curves 10.1 Proposition 1.8][8].

Definition 3.1.10: Let \(X \to S\) be a minimal arithmetic surface with \(P_a(X_\eta) \geq 2\), where \(P_a(X_\eta)\) denotes the arithmetic genus of a vertical divisor \(X_\eta\) (We can see the definition of vertical divisor in Qingliu [Algebraic Geometry and Arithmetic Curves, 8.3.1, Definition 3.5][8]). Let \(f : X \to Y\) be the contraction of the vertical prime divisors \(\Gamma\) such that \(K_{X/S} \cdot \Gamma = 0\) (The definition of intersection product operation \(\cdot\) is found in Qingliu [Algebraic Geometry and Arithmetic Curves 9.1 Definition 1.15][8], \(K_{X/S}\) denotes the canonical divisor on a fibered surface \(X \to S\)). The surface \(Y \to S\) is called the canonical model of \(X\). It is called singular as soon as there exists at least one contracted component.

Definition 3.1.11: Let \(C\) be a smooth projective curve over \(k\) of genus \(g \geq 2\). Let us suppose that \(C\) admits a minimal regular model \(C_{\text{min}}\) over \(S\) (which is the case if \(S\) is affine). We call the canonical model \(C_{\text{can}}\) of the minimal surface \(C_{\text{min}}\) the canonical model of \(C\) over \(S\).

Proposition 3.1.12: \(S\) is a Dedekind scheme of dimension 1. Let \(C\) be a smooth projective curve \(K = K(S)\) of genus \(g \geq 1\) (resp. \(g \geq 2\)) admitting a minimal regular model \(C_{\text{min}}\) (resp. a canonical model \(C_{\text{can}}\)) over \(S\). Then any automorphism of \(C\) extends in a unique way to an isomorphism \(\sigma\) of \(C_{\text{min}}\) (resp. of \(C_{\text{can}}\)).

Proof: For the minimal regular model, see Qingliu [Algebraic Geometry and Arithmetic Curves 9.3.2 Proposition 3.13][8]. Let \(\Gamma\) be a vertical prime divisor of \(C_{\text{min}}\) such that \(K_{C_{\text{min}}/S} \cdot \Gamma = 0\). Fact: \(f : X \to Y\) is a dominant morphism of regular fibered surfaces over \(S\). \(C\) (resp. \(D\)) is a divisor on \(X\) (resp. \(Y\)). We have the following property: The extension \(K(X)/K(Y)\) is finite. Let \(C\) be a vertical divisor on \(Y\), then \(f^*C\) is vertical and \(f^*C \cdot f^*D = [K(X) : K(Y)]C \cdot D\). We can see the proof from Qingliu [Algebraic Geometry and Arithmetic Curves, 9.2.2, Theorem 2.12(c)][8]. According to the Fact, \(\sigma(\Gamma)^2 = \Gamma^2\) which implies \(K_{C_{\text{min}}/S} \cdot \sigma(\Gamma) = 0\). See Qingliu [Algebraic Geometry and Arithmetic Curves 9.4.1 Proposition 4.8][8]. Thus \(\sigma\) acts on the set of divisors contracted by the morphism \(C_{\text{min}} \to C_{\text{can}}\). By the uniqueness of the contraction, \(\sigma\) induces an automorphism of \(C_{\text{can}}\). Then we get the desired result.
Definition 3.1.13: Let \( f : X \to Y \) be a morphism of Dedekind schemes, we say that \( f \) is \( \text{étale} \) at \( x \) if it is unramified and flat at \( x \). We call \( f \text{étale} \) if it is \( \text{étale} \) at every point of \( X \).

Proposition 3.1.14: Let \( C, S \) be as in Proposition 4.12. Let \( S' \) be a Dedekind scheme of dimension 1 that is \( \text{étale} \) over \( S \). Let \( K' = K(S') \). We suppose \( C \) is of genus \( g \geq 1 \) (resp. \( g \geq 2 \)) and admitting a minimal regular (resp. canonical) model \( C \) over \( S \). Then \( C \times_S S' \) is the minimal regular (resp. canonical) model of \( C_{K'} \) over \( S' \).

Proof: See Qingliu [Algebraic Geometry and Arithmetic Curves, 10.4.1, Proposition 1.17][8].

2. Reduction

\( S \) denotes a Dedekind scheme of dimension 1. \( K = K(S) \) in this section.

Definition 3.2.1: Let \( C \) be a normal projective curve over \( K \). We fix a closed point \( s \in S \). We say that the fiber \( C_s \) of a model \( C \) of \( C \) is a reduction of \( C \) at \( S \). If \( S \) is the spectrum of a Dedekind ring \( A \), and if \( p \) is maximal ideal of \( A \) corresponding to \( s \). We also call \( C_s \) a reduction of \( C \) modulo \( p \).

Definition 3.2.2: Let \( C \) be as above. We say that \( C \) has good reduction at \( s \in S \) if it admits a smooth model over \( \text{Spec} \, \mathcal{O}_{S,s} \), which implies that \( C \) is smooth over \( K \). If \( C \) does not have any good reduction at \( s \), we say that \( C \) has bad reduction at \( s \). \( C \) has good reduction over \( S \) if it has good reduction at every \( s \in S \).

Proposition 3.2.3: Let \( S \) be a Dedekind scheme of dimension 1, \( C \) is a smooth projective curve over \( K = K(S) \) of genus \( g \geq 1 \).

(a) The curve \( C \) has good reduction at \( s \in S \) except perhaps for a finite number of \( S \).
(b) We suppose that \( S \) is affine. Then \( C \) has good reduction over \( S \) if and only if the minimal regular model \( C_{\text{min}} \) of \( C \) over \( S \) is smooth. Furthermore, this implies that \( C_{\text{min}} \) is the unique smooth model of \( C \) over \( S \).
(c) (Étale base change) Let \( S' \to S \) be as in Proposition 3.1.14. Let \( s \in S' \) and Let \( s \) be its image in \( S \). Then \( C_{K'} \) has good reduction at \( s' \) if and only if \( C \) has good reduction at \( s \).

Proof: (a) The curve \( C \to \text{Spec} \, K \) extends to a Projective scheme \( C \to U \) over a non-empty open subscheme \( U \) of \( S \). This construction follows from Qingliu [Algebraic Geometry and Arithmetic Curves, 10.1.1, Example 1.4][8]. We can suppose that \( C \) is integral, and then flat over \( U \). In fact, we just need to take the irreducible component of \( C \) containing \( C \) and endow it with the reduced scheme structure. Now, we need to give a Lemma.

Lemma 3.2.4: Let \( \pi : X \to S \) be a fibered surface over a Dedekind scheme \( S \). We suppose that the generic fiber \( X_{\eta} \) is smooth. Then \( \exists \) a non-empty open subset \( V \) of \( S \) s.t. \( \pi^{-1}(V) \to V \) is smooth.

We can see the proof from Qingliu [Algebraic Geometry and Arithmetic Curves, 8.3.1, Proposition 3.11][8].
According to lemma 3.2.4, there \( \exists \) a non-empty open subscheme \( V \) of \( U \) s.t. \( C_V \rightarrow V \) is smooth. Thus \( C \) has good reduction over \( V \). More, \( S \) has dimension 1, \( S/V \) is finite which means that (a).

(b) First, supposing that \( C \) has good reduction over \( S \), Proposition 4.9 tells us that there exists the minimal regular model of \( C \) over \( S \) called \( C_{min} \). Let \( s \in S, \chi \rightarrow \text{Spec } O_{S,s} \) be a smooth model of \( C \). The \( C_{min} \times_S \text{Spec } O_{S,s} \simeq \chi \) which implies that \( C_{min} \) is smooth. Conversely, it is trivial. More, any smooth model of \( C \) over \( S \) is relatively minimal, and then isomorphic to \( C_{min} \).

(c) We also need a Lemma.

**Lemma 3.2.5:** \( X \rightarrow S \) is an arithmetic surface such that \( P_a(X_\eta) \geq 1 \), \( S' \rightarrow S \) is a morphism. We suppose that \( S' \rightarrow S \) is \( \text{\acute{e}tale} \) surjective. Then \( X \rightarrow S \) is minimal if and only if \( X \times_S S' \rightarrow S' \) is minimal.

This proof is in Qingliu [Algebraic Geometry and Arithmetic Curves, 9.3.3, Proposition 9.3.28][8].

According to lemma 3.2.5 and property (b), (c) is clear.

**Definition 3.2.6:** A morphism \( f : X \rightarrow Y \) is said to be **of finite type** if it is quasi-compact and if for every affine open subset \( V \) of \( Y \), and for every affine open subset \( U \) of \( f^{-1}(V) \), the canonical homomorphism \( O_Y(V) \rightarrow O_X(U) \) makes \( O_X(U) \) into a finitely generated \( O_Y(V) \)-algebra. A \( Y \)-scheme is said to be **of finite type** if the structural morphism is of finite type.

**Definition 3.2.7:** A morphism \( f : X \rightarrow Y \) is **quasi-projective** if it can be decomposed into an open immersion of finite type \( X \rightarrow Z \) and a projective morphism \( Z \rightarrow Y \). We say that \( X \) is a quasi-projective scheme over \( Y \).

**Proposition 3.2.8:** Let \( C \) be a smooth, connected, projective curve over \( K \) of arithmetic genus \( P_a(C) \geq 2 \). Let \( C \) be a model of \( C \) over \( S \). We suppose that \( C_S \) contains an irreducible component \( \Gamma \) whose normalization \( \Gamma' \) is smooth of arithmetic genus \( P_a(\Gamma') \geq P_a(C) \). Then \( C \) has good reduction at \( S \). Furthermore, the reduction at \( s \) of the smooth model of \( C \) over \( \text{Spec } O_{S,s} \) is isomorphic to \( \Gamma' \).

**Proof:** See Qingliu [Algebraic Geometry and Arithmetic Curves, 10.1.2, Lemma 1.24][8]

**Definition 3.2.9:** Let \( C \) be a smooth projective curve over \( K = K(S) \). We say \( C \) has **potential good reduction** at \( s \in S \) if there exists a morphism \( S' \rightarrow S \) from a Dedekind scheme \( S' \) to \( S \) and a point \( s' \in S' \) lying above \( s \) such that \( C_{K(S')} \) has good reduction at \( S' \). A simple observation, if \( C \) has good reduction at \( S \), then it has potential good reduction at \( s \).

**Proposition 3.2.10:** Let \( C \) be a smooth, geometrically connected, projective curve over \( K = K(S) \). Let us fix \( s \in S \). Let \( W^0 \) be a quasi-projective scheme over \( S \) such that \( W^0 \) is an open subscheme of \( C \), that \( W^0_s \) is integral and \( K(W^0_s) \) is the function field of a normal curve \( \Gamma \) of arithmetic genus \( 1 \leq P_a(\Gamma) < P_a(\Gamma') \). Then \( C \) has bad reduction. If
Γ × Spec k(s) Spec k(s) is integral and non-rational, then C does not have potential good
reduction.

Proof: See Qingliu [Algebraic Geometry and Arithmetic Curves, 10.1.2, Proposition 1.29][8].

3. Reduction map

**Definition 3.3.1:** A is a Noetherian local ring. We say that A is **Henselian** if every
finite A-algebra is a direct sum of local A-algebras.

**Definition 3.3.2:** A morphism of schemes f : X → Y is **proper** if it is of finite type,
separated and universally closed. Then we say that a Y-scheme is proper if the structural
morphism is proper.

**Definition 3.3.3:** Let S be the spectrum of a Henselian discrete valuation ring O_K.
Let χ be a proper scheme over S with generic fiber X. Let X^0 denote the set of closed
points of X. The map r : X^0 → χ_s, which to every closed point x ∈ X associates the
point {x} ∩ χ_s is called the **reduction map** of X. We say that x reduces or specializes
to r(x). For fixed X, the map r depends on the choice of χ. We denote this map by r_χ.

**Proposition 3.3.4:** Let S be a Dedekind scheme of dimension 1. χ → S a dominant
morphism of finite type with χ irreducible. Let ˜x := r_χ(x) ∈ χ_s be a closed point of a
closed fiber. Then there exists a closed point x of χη such that ˜x ∈ {x}.

**Proof:** See Qingliu [Algebraic Geometry and Arithmetic Curves, 10.1.3, Proposition
1.36][8].

**Definition 3.3.5** Let S be the spectrum of a complete discrete valuation ring O_K.
Let χ → S be a surjective proper morphism, and ˜(x) ∈ χ_s a closed point. Let L be a finite
extension of K. Then the following properties are true.
(a) There exist canonical bijections

\[ X_+(\tilde{x})(L) ≃ Mor_S(Spec O_L, Spec O_{\chi, \tilde{x}}) ≃ Mor_S(Spec O_L, Spec \hat{O}_{\chi, \tilde{x}}). \]  

(b) Let f_1, f_2, ..., f_q ∈ O_{\chi, \tilde{x}} be such that their images in O_{\chi, \tilde{x}} generate its maximal
ideal. Let us suppose that \( \tilde{x} \) is rational over \( k(s) \). Then the homomorphism
\[
\varphi : O_K[[T_1, T_2, \ldots, T_q]] \to \hat{O}_{\chi, \tilde{x}}, \ T_i \to f_i
\]
is surjective and induces a bijection
\[
X_+(\tilde{x})(L) \simeq \{(t_1, t_2, \ldots, t_q) \in m_L^q | F(t_1, t_2, \ldots, t_q) = 0, \exists F \in \text{Ker} \varphi \}
\]
where \( m_L \) is the maximal ideal of \( O_L \), and \( m_L^q = m_L \times \ldots \times m_L(q \text{ times}) \). In particular, if \( \chi \to S \) is smooth at \( \tilde{x} \) and if \( q = \text{dim}_k \chi_s \), then \( \varphi \) induces a bijection
\[
X_+(\tilde{x})(L) \simeq \mathbb{A}^n_{O_L}(L) = L^n.
\]

\( X \) and Semi-stable Reduction

**Definition 3.4.1:** Let \( C \) be an algebraic curve over an algebraically closed field \( K \). \( C \) is called semi-stable if it is reduced, and if its singular points are ordinary double points (See Qingliu [Algebraic Geometry and Arithmetic Curves, 7.5, Definition 5.13][8]).

\( C \) is stable if the following conditions are verified: (1) \( C \) is connected and projective, of arithmetic genus \( P_a(C) \geq 2 \). (2) Let \( \Gamma \) be an irreducible component of \( C \) that is isomorphic to \( \mathbb{P}^1_k \). Then \( \Gamma \) intersects the other irreducible components at at least three points.

We call a curve \( C \) over a field \( K \) semi-stable (resp. stable) if its extension \( C_K \) to the algebraic closure \( K \) of \( K \) is semi-stable (resp. stable) curve over \( K \).

**Definition 3.4.2:** Let \( f : X \to S \) be a morphism of finite type to a scheme \( S \). We call \( f \) semi-stable, or we say that \( X \) is a semi-stable curve over \( S \), if \( f \) is flat and if for any \( s \in S \), the fiber \( X_s \) is a semi-stable curve over \( k(s) \). We say that \( f \) is stable of genus \( g \geq 2 \) or that \( X \) is a stable curve over \( S \) of genus \( g \geq 2 \) if \( f \) is proper, flat, with stable fibers of arithmetic genus \( g \).

**Definition 3.4.3:** Let \( S \) be a Dedekind scheme of dimension 1. Let \( C \) be a smooth projective curve over \( k(s) \). We say that \( C \) has semi-stable reduction (resp. stable reduction) at \( s \in S \) if there exists a model \( \mathcal{C} \) of \( C \) over \( \text{Spec} \ O_{S,s} \) that is semi-stable (resp. stable) over \( \text{Spec} \ O_{S,s} \). The special fiber \( \mathcal{C}_S \) of a stable model over \( \text{Spec} \ O_{S,s} \) is called the stable reduction of \( C \) at \( s \). We say that \( C \) has semi-stable (resp. stable) reduction over \( S \) if the property is true for every \( s \in S \). A model \( \mathcal{C} \) of \( C \) over \( S \) is called a stable...
Theorem 3.4.4: Let $S$ be an affine Dedekind scheme of dimension 1. Let $C$ be a smooth projective curve over $K = K(S)$, of genus $g \geq 1$. Let us suppose that $C$ has semi-stable reduction over $S$.

(a) The minimal regular model $C_{\text{min}}$ of $C$ over $S$ is semi-stable over $S$.
(b) Let us suppose $g \geq 2$ and that $C$ is geometrically connected over $K$. Then the canonical model $C_{\text{can}}$ of $C$ over $S$ is a stable curve over $S$, and it is the unique stable model of $C$ over $S$.
(c) The curve $C$ admits a semi-stable model over $S$. If $C$ has stable reduction over $S$, then it admits a stable model over $S$.

Proof: See Qingliu [Algebraic Geometry and Arithmetic Curves, 10.3, Theorem 3.34][8].

Corollary 3.4.5: Let $S$ be a Dedekind scheme of dimension 1, $C$ a smooth projective curve over $K(S)$ with $p_a(C) \geq 1$, and $S'$ a Dedekind scheme of dimension 1 that dominates $S$. Let $K' = K(S')$.

(a) If $C$ has semi-stable(resp.stable) reduction over $S$, then $C_{K'}$ has semi-stable(resp.stable) reduction over $S'$. If $C$ has semi-stable(resp.stable) model of $C$ over $S$, then $C \times_S S'$ is a semi-stable(resp.stable) model of $C_{K'}$ over $S'$.
(b) Let $C_s$ be the stable reduction(of which we suppose the existence) of $C$ at a point $s \in S$. Let $s' \in S'$ lie over $s$. Then the stable reduction of $C_{K'}$ at $s'$ is isomorphic to $C_s \times_{\text{Spec } k(s)} \text{Spec } k(s')$.
(c) Let us moreover suppose either that $S' \to S$ is étale surjective, or that $S = \text{Spec } O_K$ is local and $S' = \text{Spec } \hat{O}_K$ or $S' = \text{Spec } O_L$ where $L$ is separable over $K$, $e_{O_L/O_K} = 1$ and $O_L$’s residue field is separable algebraic over $O_K$’s residue field. If $C_{K'}$ has semi-stable (resp.stable) reduction over $S'$, then $C$ has semi-stable(resp.stable) reduction over $S$.

Proof: See Qingliu [Algebraic Geometry and Arithmetic Curves, 10.3, Corollary 3.36][8].

5. Graphs

Let $X \to S$ be an arithmetic surface. It is convenient to represent the singular closed fiber $X_s$ of $X$ by its dual graph.

Definition 3.5.1: A finite graph $G$ consists of a finite set $V$ whose elements are called vertices, and for every pair of vertices $v_1, v_2$, a finite (possibly empty) set whose elements are called the edges. We say they $v_1, v_2$ are the end vertices of these edges. A subgraph of $G$ consists of a subset $V'$ of the vertices of $G$ and of a subset of edges with end vertices belonging to $V'$. A textbf{path} between vertices $v_1, v_n$ is a set of vertices $v_1, v_2, \ldots, v_n$ that are pairwise distinct, with $v_i, v_{i+1}$ joined by an edge $e_i$. The integer $n - 1$ is called the length of the path. We say that $G$ is connected if two arbitrary vertices of $G$ are joined by a path. A connected component is a maximal connected subgraph. A vertex is called a node if it is an end vertex of at least three edges.
Definition 3.5.2: Let $G$ be a graph. A **circuit** in $G$ is a path $v_1, v_2, \ldots, v_n$ to which we add an edge $e$ joining the vertices $v_1, v_n$, the edge $e$ being different from those already appearing in the path. A connected graph is a **tree** if it does not contain any circuit.

Definition 3.5.3: Let $C > 0$ be a vertical divisor contained in a closed fiber $X_s$ of a regular fibered surface $X \to S$. Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ be its irreducible components. We associate a graph $G$ to $C$ in the following manner. The vertices of $G$ are the irreducible components $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$, and there are $\Gamma_i \cdot \Gamma_j$ edges between $\Gamma_i$ and $\Gamma_j$ if $i \neq j$. This graph is called the **dual graph** of $C$.

Proposition 3.5.4: Let $C$ be a divisor as in Definition 8.3 and such that $C \leq X_s$. Let $G$ be the dual graph of $C$. Then the following properties are true.

(a) The curve $C$ is connected if and only if $G$ is connected.

(b) Let us suppose $C$ is reduced with irreducible components $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$. Then $\beta(\Gamma) = p_a(C) - \sum_{1 \leq i \leq n} p_a(\Gamma_i)$ where $\beta(G)$ denotes the first Betti number of $G$.

(c) Let us suppose $k$ is algebraically closed and $C$ is connected. Let $t, u$ be the toric and unipotent ranks of $C$. Then we have $\beta(G) \leq t + u$.

Proposition 3.5.5: Let $R > 0$ be a reduced, connected, vertical divisor on $X$. We suppose that the irreducible components $\Gamma$ of $R$ all verify $K_{X/S} \cdot \Gamma = 0$, and that $R$ does not contain all of irreducible components of $X_s$. Let $G$ be the dual graph of $R$. Then $G$ has only finite types of the forms.

Proof: Qingliu [Algebraic Geometry and Arithmetic Curves][8].

4 Background knowledge of metric graphs and some notations

4.1 Metric graph

Definition 4.1.1: A **metric graph** $\Gamma$ is a finite connected graph equipped with a distinguished parametrization of each of its edges i.e. for each $p \in \Gamma$, $\exists r_p > 0$ and an integer $n_p \geq 1$ s.t. $p$ has a neighborhood $V_p(r_p)$ isometric to the star-shaped set

$$S(n_p, r_p) = \{ z \in \mathbb{C} | z = te^{2\pi i/n_p} \text{ for some } 0 \leq t \leq r_p \text{ and some } k \in \mathbb{Z} \} \quad (43)$$

equipped with path metric.

A **finite weighted graph** $G$ is a connected, weighted combinatorial graph equipped with a collection $V = \{ v_1, \ldots, v_n \}$ of vertices, $E = \{ e_1, e_2, \ldots, e_m \}$, and $W = \{ w_{ij} \}$ of nonnegative weights ($1 \leq i, j \leq n$) satisfying:

(1) If $w_{ij} = 0$ then there is no edge connecting $v_i$ and $v_j$. 

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(2) If $w_{ij} > 0$ then there is a unique edge $e_k$ connecting $v_i$ and $v_j$.
(3) $w_{ij} = w_{ji}$ for all $i, j$.
(4) $w_{ii} = 0$ for all $i$.

There is a natural partial ordering on the collection of finite weighted graphs, where $G' \leq G$ if we can refine $G$ to $G'$ by a sequence of length-preserving subdivisions. We write $G \sim G'$ if there exists a finite weighted graph $G''$ with $G'' \sim G$ and $G'' \sim G'$.

Notation: There is a bijective correspondence between metric graphs and equivalence classes of finite weighted graphs.

For any given $p \in \Gamma$, the number of directions emanating from $p$ will be called the \textbf{valence} of $p$ and will be denoted by $v(p)$. By compactness, $\Gamma$ can be covered by a finite number of neighborhoods $V_p(r_p)$. Thus there are only finitely many points $p \in \Gamma$ for which $n_p \neq 2$.

For a given metric graph $\Gamma$, we will denote its set of vertices by $V(\Gamma)$. We need to require that $V(\Gamma)$ is non-empty and $p \in V(\Gamma)$ for each $p \in \Gamma$ with $v(p) \neq 2$.

For a given metric graph $\Gamma$ with vertex set $V(\Gamma)$, the set of edges of $\Gamma$ is the set of closed line segments with end points in $V(\Gamma)$. We denote the set of edges of $\Gamma$ by $E(\Gamma)$.

If $e_i$ is an edge, by $\Gamma - e_i$ we mean the graph obtained by deleting the interior of $e_i$. Let $V := \#(V(\Gamma))$, $e := \#(E(\Gamma))$, $g(\Gamma) := e - v + 1$. We simply use $g$ to show $g(\Gamma)$ when there is no confusion.

We use $L_i$ to denote the length of an edge $e_i \in E(\Gamma)$. Total length of $\Gamma$, which will be denoted by $l(\Gamma)$, is given by $l(\Gamma) = \sum_{i=1}^{e} L_i$.

\textbf{Definition 4.1.2: Vertex connectivity} is the minimum number of vertices whose deletion disconnects $\Gamma$, which is denoted by $\kappa(\Gamma)$. \textbf{Edge connectivity} is similarly defined which is defined by $\Lambda(\Gamma)$. The minimum of valences of the vertices denoted by $\overline{\delta}(\Gamma) := min\{v(p) | p \in v(\Gamma)\}$. We have the inequality $\kappa(\Gamma) \leq \Lambda(\Gamma) \leq \overline{\delta}(\Gamma)$.

We call a metric graph $\Gamma$ \textbf{irreducible}, if it can be disconnected by deleting any simple point. We say that $\Gamma$ is a \textbf{bridgeless metric graph}, if the edge connectivity of a metric graph $\Gamma$ is at least two. From definition, every irreducible graph is bridgeless, but there can be bridgeless graphs which are not irreducible. For example, union of two copies of the circle graph along a vertex is a bridgeless metric graph but not an irreducible metric graph.

\textbf{Definition 4.1.3:} In Baker and Rumely’s paper [Harmonic analysis on metric graphs], the following measure valued Laplacian on a given metric graph is defined.

$$\Delta_x(f(x)) = -f(x)''dx - \sum_{p \in V(\Gamma)} \left[ \sum_{\vec{v} \at \ p} d\delta(p)\right] \delta_p(x)$$

for a continuous function $f : \Gamma \to \mathbb{C}$ such that $f$ is $C^2$ on $\Gamma \setminus V(\Gamma)$ and $f'' \in L^1(\Gamma)$.

\textbf{Definition 4.1.4:} In Chinburg and Rumely’s paper [The capacity pairing][2], there is a detailed study of harmonic kernel function $j_z(x, y)$ having the following properties:

(A) It is jointly continuous as a function of three variables.
(B) It is nonnegative with $j_z(z, y) = j_z(x, z) = 0$ for all $x, y, z \in \Gamma$.

(C) It is symmetric in $x$ and $y$ i.e. for fixed $z$, $j_z(x, y) = j_z(y, x)$.

(D) For fixed $z$ and $y$, the function $j_{z, y}(x) = j_z(x, y)$ is in $CPA(\Gamma)$ and satisfies the Laplacian equation $\Delta_x(j_z(x, y)) = \delta_y(x) - \delta_z(x)$ where $CPA(\Gamma)$ means the space of continuous, piecewise affine complex valued functions on $\Gamma$.

For fixed $z$ and $y$, $j_z(x, y)$ has the following physical interpretation, when $\Gamma$ is viewed as a resistive electric circuit with terminals at $z$ and $y$ with the resistance in each edge given by its length, then $j_z(x, y)$ is the voltage difference between $x$ and $z$ when unit current enters at $y$ and exits at $z$ (with reference voltage $0$ at $z$). The effective resistance between two points $x$, $y$ of a metric graph $\Gamma$ is given by $r(x, y) = j_y(x, x)$.

For fixed $z$ and $y$, $j_z(x, y)$ is the voltage function in $\Gamma - e_i$ of a graph $\Gamma$ when the interior of the edge $e_i$ is deleted from $\Gamma$.

We will denote by $R_{ai, p}$ or $R_{bi, p}$ if there is no confusion, the resistance between the points of an edge $e_i$ of a graph $\Gamma$ when the interior of the edge $e_i$ is deleted from $\Gamma$.

Let $\Gamma$ be a metric graph with $p \in V(\Gamma)$ and let $e_i \in E(\Gamma)$ having end points $p_i$ and $q_i$. If $\Gamma - e_i$ is connected, then $\Gamma$ can be transformed to be the following graph:

$$
\begin{array}{c}
\vdots \\
p_i \xrightarrow{e_i} q_i \\
R_{ai, p} \\
p \
\end{array}
$$

Notation: $R_{ai, p} = \hat{j}_{p_i}(p, q_i)$, $R_{bi, p} = \hat{j}_{q_i}(p, p_i)$, $R_{ci, p} = \hat{j}_p(p_i, q_i)$ where $\hat{j}_x(y, z)$ is the voltage function in $\Gamma - e_i$. We have $R_{ai, p} + R_{bi, p} = R_i$ for each $p \in \Gamma$.

Details can be found in the article Chinkir [The tau constant of a metric graphs][4].

Now, we state two reductions, and three transformations in the circuit theory.

**Series Reduction:** Let $\Gamma$ be a graph with vertex set $\{p, q, s\}$. Suppose that $p$ and $s$ are connected by an edge of length $A$, and that $s$ and $q$ are connected by an edge of length $B$. Let $\beta$ be a graph with vertex set $\{p, q\}$, where $p$ and $q$ are connected by an edge of length $A + B$. Then the effective resistance in $\Gamma$ between $p$ and $q$ is equal to the effective resistance in $\beta$ between $p$ and $q$.

**Parallel Reduction:** Suppose $\Gamma$ and $\beta$ be two graphs with vertex set $\{p, q\}$. Suppose $p$ and $q$ in $\Gamma$ are connected by two edges of lengths $A$ and $B$, respectively, and let $p$ and $q$ in $\beta$ be connected by an edge of length $\frac{AB}{A+B}$. Then the effective resistance in $\Gamma$ between $p$ and $q$ is equal to the effective resistance in $\beta$ between $p$ and $q$.

Three transformations are **Delta-Wye transformation**, **Wye-Delta transformation** and **Star-Mesh transformation**.

In the article Chinkir [The tau constant of a metric graphs][4], it was shown that the theory of harmonic functions on metric graphs is equivalent to the theory of resistance electric circuits with terminals.

Indeed, we have the following dictionary.
Connected metric graph \( \Gamma \) 

Electric circuit \( \Gamma \) 

<table>
<thead>
<tr>
<th>Length of edge</th>
<th>Resistance</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) \in \text{CPA}(\Gamma) )</td>
<td>Voltage at ( x )</td>
</tr>
</tbody>
</table>

Negative of slope of \( f(x) \) 
Current at \( x \)

Point where \( \Delta_p(f) > 0 \) 
Terminal; Current sink

Point where \( \Delta_p(f) < 0 \) 
Terminal; Current source

\[
\sum_p \Delta_p(f) = 0 \quad \text{Conservation of current}
\]

If \( \Gamma - e_i \) is not connected, we set \( R_{b_{i,p}} = R_i = \infty \) and \( R_{a_{i,p}} = 0 \) if \( p \) belongs to the component of \( \Gamma - e_i \) containing \( P_i \) and we set \( R_{a_{i,p}} = R_i = \infty \) and \( R_{b_{i,p}} = 0 \) if \( p \) belongs to the component of \( \Gamma - e_i \) containing \( q_i \).

For any real-valued, signed Borel measure \( \mu \) on \( \Gamma \) with \( \mu(\Gamma) = 1 \) and \( |\mu|(\Gamma) < \infty \), define the function \( j_{\mu}(x,y) = \int_\Gamma j_z(x,y) d\mu(z) \). It is clear that \( j_{\mu}(x,y) \) is symmetric, and is jointly continuous in \( x \) and \( y \). In the paper Chinburg and Rumely, [The capacity pairing], it is discovered that there is a unique real-valued, signed Borel measure \( \mu = \mu_{\text{can}} \) such that \( j_{\mu}(x,x) \) is constant on \( \Gamma \).

Let \( \mu \) be a real-valued signed Borel measure of total mass 1 on \( \Gamma \). In the article Bollabás [Harmonic analysis on metric graphs], the function \( g_{\mu}(x,y) \) associated to \( \mu \) is defined to be

\[
g_{\mu}(x,y) = \int_\Gamma j_z(x,y) d\mu(z) - \int_\Gamma j_z(x,y) d\mu(z) d\mu(x) d\mu(y)
\]

where the latter integral is a constant that depends on \( \Gamma \) and \( \mu \).

As shown in the article Baker[Harmonic analysis on metric graphs][1], one can characterize \( g_{\mu}(x,y) \) as the unique function on \( \Gamma \times \Gamma \) such that

(1) \( g_{\mu}(x,y) \) is jointly continuous in \( x, y \) and belongs to \( BDV_{\mu}(\Gamma) \) as a function of \( x \) for each fixed \( y \) where \( BDV_{\mu}(\Gamma) := \{ f \in BDV(\Gamma), \int_{\Gamma} f d\mu = 0 \} \). The meaning of \( BDV(\Gamma) \) is defined in the article Bollabás [Harmonic analysis on metric graphs].

(2) For fixed \( y \), \( g_{\mu} \) satisfies the identity \( \Delta_x g_{\mu}(x,y) = \delta_y(x) - \mu(x) \).

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∫∫ \Gamma \times \Gamma g_\mu(x,y)d\mu(x)d\mu(y) = 0.

**Theorem 4.1.5:** (1) The probability measure \( \mu_{can} = \Delta_x(\frac{1}{2}r(x,y)) + \delta_y(x) \) is dependent of \( y \in \Gamma \).

(2) \( \mu_{can} \) is the unique measure \( \mu \) of the total mass 1 on \( \Gamma \) for which \( g_\mu(x,x) \) is a constant independent of \( x \).

(3) There is a constant \( \tau(\Gamma) \in \mathbb{R} \) such that \( g_{\mu_{can}}(x,y) = -\frac{1}{2}r(x,y) + \tau(\Gamma) \). We call \( \mu_{can} \) the canonical measure on \( \Gamma \), and we call \( \tau(\Gamma) \) the tau constant of \( \Gamma \). In Chinburg and Rumely, [The capacity pairing], the following explicit formula is given:

\[
\mu_{can}(x) = \sum_{p \in V(\Gamma)} (1 - \frac{1}{2}v(p))\delta_{p}(x) + \sum_{e_i \in E(\Gamma)} \frac{dx}{L_i + R_i}. \tag{46}
\]

The following two lemmas express \( \tau(\Gamma) \) in terms of the resistance functions and the canonical measure.

**Lemma 4.1.6:** For any metric graph \( \Gamma \) and its resistance function \( r(x,y) \), and for each \( x \in \Gamma, \tau(\Gamma) = \frac{1}{2} = \int_{\Gamma} r(x,y)d\mu_{can}(y) \).

**Lemma 4.1.7:** For any fixed \( y \in \Gamma \), we have \( \tau(\Gamma) = \frac{1}{4} \int_{\Gamma} (\frac{dr(x,y)}{dx})^2 dx \).

Lemma 4.1.7 implies that \( \tau(\Gamma) \geq 0 \) for any metric graph \( \Gamma \). We use circuit reductions to obtain the following equalities:

\[
r(p_i,p) = \frac{(L_i + R_{b_i,p})R_{a_i,p}}{L_i + R_i} + R_{c_i,p} \tag{47}
\]

\[
r(q_i,p) = \frac{(L_i + R_{b_i,p})R_{b_i,p}}{L_i + R_i} + R_{c_i,p} \tag{48}
\]

Therefore,

\[
r(p_i,p) - r(q_i,p) = \frac{(R_{a_i,p} - R_{b_i,p})L_i}{L_i + R_i} \tag{49}
\]

\[
r(p_i,p) + r(q_i,p) = \frac{L_i R_i}{L_i + R_i} + 2\frac{R_{a_i,p}R_{b_i,p}}{L_i + R_i} + 2R_{c_i,p}. \tag{50}
\]

**Definition 4.1.8:** Let \( \Gamma \) be a metric graph and let \( q : \Gamma \rightarrow \mathbb{N} \) be a function on the set of vertices of \( \Gamma \). The canonical divisor \( K \) of \((\Gamma, q)\) is defined to be the following divisor on \( \Gamma \).

\[
K = \sum_{p \in V(\Gamma)}(v(p) - 2 + 2q(p))p \quad \text{and} \quad \delta_K(x) = \sum_{p \in V(\Gamma)}(v(p) - 2 + 2q(p))\delta_p(x).
\]

The pair \((\Gamma, q)\) will be called a weighted metric graph (wm-graph) if \( q \) is nonnegative and \( K \) is an effective divisor. The genus \( \overline{g}(\Gamma) \) of a wm-graph \((\Gamma, q)\) is defined to be \( \overline{g}(\Gamma) = 1 + \frac{1}{2}\deg K = g(\Gamma) + \sum_{p \in V(\Gamma)} q(p) \).

Note that \( \overline{g} \geq 1 \) for a wm-graph. We call a wm-graph \((\Gamma, q)\) irreducible if the underlying metric graph \( \Gamma \) is irreducible.
Let $\mu_{ad}(x)$ be the admissible metric associated to $K$ (as defined in the article Zhang[Admissible pairing on a curve, Lemma 3.7]). Then

$$\mu_{ad}(x) = \frac{1}{g} \left( \sum_{p \in V(\Gamma)} q(p) \delta_p(x) + \sum_{e_i \in E(\Gamma)} \frac{dx}{L_i + R_i} \right).$$

(51)

Then we have $\mu_{ad}(x) = \frac{1}{2g}(2\mu_{can}(x) + \delta_K(x))$. Moreover $\delta_K(\Gamma) = \text{deg}(K) = 2g - 2$ and $\mu_{can}(\Gamma) = 1 = \mu_{ad}(\Gamma)$.

Set $\theta(\Gamma) := \sum_{p,q \in V(\Gamma)} (v(p) - 2 + 2q(p))(v(q) - 2 + 2q(q))r(p,q)$, and define

$$\epsilon(\Gamma) = \int \int \Gamma \times \Gamma r(x,y)\delta_K(x)\mu_{ad}(x)$$

(52)

$$a(\Gamma) = \frac{1}{2} \int \int \Gamma \times \Gamma r(x,y)\mu_{ad}(x)\mu_{ad}(y)$$

(53)

$$\varphi(\Gamma) = 3g \cdot a(\Gamma) - \frac{1}{4}(\epsilon(\Gamma) + l(\Gamma))$$

(54)

$$\lambda(\Gamma) = \frac{7}{6}(2g + 1)\varphi(\Gamma) + \frac{1}{12}(\epsilon(\Gamma) + l(\Gamma)).$$

(55)

We have $\theta(\Gamma) \geq 0$ for any wm-graph $\Gamma$ since the corresponding divisor $K$ is effective.

Now, we give some easy relationships between these notations.

**Proposition 4.1.9:** Let $\Gamma$ be wm-graph. Then we have

$$\epsilon(\Gamma) = \frac{(4g - 4)\tau(\Gamma)}{g} + \frac{\theta(\Gamma)}{2g}.\quad (56)$$

**Proof:** By definition, we have

$$\epsilon(\Gamma) = \int \int \Gamma \times \Gamma r(x,y)\delta_K(x)\mu_{ad}(x)$$

(57)

$$= \sum_{p \in V(\Gamma)} (v(p) - 2 + 2q(p)) \int \Gamma r(p,y)\mu_{ad}(y)$$

(58)

$$= \sum_{p \in V(\Gamma)} (v(p) - 2 + 2q(p)) \int \Gamma r(p,y)(\frac{1}{2g}(2\mu_{can}(y) + \delta_K(y))).$$

(59)

Then we get the desired result from Lemma 9.6 and the fact $\text{deg}(K) = 2g - 2$.

**Proposition 4.1.10:** Let $\Gamma$ be wm-graph. Then we have

$$a(\Gamma) = \frac{(2g - 1)\tau(\Gamma)}{g^2} + \frac{\theta(\Gamma)}{8g^2}.\quad (60)$$
Proof: By definition, we have

\[ 2a(\Gamma) = \int \int_{\Gamma \times \Gamma} r(x, y) \mu_{ad}(x) \mu_{ad}(y) \]

\[ = \int_{\Gamma} \left( \int_{\Gamma} r(x, y) \frac{1}{2g} (2\mu_{can}(x) + \delta_K(x)) \mu_{ad}(y) \right) \mu_{ad}(x) \]

\[ = \int_{\Gamma} \left( \frac{2\tau(\Gamma)}{g} + \frac{1}{2g} \sum_{p \in V(\Gamma)} (v(p) - 2 + 2q(p)r(p, y)) \right) \mu_{ad}(y) \mu_{ad}(x) \]

\[ = \frac{2\tau(\Gamma)}{g} + \frac{1}{2g} \sum_{p \in V(\Gamma)} (v(p) - 2 + 2q(p)) \int_{\Gamma} r(p, y) \mu_{ad}(y). \]

Since \( deg(K) = 2g - 2 \), then we get the desired result.

**Proposition 4.1.11:** Let \( \Gamma \) be wm-graph. Then we have

\[ \varphi(\Gamma) = \frac{(5g - 2)\tau(\Gamma)}{g} + \frac{\theta(\Gamma)}{4g} - \frac{l(\Gamma)}{4}. \] (61)

Proof: By definition, we have

\[ \varphi(\Gamma) = 3g \cdot a(\Gamma) - \frac{1}{4} (\epsilon(\Gamma) + l(\Gamma)) \]

\[ = 3g \left( \frac{(2g - 1)\tau(\Gamma)}{g^2} - \frac{1}{4} \left( \frac{4g - 4}{g} \right) \tau(\Gamma) \right) + \frac{\theta(\Gamma)}{2g} + l(\Gamma). \] (62)

Now we get the result.

**Proposition 4.1.12:** Let \( \Gamma \) be wm-graph. Then we have

\[ \lambda(\Gamma) = \frac{(3g - 3)\tau(\Gamma)}{4g + 2} + \frac{\theta(\Gamma)}{16g + 8} + \frac{(g + 1)(l(\Gamma))}{16g + 8}. \] (64)

Proof: It is clear from the above three propositions.

**Definition 4.1.13:** We will call a wm-graph \( (\Gamma, q) \) a **simple** wm-graph if \( q \equiv 0 \). We will denote a simple wm-graph \( (\Gamma, o) \) simply by \( \Gamma \). Note that \( g = \overline{g} \) (i.e., \( \overline{g}(\Gamma) = g(\Gamma) \)) when \( q(p) = 0 \) for each \( p \in V(\Gamma) \).

**Definition 4.1.14:** We call \( \Gamma \) a **bouquet graph** if \( \Gamma \) is a simple wm-graph with \( V(\Gamma) = \{p\} \) and \#(E(\Gamma)) = \( e \geq 2 \).

**Definition 4.1.15:** We call \( \Gamma \) a **banana graph** if \( \Gamma \) is a simple wm-graph with \( V(\Gamma) = \{p, q\} \) and \#(E(\Gamma)) = \( e \geq 2 \).

**Definition 4.1.16:** For any given simple bridgeless wm-graph \( \Gamma \) with genus \( g \geq 2 \) and a vertex set \( V(\Gamma) \), we can always find a non-empty vertex subset \( V(\Gamma)' := \{p \in \)
$V(\Gamma)|v(\rho) \geq 3}$ by removing vertices of valence 2 if there is any. By valence property of $\varphi(\Gamma)$ (See Cinkir [The tau constant of a metric graph, Remark2.10]), this does not change $\varphi(\Gamma)$. We call $V(\Gamma)'$ be the minimal vertex set of $\Gamma$. Note that $g \geq \frac{v}{2} + 1$ where $\#(V(\Gamma)) = v$.

Now, we give an important theorem.

**Theorem 4.1.17:** Let $\Gamma$ be an irreducible wm-graph of genus $g \geq 2$. Then we have $\varphi \geq t(\overline{g})l(\Gamma)$, where $t(2) = \frac{1}{27}$, $t(3) = \frac{1}{30}$, and $t(\overline{g}) = \frac{(g-1)^2}{2g(7g+3)}$ for $g \geq 4$.

**Proof:** First, we claim that it will be enough to prove the desired lower bound inequalities for irreducible simple wm-graph. Thus we give a lemma.

**Lemma 4.1.18:** Let $(\Gamma, q)$ be a given wm-graph with genus $g$. For any given $\varepsilon > 0$, there exist a wm-graph $(\Gamma_0, 0)$ of genus $g$ such that $\varphi(\Gamma_0) \leq \varphi(\Gamma) + \varepsilon$, $\epsilon(\Gamma_0) \leq \epsilon(\Gamma) + \varepsilon$, $a(\Gamma_0) \leq a(\Gamma) + \varepsilon$, $\lambda(\Gamma_0) \leq \lambda(\Gamma) + \varepsilon$.

**Proof:** Suppose that $(\Gamma, q)$ is a wm-graph of genus $g \geq 2$. If there is a vertex $p \in V(\Gamma)$ with $q(p) > 0$, we attach $q(p)$ circles of length $\varepsilon > 0$ to $\Gamma$ at the vertex $p$. By repeating the process for each such vertex, we obtain a new metric graph, which we denote by $\Gamma_0$. By choosing $q_0 = 0$ as the polarization on $\Gamma_0$, we have a wm-graph $(\Gamma_0, 0)$. Note that $V(\Gamma_0) = V(\Gamma)$, $l(\Gamma_0) = l(\Gamma) + \varepsilon \sum_{p \in V(\Gamma)} q(p)$. Since $g(\Gamma_0) = g(\Gamma) + \sum_{p \in V(\Gamma)} q(p)$, $\overline{g}(\Gamma_0) = \overline{g}(\Gamma)$. Moreover, $v_{\Gamma_0}(p) = v_{\Gamma}(p) + 2q(p)$, for each $p \in V(\Gamma)$ and $r_{\Gamma_0}(p, q) = r_{\Gamma}(p, q)$ for each $p$ and $q$ in $V(\Gamma)$, where $v_{\Gamma}(p)$ is valence of $p$ in $\Gamma$, $r_{\Gamma}(x, y)$ is the resistance function on $\Gamma$. These equations imply $\theta(\Gamma_0) = \theta(\Gamma)$. By applying the additive property of the tau constant and the fact that $\tau(\beta) = \frac{l(\beta)}{12}$ for a circle graph $\beta$ (See Cinkir [The tau constant of Metric Graphs]), we have $\tau(\Gamma_0) = \tau(\Gamma) + \frac{\varepsilon}{12} \sum_{p \in V(\Gamma)} q(p)$. After some simple calculations, we have the following equations. $\varphi(\Gamma_0) = \varphi(\Gamma) + \varepsilon Q\frac{g-1}{6g}$, $\epsilon(\Gamma_0) = \epsilon(\Gamma) + \varepsilon Q\frac{g-1}{6g}$, $a(\Gamma_0) = a(\Gamma) + \varepsilon Q\frac{g-1}{12g}$, $\lambda(\Gamma_0) = \lambda(\Gamma) + \varepsilon Q\frac{g}{8g+4}$. Then we obtain the desired result by choosing an appropriate $\varepsilon$ as we construct $\Gamma_0$.

Since $\varepsilon$ in Lemma 4.1.17 can be taken arbitrarily small for a given wm-graph $\Gamma$, it will be enough to consider wm-graphs with polarization $q \equiv 0$ in order to give lower bounds for $\varphi(\Gamma)$, $\epsilon(\Gamma)$, $a(\Gamma)$, and $\lambda(\Gamma)$.

Furthermore, since every irreducible wm-graph is bridgeless, proving these lower bounds for bridgeless simple wm-graphs will implies that these lower bounds hold for irreducible wm-graphs.

Let $\Gamma$ be a bridgeless simple wm-graph. Then we can work with a minimal vertex set $V(\Gamma)$ by Definition 4.1.16. Let $\#(V(\Gamma)) = v$.

If $v = 1$, then $\Gamma$ is a bouquet graph. We claim that $\varphi(\Gamma) = \frac{g-1}{6g} l(\Gamma)$. To prove the claim, we recall that $\tau(\Gamma) = \frac{l(\Gamma)}{12}$ if $\Gamma$ is a circle graph. Now, we give an important Proposition.
Proposition 4.1.19: Let \( \Gamma \) be a bridgeless \( \text{wm}\)-graph. Then we have
\[
\varphi(\Gamma) = \frac{2g + 1}{g} - \frac{l(\Gamma)}{4g} + \frac{1}{2g} \sum_{p, q \in V(\Gamma)} (v(p) - 2 + 2q(p))q(q)r(p, q)
+ \frac{1}{2g} \sum_{p, q \in V(\Gamma)} (v(p) - 2 + 2q(p)) \sum_{e_i \in E(\Gamma)} L_i \frac{L_i}{L_i + R_i} R_{c_i, p}.
\]

Proof: We have this equation
\[
\tau(\Gamma) = \frac{l(\Gamma)}{12} - \frac{1}{6} \sum_{q \in V(\Gamma)} (v(q) - 2)(v(p) - 2 + 2q(p))r(p, q)
\]
\[
+ \frac{1}{3} \sum_{p \in V(\Gamma)} (v(p) - 2 + 2q(p)) \sum_{e_i \in E(\Gamma)} \frac{L_i}{L_i + R_i} R_{c_i, p}.
\]

Since \( v((q) - 2) = (v(q) - 2 + 2q(p) - 2q(q)), \)
\[
(2g - 2)\tau(\Gamma) = \frac{(2g - 2)l(\Gamma)}{12} - \frac{\theta(\Gamma)}{6} + \frac{1}{3} \sum_{p, q \in V(\Gamma)} (v(p) - 2 + 2q(p))q(q)r(p, q)
\]
\[
+ \frac{1}{3} \sum_{p \in V(\Gamma)} (v(p) - 2 + 2q(p)) \sum_{e_i \in E(\Gamma)} \frac{L_i}{L_i + R_i} R_{c_i, p}.
\]

Equivalently,
\[
12\tau(\Gamma) + \frac{\theta}{g - 1} = l(\Gamma) + \frac{2}{g - 1} \sum_{p, q \in V(\Gamma)} (v(p) - 2 + 2q(p))q(q)r(p, q)
\]
\[
+ \frac{2}{g - 1} \sum_{p \in V(\Gamma)} (v(p) - 2 + 2q(p)) \sum_{e_i \in E(\Gamma)} \frac{L_i}{L_i + R_i} R_{c_i, p}.
\]

By multiplying both sides of equation by \( \frac{g - 1}{12g} \) and by the Proposition 4.1.11, we obtain
the equality.
\[
\varphi(\Gamma) = \frac{2g + 1}{g} - \frac{l(\Gamma)}{4g} + \frac{1}{2g} \sum_{p, q \in V(\Gamma)} (v(p) - 2 + 2q(p))q(q)r(p, q)
+ \frac{1}{2g} \sum_{p, q \in V(\Gamma)} (v(p) - 2 + 2q(p)) \sum_{e_i \in E(\Gamma)} \frac{L_i}{L_i + R_i} R_{c_i, p}.
\]
According to Proposition 4.1.19, when $\Gamma$ is a simple bouquet graph, we obtain the equation $\varphi(\Gamma) = \frac{g-1}{6g} l(\Gamma)$, which is stronger than the desired results in Theorem 4.1.17.

We are back to Theorem 4.1.17. If $v = 2$, then $\Gamma$ is a banana graph. Furthermore, we obtain

$$\varphi(\Gamma) = \frac{g-1}{6g} l(\Gamma) - \frac{(g-1)(2g+1)}{6g} \sum_{i=1}^{e} \frac{1}{L_i}. \quad (66)$$

In particular, we have $\varphi(\Gamma) \geq \frac{g-1}{6g} l(\Gamma)$. We can see the proof from Zubeyir cinkir [Zhang’s Conjecture and The Effective Bogomolov Conjecture over Function Field, Proposition 5.4][5]. This result is stronger than what we wanted. When $g = 2$, this gives $t(2) = \frac{1}{27}$.

If $v \geq 3$, we have $g \geq 3$ because we work with a minimal vertex set for $\Gamma$. If $g = 3$, we have $\varphi(\Gamma) \geq \frac{1}{27} l(\Gamma)$. In fact, if $\Gamma$ is a bridgeless simple w.m-graph, the formula in Proposition 4.1.19 reduces to the following formula.

$$\varphi(\Gamma) = \frac{(2g+1)\tau(\Gamma)}{g} - \frac{l(\Gamma)}{4g} + \frac{1}{2g} \sum_{p \in V(\Gamma)} (v(p) - 2) \sum_{c_i \in E(\Gamma)} \frac{L_i}{L_i + R_i} R_{c_i,p}. \quad (67)$$

We obtain the inequality $\varphi(\Gamma) \geq \frac{(g-1)l(\Gamma)}{6g(g+2)}$. See Zubeyir cinkir [Zhang’s Conjecture and The Effective Bogomolov Conjecture over Function Field, Theorem 5.1][5].

If $g \geq 4$, we have $\varphi(\Gamma) \geq \frac{(g-1)^2}{2g(g+3)} l(\Gamma)$. To obtain this claim, we need the following Theorem.

**Theorem 4.1.20:** Let $\Gamma$ be a bridgeless simple w.m-graph with genus $g$ and with $\#(V(\Gamma)) = v \geq 3$. Then we have the following inequalities. If $v(p) \geq 3$ for each $p \in V(\Gamma)$, then we have

$$\varphi(\Gamma) \geq \frac{g^2(v+14) - 2g(3v+2) - 7v - 10}{2g(7g+5)(v+6)} l(\Gamma). \quad (68)$$

In particular, since $2(g-1)v \geq v$, we have $\varphi(\Gamma) \geq \frac{(g-1)^2}{2g(g+5)} l(\Gamma)$.

**Proof:** See Zubeyir cinkir [Zhang’s Conjecture and The Effective Bogomolov Conjecture over Function Field, Theorem 5.19][5].

Now, we finish the proof of Theorem 4.1.17.

**Remark 4.1.21:** The lower bounds to $\varphi(\Gamma)$ were given in Theorem 4.1.17. When $g = 3$, the previous lower bound can be improved as follows: Let $\Gamma$ be an irreducible w.m-graph with genus $g = 3$. Then we have $\varphi(\Gamma) \geq \frac{892 - 11v^{1/2}}{14580} l(\Gamma)$. See Zubeyir cinkir [Zhang’s Conjecture and The Effective Bogomolov Conjecture over Function Field, Theorem 7.8][5].
5 The Bogomolov Conjecture

5.1 Proof of the Bogomolov Conjecture

**Definition 5.1.1:** Let $X$ be a smooth projective surface over a field $k$, and let $Y$ be a smooth projective curve over $k$. A fibration $f : X \rightarrow Y$ over $Y$ is called **isotrivial** if all smooth fibers are isomorphic to a fixed curve.

Let $k$ be a field. Let $X$ be a smooth projective surface over $k$, let $Y$ be a smooth projective curve over $k$. Let $f : X \rightarrow Y$ be a semi-stable fibration such that the generic fiber of $f$ is smooth and of genus $g \geq 2$. Let $K$ be the function field of $Y$, with algebraic closure $\overline{K}$, and let $C$ be the generic fiber of $f$. The height pairing on the Jacobian variety (See Qingliu [Algebraic Geometry and Arithmetic Curves, 7.4.4, Definition 4.40][1]) $Jac(C)(\overline{K}) = Pic^0(C)(\overline{K})$, we have a canonical inclusion $j : C(\overline{K}) \rightarrow Jac(C)(\overline{K})$ defined by $j(x) = (2g - 2)x - \omega_C$.

We define $B_C(P, r) = \{x \in C(\overline{K})\| j(x) - P\|_{NT} \leq r\}$, where $P \in Pic^0(C)(\overline{K})$ and $r \geq 0$, and if we set $r_C(P) = -\infty$, if $\#(B_C(P, 0)) = \infty$. Otherwise $r_C(P) = \sup\{r \geq 0\| \#(B_C(P, r)) < \infty\}$.

Then Bogomolov’s conjecture can be stated as follows:

**Conjecture 5.1.2 (Bogomolov Conjecture):** If $f$ is non-isotrivial, then $r_C(P)$ for all $P$.

**Conjecture 5.1.3 (Effective Bogomolov Conjecture):** If $f$ is non-isotrivial, then there exists an effectively calculable positive number $r_0$ such that

$$\inf_{P \in Pic^0(C)(\overline{K})} r_C(P) \geq r_0. \quad (69)$$

We now describe how metric graph can be related to above conjectures by the articles Moriwaki [Bogomolov conjecture for curves of genus 2 over function fields][9] and Zhang [Admissible pairing on a curve][10].

For the semistable fibration $f : X \rightarrow Y$, let $CV(f) = \{y_1, y_2, \ldots, y_s\}$ be the set of critical values of $f$, where $s$ is the number of singular fibers. That is, $y \in CV(f)$ if and only if $f^{-1}(y)$ is singular. For any $y_i \in CV(f)$, Let $\Gamma_{y_i}$ be the dual graph of the fiber $C_{y_i} := f^{-1}(y_i)$, for each $1 \leq i \leq s$. The metric graph $\Gamma_{y_i}$ is obtained as follows. The set of vertices $V_{y_i}$ of $\Gamma_{y_i}$ is indexed by irreducible components of the fiber $f^{-1}(y_i)$ and the singularities of $f^{-1}(y_i)$ correspond to edges of length 1. Let $I(C_{y_i}) := \{C_{1, y_i}, C_{2, y_i}, \ldots, C_{v_i, y_i}\}$ be the set of irreducible components of the fiber $C_{y_i}$, where $v_i$ is the number of irreducible components in $C_{y_i}$. Then the irreducible curve $C_{j, y_i}$ corresponds to the vertex $p_j \in V_{y_i}$ for each $1 \leq j \leq v_i$. Let $\delta_{y_i}$ be the number of singularities in $C_{y_i}$. By our construction, $\delta_{y_i} = l(\Gamma_{y_i})$, the length of $\Gamma_{y_i}$, for each $1 \leq i \leq s$. Let $\delta := \sum_{i=1}^{s} \delta_{y_i}$, the total number of singularities in the fibration. For any $p_j \in V_{y_i}$, let $q(p_j) := g(C_{j, y_i})$, where $g(C_{j, y_i})$ is the arithmetic genus of $C_{j, y_i}$. Let $g(Y)$ be the genus of $Y$. We have $K_X, K_Y, \omega_X, \omega_Y$, and $\omega_{X/Y}$, which are the canonical divisors of $X$, the canonical divisor of $Y$, the dualizing sheaf of $X$, the dualizing sheaf of $Y$, and the relative dualizing sheaf, respectively. For the
admissible dualizing sheaf $\omega_a$ (See Zhang [Admissible pairing on a curve][10]), we have the following inequalities $\omega^2_{X/Y} \geq \omega_a^2 \geq 0$.

In the paper Zhang [Admissible pairing on a curve][10], it is showed that $\omega_a^2 > 0$ is equivalent to the Bogomolov conjecture.

We will use the following notation for the singularities that are in the fibers of $f$:

Let $y \in CV(f)$, and $p \in f^{-1}(y)$ be a node. If the partial normalization of $f^{-1}(y)$ at $p$ is connected, we say that $p$ is of type 0. If it is disconnected, then it has two components, in which case $p$ will be said to be of type $i$, where $i$ is the fiber $f^{-1}(y)$ by $\delta_i(\Gamma_y)$, and we set $\delta_i(X) = \sum_{j=1}^{s} \delta_i(\Gamma_{y_j})$. We have $\delta_{y_j} = \sum_{i \geq 0} \delta_i(\Gamma_{y_j})$ and $\delta = \sum_{i \geq 0} \delta_i(X)$.

In Zhang [Gross-Schoen cycles and dualising sheaves][11], the following conjecture implies Bogomolov Conjecture and Effective Bogomolov Conjecture.

**Conjecture 5.1.4:** For any $y \in CV(f)$, there is a positive continuous function $c(\overline{g})$ of $\overline{g} \geq 2$ such that the following inequalities holds:

$$\varphi(\Gamma_y) \geq c(\overline{g})\delta_0(\Gamma_y) + \sum_{i \geq 1} \frac{2i(\overline{g} - 1)}{\overline{g}} \delta_i(\Gamma_y).$$  \hfill (70)

We prove that Conjecture 5.1.4 holds as follows:

**Theorem 5.1.5:** Let $\Gamma$ be a wm-graph with genus $\overline{g}$. Then we have

$$\varphi(\Gamma) \geq t(\overline{g})\delta_0(\Gamma) + \sum_{i \geq 1} \frac{2i(\overline{g} - 1)}{\overline{g}} \delta_i(\Gamma)$$  \hfill (71)

where $t(2) = \frac{1}{27}$, $t(3) = \frac{892 - 11\sqrt{79}}{14580}$, and $t(\overline{g}) = \frac{(\overline{g} - 1)^2}{2^3(\overline{g} + 3)}$ for $\overline{g} \geq 4$.

**Proof:** This result follows from Theorem 4.1.17, Remark 4.1.21 and the article Zhang [Gross-Schoen cycles and dualising sheaves, Corollary 4.4.2][11].

**References**


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Appendix A Translation 本文主要研究证明了张的猜想以及在函数域中的有效伯格莫诺夫猜想。

根据文，我们先介绍了高度函数以及半稳定简并。然后我们重写了伯格莫诺夫猜想，使之简化为度量图中的猜想。

我们研究了几种关于度量图的电位和度量图，并且我们导出了这些不变量关于τ常数的公式。这使得我们可以进一步研究τ常数，从而可以证明张的猜想。

纵观全文，我们用到了很多关于度量图的物理解释，比如说电路原理。我们在考虑度量图的时候，只是单纯地研究利用了它的组合图结构。我们将用度量图上的连续拉普拉斯算子来研究τ常数。紧接着，我们给了关于度量图的进一步解释，并且提到了电路原理和组合图论的相关知识。

在这一系列准备工作完成后，我们最终证明了猜想。

具体说来，我们将如下证明此猜想。

为此，我们首先引入了射影空间中的高度函数。当然，这一引入不是平凡的。我们回忆了一系列相关定义，比如说，数域，赋值域等等。然后我们需要验证定义的合理性。最重要的是，我们需要研究这个高度函数的相关性质。具体说来是有限性和在伽罗华作用下的不变性。

在此之后，我们又引入了簇上的高度函数，并且进一步研究了它的性质。当然，这绝不是平凡的性质。在这里我们引入了威尔高度工具。这一工具也在事实上为我们证明了高度函数的诸多性质。正确性，可加性，正定性，有限性，代数等价性等等。显然，这一系列性质的证明都不是平凡的，需要很多引理的铺垫，在论文正文部分，我们已经详细论述过了，在此我就不赘述了。
为了行文的完整性，我紧接着介绍了典型高度函数。虽然这一部分对我们的猜想证明
没有直接的贡献，但是不可否认的是它在代数几何领域中的重大作用。然后我们具体考
量了典型高度函数的性质，比较重要的两个性质是平行四边形法则以及唯一性。

以上的高度函数主要是从代数观点来考察的。为了与伯格莫夫猜想相关，我们需要以几
何观点来研究高度函数。这时，函数域上的高度函数就应运而生了。这一部分跟我们所
要证明的猜想直接相关，但是由于篇幅有限，我们不会太过详细介绍，只是引入了具体
的定义，并且陈述了几个重要定理。

在高度函数之后，我们又介绍了代数曲面上的模。这一定义的引入相对比较复杂。首
先我们要介绍体系，以此我们来定义代数模。注意到此处代数模的定义与之前代数模的
定义是等价的。具体的证明我们可以参见代数几何相关文献。

除了代数模的定义，我们还介绍了纤维，纤维积，代数曲面等等。在此之后，我们可以
引入正模，以及半正模。这一系列的定义都是为了后来的猜想证明而做准备。

在有了关于模的定义之后，我们可以介绍约简的概念。约简概念相对复杂，我们
在此也不赘述，读者可直接参看正文部分。约简分为很多类型。比较重要的有约简
和约简。关于约简的存在性定理是非常重要的。在此之后，我们又进一步探讨了约
简映射。这一概念与猜想证明并不直接相关，因此，我们只是单纯地给出了相关定义和
关键性的定理。

之后，我们着重介绍了稳定以及半稳定的约简。这一系列定义与我们的猜想证明密
切相关。我们主要利用了稳定性约简定理。这一定理主要是说明了半稳定的约简的存在
性。除此之外，我们还提到了与此相关的部分推论以及命题。虽然这一些命题不能直接用来证
明我们的猜想，但是事实证明这一些结果对代数几何领域的重要性的。特别需要注意的是，对于初学者来说，我们一定要搞清楚相关概念与结论。否则，将极大
影响我们的阅读。

接下来，我们想要建立该猜想与度量图的联系。

为了便于读者理解，我们先回顾了有限图的定义。这一些定义都是容易理解的。
然后，我们具体证明了纤维，代数模等等概念如何与有限图相联系。这为我们的定理证
明做了铺垫。

最后我们进入了猜想证明的主体部分。一开始，我们先说明了有限图与度量图之
间的联系，并且介绍了度量图的相关基本知识。这一过程是非常容易的。毕竟只是平凡的
定义，不涉及高深内容，只需要读者有高中水平即可。

关于度量图，我们还给出了几种分类，尤其是重要的几种，不可约、无桥度量图等等。
在此之后，我们需要进一步研究度量图上的度量。虽然给出一个度量图，它上面会有很
多种度量，但是对我们猜想证明有帮助的度量比较少。事实上，我们主要研究了两种，
一种是典型度量，一种是容许度量。

紧接着，我们引入了最为重要的两个函数。这两个函数的存在性唯一性证明是相
当复杂的。在论文主体部分我们只是列出来相关的参考文献，而不会去具体证明，有兴
趣的读者可以阅读文献。接着，为了便于读者更好地理解这两大函数，我们将它们赋
予了具体的物理含义。在这一部分，我们将提到电路原理的相关知识。比较重要的关
于电路的两个约简和三个变形。通过这些准备，我们可以将度量图简化为非常简单的
情形。在此过程中，我们还会具体给出数学符号与物理符号的对应关系。之后，我们介绍
了度量图上的典型例子。与此相关，我们又定义了度量图。这一些定义都是为了我们证
明定理的方便。紧接着，为了进一步证明猜想，我们引入了四个相关记号。这些记号是
有具体的几何意义的，但是考虑到与论文证明直接关系不明显，我们并没有直接说明这
一些几何意义，有兴趣的读者可以参考相关文献。最后，我们讨论了一个相关恒等式，

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以此来进一步简化我们的猜想。而且我们还引入了简单权度量图的概念，并且把某些定理的证明直接简化为简单权度量图的情况。

我们在论文的最后，具体阐释了张的猜想和在函数域中的有效伯格霍夫猜想之间的关系。我们在详细说明这一层关系的时候，利用了文章开头提到的各种概念和定理。我们考虑了雅克比的这种函数，然后以此来讨论代数曲线与雅克比之间的典型嵌入。之后我们介绍了伯格诺夫猜想的具体陈述。接着，我们描述了度量图如何与该猜想相联系。在完成这过程中，我们又介绍了奇异纤维和对偶图。接下来，典型除子的概念就又成为我们所必须了解的了。在这之后，我们就可以把伯格诺夫猜想简化为度量图中的张的猜想。进一步，我们可以利用度量图中的知识来证明张的猜想。

在证明度量图中的张的猜想的过程中，我们详细研究了论文前段所引入的四个参数。这四个参数非常重要，不仅他们之间有着重要的联系，而且它们与^{\tau}常数之间也有着密切的联系。这些关系我们会在论文中具体阐述。在具体说明这些关系时，我们会利用很多恒等式。在这些恒等式中，很多不是平凡的。为了证明这些非平凡的恒等式，我们参考了很多文献，并且直接引用了很多定理。这一些定理在研究^{\tau}常数时非常有用，但是的这类定理与论文主题关系不大，因此我们并没有在论文正文中详细证明。依据这一些定理，我们可以证明出度量图中的张的猜想，至此我们也就证明了函数域中的有效伯格诺夫猜想。

这篇文章中，我们最主要的目的就是证明了函数域中的有效伯格诺夫猜想。为了证明这一猜想，我们引入了一系列的概念以及重要工具。通过这篇文章，我们可以看到代数和几何的紧密联系。这也给了我们很多研究的启示。通过代数来研究几何，通过几何来研究代数都不失为一种研究方式。这一篇文章，我们用到了很多代数几何中的概念，这也是目前数学研究的主流。
Reducibility of \( f(x) - g(y) \)

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Abstract

In this thesis we study the reducibility of \( f(x) - g(y) \), where \( f, g \in \mathbb{C}[x] \) are indecomposable. We prove that if \( f(x) - g(y) \) is reducible, then either \( g = f \circ u \) for some \( u \in \mathbb{C}(X) \) of degree 1 or \( m = n(n - 1)/2 \) with \( n \geq 4 \) (\( n := \deg(f) \) and \( m := \deg(g) \)) or some small cases (\( \max\{m, n\} \leq 120 \)). We also prove that if \( m = n(n - 1)/2 \), then it is reduced to the problem (namely the hypothesis of Theorem 1.1.1 in their paper[4]) that Guralnick-Shareshian dealt with in one of their paper[4]. And we will give a much simpler proof of one of Guralnick-Shareshian’s two main theorems (namely Theorem 2.7 in my thesis) in their paper.

1 Introduction

Many authors have studied the problem of classifying all complex polynomials \( f, g \in \mathbb{C}[X] \) for which the polynomial \( f(X) - g(Y) \) is reducible in \( \mathbb{C}[X,Y] \). This problem has been studied both for its intrinsic interest and for applications to number theory and complex analysis[1, 2, 4, 5, 6]. Although much progress has been made, still the problem has not been completely solved. One instance when it has been solved is the case that \( f \) and \( g \) are indecomposable, which means that they are polynomials of degree at least 2 which cannot be written as the composition of lower-degree polynomials. In this case Feit proved the following result [3]:

**Theorem 1.1.** (Feit). If \( f, g \in \mathbb{C}[x] \) are indecomposable polynomials, \( f(x) - g(y) \) is reducible in \( \mathbb{C}[X,Y] \), then \( f = g \circ u, u \in \mathbb{C}[x], \deg(u) = 1 \) or \( \deg(f) = \deg(g) \in \{7, 11, 13, 15, 21, 31\} \).

The exceptional polynomials of degree at most 31 were determined recently by Halllett, Wells, Xia and Zieve, building on previous work of Feit, Fried, Cassou-Noguës and Couveignes. The hypothesis that \( f \) and \( g \) are indecomposable plays a crucial role in the above result, and only much weaker conclusions are known if this hypothesis is removed.

In this thesis we study a variant of the above question, in which \( f(X) \) is a polynomial and \( g(X) \) is a rational function. We will prove the following result.

**Theorem 1.2.** If \( f \in \mathbb{C}[X] \) and \( g \in \mathbb{C}(X) \) are indecomposable and \( f(X) - g(Y) \) is reducible in \( \mathbb{C}(Y)[X] \), then one of the following holds, where we write \( n := \deg(f) \) and \( m := \deg(g) \):

*基数01*
1. \( g = f \circ \mu \) for some \( \mu \in \mathbb{C}(X) \) of degree 1
2. \( m = n(n-1)/2 \) with \( n \geq 4 \)
3. \( m = n \in \{7, 11, 13, 15, 21, 31\} \)
4. \( n = 5 \) and \( m = 6 \)
5. \( n = 6 \) and \( m \in \{5, 6, 10\} \)
6. \( n = 7 \) and \( m \in \{8, 35\} \)
7. \( n = 8 \) and \( m \in \{21, 35\} \)
8. \( n = 9 \) and \( m \in \{28\} \)
9. \( n = 10 \) and \( m = 120 \)
10. \( n = 11 \) and \( m = 12 \)
11. \( n = 15 \) and \( m \in \{8, 28, 35\} \)

Conversely, there exist examples for each pair \((n, m)\) in the above list.

In addition to determining the degrees of \( f \) and \( g \), we also know the shape of the factorization of \( f(X) - g(Y) \) in each case. For instance, in case (2), if \( n > 9 \) then \( f(X) - g(Y) \) is the product of two irreducible factors in \( \mathbb{C}(Y)[X] \), of which one has degree 2. Although we do not know explicit expressions for \( f \) and \( g \), we are able to determine all possibilities for the ramification types of \( f \) and \( g \), by which we mean the set contains the ramification index of a branch point. For \( n > 9 \), all possible ramification types for \( f \) from case (2) are listed in Theorem 2.7. We will also explain how the ramification type of \( g \) in this case can be determined from the ramification type of \( f \).

Our work builds on previous results due to Fried, Müller, Guralnick and Shareshian. In particular, a result of Guralnick and Shareshian can be used to determine all possible ramification types in case (2) of the above theorem. Their proof of this result is extremely long and complicated. We will give a much shorter and simpler proof of this Guralnick–Shareshian result in case \( n \) is sufficiently large. The key to our proof is the following new result.

**Theorem 1.3.** If \( f \in \mathbb{C}[x] \) has sufficiently large degree, and \( \frac{f(x)-f(y)}{x-y} \) is irreducible in \( \mathbb{C}[x, y] \), then genus of the curve \( \{(x, y) \in \mathbb{C}^2 | \frac{f(x)-f(y)}{x-y} = 0\} \) is at most \( n \) if and only if \( f \) has one of 17 known ramifications types (listed in Theorem 2.7 of my thesis).

There are many application of the above result.

Carney, Hortsch, Zieve proved

**Theorem 1.4.** (Carney-Hortsch-Zieve). For any \( f(x) \in \mathbb{Q}[x] \), all but finitely many rational numbers have at most 6 rational preimages under \( f(x) \).
Applying Theorem 1.3, it is promising to generalize Carney-Hortsch-Zieve’s theorem. More specifically, we will determine whether there is an analogous result in which \( \mathbb{Q} \) is replaced by the union of all quadratic number fields.

First, we will determine all \( f \) such that \( f : E \to E \) (\( E \) is the union of all quadratic number fields) is no less than 2 to 1 over infinitely many values. Abramovich-Harris’s theorem implies \( \frac{f(x)-f(y)}{x-y} \) has an irreducible factor \( H(x, y) \) such that curve \( H(x, y) = 0 \) admits a map of degree 1 or 2 over infinitely many values. Abramovich-Harris’s theorem implies

\[
\frac{f(x)-f(y)}{x-y} - \frac{f(x)-f(y)}{x-y} = 0
\]

admits a map of degree 1 or 2 to an curve with genus 0 or 1. If \( \frac{f(x)-f(y)}{x-y} \) is irreducible, curve \( \frac{f(x)-f(y)}{x-y} = 0 \) admits a map of degree 1 or 2 to an curve with genus 0 or 1. By Castelnuovo’s inequality (stated in section 2), the function field \( \mathbb{Q}(x, y) \) of \( \frac{f(x)-f(y)}{x-y} = 0 \) has genus at most \( \deg(f) \). Then we can determine the ramification type of \( f \). When \( \frac{f(x)-f(y)}{x-y} \) is reducible, we can use the result of irreducible case to discuss and go on. Therefore, it’s very promising to have a generalization of Carney-Hortsch-Zieve’s theorem.

2 Tools used

**Theorem 2.1.** (Riemann-Hurwitz). Suppose that \( F_1/F_2 \) is a finite extension of algebraic function fields having the same constant field \( C \). Then

\[
2\text{genus}(F) - 2 = [F_1:F_2](2\text{genus}(F_2) - 2) + \sum_{i=1}^{r} \sum_{P | P_i} (e(P|P_i) - 1)
\]

where \( P_i \) (\( i = 1, 2, \ldots, r \)) are all the branch points.

**Theorem 2.2.** (Riemann-Hurwitz). Let \( f \in \mathbb{C}[x], \mathbb{C}(x, y) \) be the function field of \( \{ (x, y) \in \mathbb{C}^2 | \frac{f(x)-f(y)}{x-y} = 0 \} \), \( A_i (i = 1, \ldots, r) \) be the ramification type of \( P_i (i = 1, \ldots, r) \), where \( P_i \) (\( i = 1, \ldots, r \)) are the whole branch points of \( f \), then

\[
2\text{genus}(\mathbb{C}(x, y)) - 2 = -2(\deg(f) - 1) + \sum_{i=1}^{r} \sum_{a, b \in A_i} (a - (a, b))
\]

Proof of this theorem can be find in Pierre Debes and Michael Fried, *Integral specialization of families of rational functions*, section 7 or F.Dorey and G.Whaples, *Prime and composite polynomials*, Lemma 1 on page 92.

**Theorem 2.3.** (Castelnuovo’s Inequality). Let \( F/K \) be a function field with constant field \( K \). Suppose there are given two subfields \( F_1/K \) and \( F_2/K \) of \( F/K \) satisfying

1. \( F = F_1F_2 \) is the compositum of \( F_1 \) and \( F_2 \) and
2. \( [F:F_i] = n_i \) and \( F_i/K \) has genus \( g_i (i=1,2) \)

Then the genus \( g \) of \( F/K \) is bounded by

\[
g \leq n_1 g_1 + n_2 g_2 + (n_1 - 1)(n_2 - 1)
\]
Theorem 2.4. (Fried). If $f, g \in \mathbb{C}[x]$ are nonconstant polynomials, $f(x) - g(y)$ is reducible, then we can write $f = f_1 \circ f_2 \ldots$, where $3 \leq a \leq b \leq c \leq d$, $\gcd(a, b, c, d) = 1$, such that $f_1(X) - g_1(Y)$ is reducible and the splitting fields of $f(X) - t$ and $g(X) - t$ over $\mathbb{C}(t)$ are the same field.

Theorem 2.5. (Müller). The monodromy group of any indecomposable $f \in \mathbb{C}$ of degree $n$ is $C_n$ ($n$ is prime) or $D_n$ ($n$ is odd prime integer) or $A_6$ ($n$ is odd) or $S_n$ twelve more specific group ($n \leq 31$). When $G$ is $C_n$, $f(x) = u(x) \circ x^n \circ v(x)$, deg$(u) = \deg(v) = 1$; when $G$ is $D_n$, $f(x) = u(x) \circ T_n (x) \circ v(x)$, deg$(u) = \deg(v) = 1$, $T_n (x)$ is Chebyshev polynomial.

Theorem 2.6. (Guralnick-Shareshian). If $G \in \{A_n, S_n\}, n \geq 21$, where $G$ is the monodromy group of $f \in \mathbb{C}[x], \Omega$ is the splitting field of $f(x) = t$. $H$ is the maximal subgroup of $G$. $H \neq A_n, \Omega^H$ is the invariant field of $H, \Omega^H$ has genus 0, then $H$ is a point stabilizer in the natural action of $G$ or $H$ is the stabilizer of a 2-set in the natural action.

This is the Theorem 1.1.2 in Guralnick-Shareshian’s paper[4], page 8.

Theorem 2.7. If $G \in \{A_n, S_n\}, n \geq 21$, where $G$ is the monodromy group of $f \in \mathbb{C}[x], \Omega$ is the splitting field of $f(x) = t$. $H$ is the stabilizer of a 2-set in the natural action of $G$. $\Omega^H$ is the invariant field of $H, \Omega^H$ has genus 0, then $f$ has one of the following ramification types:

1. $\{1^3, 4^1, 2^{n-7}\}, \{1^1, 2^{n-1}\}
2. $\{1^2, 3^1, 2^{n-5}\}, \{1^1, 2^{n-1}\}
3. $\{1^2, 2^{n-2}\}, \{1^1, 4^1, 2^{n-6}\}
4. $\{1^2, 2^{n-2}\}, \{1^1, 1^3, 2^{n-4}\}
5. $\{1^3, 2^{n-3}\}, \{3^1, 2^{n-2}\}
6. $\{1^3, 2^{n-3}\}, \{1^1, 4^1, 2^{n-5}\}
7. $\{n^{-2}, 2^1\}, \{1^2, 2^{n-2}\}, \{1^2, 2^{n-2}\}
8. $\{1^3, 2^{n-3}\}, \{1^1, 2^{n-1}\}, \{1^3, 2^{n-3}\}
9. $\{n^{-3}, 3^1\}, \{a, b, c\}, \text{where } 3 \leq a \leq b \leq c, \gcd(a, b, c) = 1
10. $\{n^{-4}, 2^2\}, \{a, b, c\}, \text{where } 3 \leq a \leq b \leq c, \gcd(a, b, c) = 1
11. $\{n^{-2}, 2^1\}, \{a, b, c\}, \text{where } 3 \leq a \leq b \leq c, \gcd(a, b, c) = 1
12. $\{n^{-4}, 4^1\}, \{a, b, c, d\}, \text{where } 3 \leq a \leq b \leq c \leq d, \gcd(a, b, c, d) = 1
13. $\{n^{-5}, 2^1, 3^1\}, \{a, b, c, d\}, \text{where } 3 \leq a \leq b \leq c \leq d, \gcd(a, b, c, d) = 1
14. $\{n^{-6}, 2^3\}, \{a, b, c, d\}, \text{where } 3 \leq a \leq b \leq c \leq d, \gcd(a, b, c, d) = 1
15. $\{n^{-3}, 3^1\}, \{1^{n-2}, 2^1\}, \{a, b, c, d\}, \text{where } 3 \leq a \leq b \leq c \leq d, \gcd(a, b, c, d) = 1
16. \{1^{n-4}, 2^2\}, \{1^{n-2}, 2^1\}, \{a, b, c, d\}, where \(3 \leq a \leq b \leq c \leq d, \gcd(a, b, c, d) = 1\)

17. \{1^{n-2}, 2^1\}, \{1^{n-2}, 2^1\}, \{1^{n-2}, 2^1\}, \{a, b, c, d\}
where \(3 \leq a \leq b \leq c \leq d, \gcd(a, b, c, d) = 1\)

3 Proofs

3.1 Proof of Theorem 1.2

Proof. Theorem 2.4 gives \(f(x) - t\) and \(g(y) - t\) have the same splitting field over \(\mathbb{C}(t)\), so we have a picture below:

![Diagram]

\[ \Omega \]
\[ \mathbb{C}(x, y) \]
\[ \mathbb{C}(x) \]
\[ f \]
\[ \mathbb{C}(t) \]
\[ g \]
\[ \mathbb{C}(y) \]

Fig. 1: : Field extension(1)

There \(t = f(x) = g(y)\), \(\Omega\) is the splitting field of \(f(x) = t\).

Theorem 2.5 implies \(G = \text{Gal}(\Omega/\mathbb{C}(t)) \in \{A_n, S_n, C_n, D_n\}\) when \(n \geq 32\).

1. If \(G = C_n, n\) is prime, by Galois theorem, \(\Omega = \mathbb{C}(x) = \mathbb{C}(y)\), then \(x = \frac{ay+b}{cy+d}\). Note that \(f(x) = t = g(y)\), we have \(f \circ h = g, h(x) = \frac{ay+b}{cy+d}\).

2. If \(G = D_n, n\) is odd prime integer, then \([\Omega : \mathbb{C}(t)] = 2n, [\Omega : \mathbb{C}(x)] = 2\). This implies \([\Omega : \mathbb{C}(y)] = 2\) or \([\Omega : \mathbb{C}(y)] = n\). On the first case, \(\mathbb{C}(x) = \mathbb{C}(y)\), the result is the same like (1). On the second case, \(f(x) = u(x) \circ T_n(x) \circ v(x), \deg(u) = \deg(v) = 1, \text{without lose of gernity, we assume that} \Omega = \mathbb{C}(w), t = w^n + w^{-n}, x = w + w^{-1}, \text{then} \mathbb{C}(y) = \mathbb{C}(w^n), w^n = \frac{ay+b}{cy+d}, f(x) = T_n(x) = w^n + w^{-1} = h(y) + \frac{1}{h(y)}.\)

Now it remains two cases \(G \in \{A_n, S_n\}\).

If \(G = S_n\), we show that \(f \in \mathbb{C}[X]\) and \(g \in \mathbb{C}(X)\) are indecomposable and \(f(X) - g(Y)\) is reducible in \(\mathbb{C}(Y)[X]\) is equivalent to:

There \(H\) is a proper maximal subgroup of \(G, H \neq A_n, \Omega^H\) is the invariant field of \(H, \Omega^H\) has genus 0.

The indecomposibility of \(g\) is equivalent to \(H = \text{Gal}(\Omega/\mathbb{C}(y))\) is a maximal subgroup of \(G\). Since \(A_n\) is normal subgroup of \(G, f(x) - t\) and \(g(y) - t\) have the same splitting field over \(\mathbb{C}(t), H \neq A_n\). Genus of \(\mathbb{C}(y)\) is obviously 0, inversely, by Lüroth’s theorem, \(\mathbb{C}(t)\) is
the intermediate field of \( \mathbb{C}(y)/\mathbb{C} \), which implies there is a rational function \( g \) such that 
\[
t = g(y).f(x) - g(y) \text{ is reducible in } \mathbb{C}(y)[x] \Leftrightarrow [\mathbb{C}(x) : \mathbb{C}(y)] < [\mathbb{C}(x,y) : \mathbb{C}(y)],
\]
by Theorem 2.6 and Galois theory, 
\[
[\mathbb{C}(x,y) : \mathbb{C}(y)] = [S_{n-2} \times S_2 : S_{n-2}] = 2 < n = [\mathbb{C}(x) : \mathbb{C}(x)].
\]
This proved the equivalence. Now continue the proof of Theorem 1.2.

Put \( H = \text{Gal}(\Omega/\mathbb{C}(y)) \), then \( H \) is a maximal subgroup of \( G \).
By Theorem 2.6, if \( G = S_n \), \( H \) must be a point stabilizer in the natural action of \( G \) or \( H \) is the stabilizer of a 2-set in the natural action.
If \( H \cong S_{n-1} \), that is to say \( \mu(y) \), for some \( \mu \in \mathbb{C}(x) \), \( \deg(\mu) = 1 \), is a root of \( f(X) - t = 0 \), so 
\[
f(y) = t = g(\mu(y)), f = g \circ \mu.
\]
If \( H \cong S_{n-2} \times S_2 \), then \( \Omega^H = \mathbb{C}(xz,x+z) \), where \( z \) is another root of \( f(X) - t = 0 \).
Since 
\[
[\mathbb{C}(x,y) : \mathbb{C}(xz,x+z)] = 2, \deg(g) = [\mathbb{C}(y) : \mathbb{C}(t)] = [\mathbb{C}(x,z) : \mathbb{C}(t)]/2 = n(n-1)/2.
\]
The case for \( G = A_n \) is the same as \( G = S_n \). If \( n \leq 20 \), all the low-degree computations can be done via computer.

3.2 Proof of Theorem 2.7

3.2.1 Key tool

**Theorem 3.1.** Let \( n \) be a sufficiently large integer, \( A = \{1^{a_1}, 2^{a_2}, \ldots, n^{a_n}\} \), such that \( \sum_{i=1}^{n} ia_i = n \), if
\[
4n - 4 \geq \sum_{a,b \in A} (a - (a,b))
\]
then 
\[
a_u \leq 4(u + 1) \text{ or } a_u \geq \frac{n}{u} - 4(u + 1), \text{ for } u = 1, 2, 3.
\]

In Guralnick-Shareshian’s paper[4], the proof of Theorem 2.7 covers over 40 pages. It is unreadable and complicate. They apply Riemann-Hurwitz formula on \( \mathbb{C}(xz,x+z)/\mathbb{C}(t) \), and then use representation theory to classify all the ramification types. But it makes the proof much complicate. In my proof, I first prove an analogous result of Thao Do and Michael Zieve, which is the key tool to make my proof much simpler. I apply Riemann-Hurwitz
formula on $\mathbb{C}(x)/\mathbb{C}(t)$ and $\mathbb{C}(x,z)/\mathbb{C}(x)$, and use Castelnuovo’s Inequality to give a bound of $\text{genus}(\mathbb{C}(x,z))$, then we get two constraints. Combining this two constraints and Theorem 3.1, we can easily give all the ramification types of $f$.

### 3.2.2 Proof of Theorem 3.1

**Proof.** We first show that for any nonnegative integer $u$, if $n$ is sufficiently large compared to $u$ then either

1. $a_i \leq 4(i + 1)$ for each $i \leq u$
2. there is some $k$ with $1 \leq k \leq u$ such that $a_k \geq n/k - 4(k + 1)$ and each $i$ with $1 \leq i < k$ satisfies $a_i \leq 4(i + 1)$.

Prove this by induction on $u$. It’s clear for $u = 0$. Now suppose that $a_i \leq 4(i + 1)$ for $1 \leq i \leq k - 1$. Note that $a_k \leq n/k$.

Then

$$4n - 4 \geq \sum_{i=1}^{k} a_i \left( \sum_{j=i+1}^{n} (j - i) a_j \right)$$

For fixed $i$ with $1 \leq i \leq u$, $(j - i)/j$ is minimized (for $j \geq k + 1$) when $j$ is as small as possible, namely when $j = k + 1$. Thus, for any $i$,

$$\sum_{j=k+1}^{n} (j - i) a_j \geq \sum_{j=k+1}^{n} j a_j \frac{k + 1 - i}{k + 1} = \frac{k + 1 - i}{k + 1} \left( n - \sum_{j=1}^{k} j a_j \right)$$

Hence

$$4n - 4 \geq \sum_{i=1}^{k} a_i \left[ \sum_{j=i+1}^{k} (j - i) a_j + \frac{k + 1 - i}{k + 1} \left( n - \sum_{j=1}^{k} j a_j \right) \right]$$

We’ll show by induction on $r$ that, for each $r = 1, 2, 3, \ldots, k$, if $a_r, a_{r+1}, \ldots, a_k$ are fixed then the right side of the above inequality is minimized when $a_1, \ldots, a_{r-1}$ are all zero. The base case $r = 1$ is vacuous, since the conclusion says nothing. So suppose that $r \geq 2$ and, if $a_{r-1}, a_r, \ldots, a_k$ are fixed, then the right side of the above inequality is minimized when $a_1, \ldots, a_{r-2}$ are all zero. Now vary $a_{r-1}$. To identify when the minimum occurs, we may assume that $a_1 = \ldots = a_{r-2} = 0$. So the right side of the inequality becomes

$$a_{r-1} \left[ \sum_{j=r}^{k} (j-r+1) a_j + \frac{k + 2 - r}{k + 1} \left( n - \sum_{j=r-1}^{k} j a_j \right) \right] + \sum_{i=r}^{k} a_i \frac{k + 1 - i}{k + 1} \left( -(r-1)a_{r-1} \right) + \text{function}(a_r, a_{r+1}, \ldots, a_k, n)$$

Ignoring terms which don’t involve $a_{r-1}$, this becomes

$$a_{r-1} * [\ldots]$$
where [...] is an expression which we’ll show is positive – once we know that positivity, then it follows that the product is minimized when \( a_{r-1} = 0 \). The expression [...] is

\[
\sum_{j=r}^{k} (j - r + 1)a_j + \frac{k + 2 - r}{k + 1}(n - \sum_{j=r-1}^{k} ja_j) + \sum_{j=r}^{k} a_j \frac{k + 1 - j}{k + 1} (- (r - 1))
\]

\[
= \frac{k + 2 - r}{k + 1}[n - (r - 1)a_{r-1}] + \frac{k}{k + 1}a_j [j - r + 1 - \frac{k + 2 - r}{k + 1} j - \frac{k + 1 - j}{k + 1} (r - 1)]
\]

\[
= \frac{k + 2 - r}{k + 1}[n - (r - 1)a_{r-1}] + \sum_{j=r}^{k-1} a_j [-2(r - 1) + \frac{2(r - 1)}{k + 1} j]
\]

\[
\geq \frac{k + 2 - r}{k + 1}[n - (r - 1)(4r)] + \sum_{j=r}^{k-1} 4(j + 1)[-2(r - 1) + \frac{2(r - 1)}{k + 1} j]
\]

\[
+ \frac{n}{k}[-2(r - 1) + \frac{2(r - 1)}{k + 1} k]
\]

\[
= n\left(\frac{k + 2 - r}{k + 1} - \frac{2(r - 1)}{k} + \frac{2(r - 1)}{k + 1}\right) + \text{function}(k, r)
\]

Since

\[
\frac{k + 2 - r}{k + 1}\frac{2(r - 1)}{k} + \frac{2(r - 1)}{k + 1} = \frac{k^2 - (r - 2)k - 2(r - 1)}{k(k + 1)} \geq \frac{rk - (r - 2)k - 2(r - 1)}{k(k + 1)} = \frac{2k - 2(r - 1)}{k(k + 1)} > 0
\]

so this last expression is positive when \( n \) is sufficiently large compared to \( u \), as desired. It follows (as noted above) that we may assume \( a_{r-1} = 0 \), completing the inductive step, so by induction we may assume that \( a_1 = a_2 = ... = a_{k-1} = 0 \), which yields the inequality

\[
4n - 4 \geq \frac{a_k(n - ka_k)}{k + 1}
\]

Rewrite this as

\[
a_k^2 - \frac{n}{k}a_k + \frac{(k + 1)(4n - 4)}{k} \geq 0
\]

which implies

\[
a_k \leq \frac{1}{2}\left(n - \sqrt{n^2 - 4k(k + 1)(4n - 4)}\right) \leq 4(k + 1)
\]

or

\[
a_k \geq \frac{1}{2}\left(n + \sqrt{n^2 - 4k(k + 1)(4n - 4)}\right) \geq \frac{n}{k} - 4(k + 1) - \frac{1}{2k}
\]

So we have proved our first claim. If \( a_k > n/k - 4(k + 1) - 1, k < u, \text{whence} ka_k > n - k(k + 1) - k \). Since \( n = \sum_j ja_j \), it follows that

\[
ua_u \leq n - ia_i < 4k(k + 1) + k \leq 4(k + 1)(k + 2) \leq 4u(u + 1)
\]

so that \( a_u \leq 4(u + 1) \).
3.2.3 Proof of Theorem 2.7

Proof. In algebraic function field language, we first have this field extension picture below:

![Field Extension Diagram](image)

Fig. 3: Field extension (3)

There we denote:
- \( \mathbb{C}(x, y) \) is the function field of the curve \( \{(x, y) \in \mathbb{C}^2 | f(x) - f(y) = 0 \} \)
- \( t = f(x) = f(y) \)

Using Riemann-Hurwitz on \( \mathbb{C}(x)/\mathbb{C}(f(x)) \), we get

\[
-2 = -2n + \sum_{i=1}^{r} (n - \#A_i)
\]

Where \( A_i \) is the ramification type set of \( p_i \), and \( p_i (i = 1, 2, \ldots, r) \) are all branch points of \( f \).

We assume \( p_r \) is the infinity point, then \( A_r = \{n\} \). So

\[
\sum_{i=1}^{r-1} (n - \#A_i) = n - 1
\]

Using Riemann-Hurwitz on \( \mathbb{C}(x, y)/\mathbb{C}(x) \), we get

\[
2g(\mathbb{C}(x, y)) - 2 = (n - 1)(0 - 2) + \sum_{i=1}^{r} \sum_{a, b \in A_i} (a - (a, b))
\]

Since \( genus(\mathbb{C}(x, y)) \leq n \), we get

\[
\sum_{i=1}^{r} \sum_{a, b \in A_i} (a - (a, b)) \leq 4n - 4
\]

For a fixed branch point of \( f \), we denote \( a_i \) to be the number of \( i \) in its ramification type set \( A \).

By Theorem 3.1, we discuss in cases:
- (1) if \( a_1 \geq n - 8 \),

\[
4n - 4 \geq \sum_{a, b \in A} (a - (a, b)) \geq (n - 8)(a_2 + 2a_3 + \cdots)
\]
Since $n$ is sufficiently large, we get
\[ a_2 + 2a_3 + \cdots \leq 4 \]

Note that
\[ a_1 + 2a_2 + 3a_3 + \cdots = n \]

we can write all possibilities:

\[
\begin{array}{|c|c|c|}
\hline
A_\infty & \text{Ramification} & n \cdot \#A_\infty \\
\hline
\{1^{n-4},4^1\} & 3(n-4) & 3 \cdot \\
\{1^{n-3},3^1\} & 2(n-3) & 2 \cdot \\
\{1^{n-5},2^1,3^1\} & 3n-12 & 3 \cdot \\
\{1^{n-2},2^1\} & n-2 & 1 \cdot \\
\{1^{n-4},2^2\} & 2(n-4) & 2 \cdot \\
\{1^{n-6},2^3\} & 3(n-6) & 3 \cdot \\
\hline
\end{array}
\]

Table 1: Ramification types(1)

(2) if $a_1 \leq 8$ and $a_2 \geq n/2 - 12$, then
\[
4n - 4 \geq \left(\frac{n}{2} - 12\right)\left(a_1 + 3a_3 + 2a_4 + 5a_5 + 4a_6 + \cdots\right)
\]
\[ \implies a_1 + 3a_3 + 2a_4 + 5a_5 + \cdots \leq 8 \]

Note that
\[ a_1 + 2a_2 + 3a_3 + \cdots = n \]

we can write all possibilities:

From this table, since any possibility has $n - \frac{1}{4}A \approx n/2$, we conclude that if one point is case(2), then there must be another point in case(2) and the other points are all in case(1).

(3) if $a_1 \leq 8$ and $a_2 \leq 12$, then $n - \frac{1}{4}A$ will roughly bigger than $n - n/3 = 2n/3$. This implies that the other point can’t be the case(2), since any possibility in case(2) has $n - \frac{1}{4}A \approx n/2$, and $n/2 + 2n/3 > n - 1$. Therefore, the other points are all case(1).

Now we can write the ramification type sets of all points:

(i) if $f$ has a point in case, from our conclusion in case(2), all the possibilities are:

1) $\{1^3,4^1,2^{\frac{n-2}{2}}\}, \{1^1,2^{\frac{n-1}{2}}\}$
2) $\{1^2,3^1,2^{\frac{n-2}{2}}\}, \{1^1,2^{\frac{n-1}{2}}\}$
3) $\{1^2,2^{\frac{n-2}{2}}\}, \{1^3,4^1,2^{\frac{n-6}{2}}\}$
4) $\{1^3,2^{\frac{n-2}{2}}\}, \{1^1,3^1,2^{\frac{n-4}{2}}\}$
5) $\{1^3,2^{\frac{n-2}{2}}\}, \{3^1,2^{\frac{2n-3}{2}}\}$
6) $\{1^3,2^{\frac{n-1}{2}}\}, \{1^1,4^1,2^{\frac{n-5}{2}}\}$
7) $\{1^{n-2},2^1\}, \{1^2,2^{\frac{n-2}{2}}\}, \{1^2,2^{\frac{n-2}{2}}\}$

45
| $\{2^{(n-5)/2}, 4, 1^1\}$ | $2(n-4)/2$ | $(n+2)/2$ |
| $\{2^{(n-6)/2}, 4, 1^1\}$ | $3(n-5)/2+3$ | $(n+1)/2$ |
| $\{2^{(n-7)/2}, 4^2, 1^1\}$ | $4(n-6)/2+6$ | $n/2$ |
| $\{2^{(n-8)/2}, 4^3, 1^4\}$ | $5(n-7)/2+9$ | $(n-1)/2$ |
| $\{2^{(n-9)/2}, 4^4, 1^8\}$ | $6(n-8)/2+12$ | $(n-2)/2$ |
| $\{2^{(n-10)/2}, 4^5, 1^{16}\}$ | $7(n-9)/2+15$ | $(n-3)/2$ |
| $\{2^{(n-11)/2}, 4^6, 1^{32}\}$ | $6(n-10)/2$ | $(n+2)/2$ |
| $\{2^{(n-12)/2}, 3^2, 1^1\}$ | $7(n-11)/2+4$ | $(n+1)/2$ |
| $\{2^{(n-13)/2}, 3^1\}$ | $3(n-12)/2$ | $(n+1)/2$ |
| $\{2^{(n-14)/2}, 3^1, 1^1\}$ | $4(n-13)/2+2$ | $n/2$ |
| $\{2^{(n-15)/2}, 3^2, 1^2\}$ | $5(n-14)/2+4$ | $(n-1)/2$ |
| $\{2^{(n-16)/2}, 3^2, 1^4\}$ | $6(n-15)/2+6$ | $(n-2)/2$ |
| $\{2^{(n-17)/2}, 3^3, 1^8\}$ | $7(n-16)/2+8$ | $(n-3)/2$ |
| $\{2^{(n-18)/2}, 1^1\}$ | $(n-1)/2$ | $(n-1)/2$ |
| $\{2^{(n-19)/2}, 1^2\}$ | $2(n-2)/2$ | $(n-2)/2$ |
| $\{2^{(n-20)/2}, 1^3\}$ | $3(n-3)/2$ | $(n-3)/2$ |
| $\{2^{(n-21)/2}, 1^4\}$ | $4(n-4)/2$ | $(n-4)/2$ |
| $\{2^{(n-22)/2}, 1^8\}$ | $5(n-5)/2$ | $(n-5)/2$ |
| $\{2^{(n-23)/2}, 1^6\}$ | $6(n-6)/2$ | $(n-6)/2$ |
| $\{2^{(n-24)/2}, 1^7\}$ | $7(n-7)/2$ | $(n-7)/2$ |

Table 2: Ramification types(2)
8) \( \{2^{n-3}, 2^1\}, \{1^1, 2^{n-1}\}, \{1^3, 2^n/3\} \)

(ii) if \( f \) has a point in case (3), then we claim that its \( n - \#A \leq 4 \).
This is because the other points are all in case (1), note that \( n - \#A \) equals the coefficient of the \( n \) in ramification. Since the ramification contribution of all the branch points add up to at most \( 4n - 4 \), and we know that any point’s ramification contribution is at least \( n/3 \), so the sum of \( n - \#A \) of points in case (1) add up to at most 3, which implies our claim.

With this claim, we can write all the possibilities:
9) \( \{1^{n-3}, 3^1\}, \{a, b, c\}, \text{where } 3 \leq a \leq b \leq c, \gcd(a, b, c) = 1 \)
10) \(\{1^{n-4}, 2^2\}, \{a, b, c\}, \text{where } 3 \leq a \leq b \leq c, \gcd(a, b, c) = 1\)
11) \(\{1^{n-2}, 2^4\}, \{1^{n-2}, 2^1\}, \{a, b, c\}, \text{where } 3 \leq a \leq b \leq c, \gcd(a, b, c) = 1\)
12) \(\{1^{n-4}, 4^3\}, \{a, b, c, d\}, \text{where } 3 \leq a \leq b \leq c \leq d, \gcd(a, b, c, d) = 1\)
13) \(\{1^{n-5}, 2^3, 3^1\}, \{a, b, c, d\}, \text{where } 3 \leq a \leq b \leq c \leq d, \gcd(a, b, c, d) = 1\)
14) \(\{1^{n-6}, 2^3\}, \{a, b, c, d\}, \text{where } 3 \leq a \leq b \leq c \leq d, \gcd(a, b, c, d) = 1\)
15) \(\{1^{n-5}, 3^1\}, \{1^{n-2}, 2^1\}, \{a, b, c, d\}, \text{where } 3 \leq a \leq b \leq c \leq d, \gcd(a, b, c, d) = 1\)
16) \(\{1^{n-4}, 2^2\}, \{1^{n-2}, 2^1\}, \{a, b, c, d\}, \text{where } 3 \leq a \leq b \leq c \leq d, \gcd(a, b, c, d) = 1\)
17) \(\{1^{n-2}, 2^1\}, \{1^{n-2}, 2^1\}, \{1^{n-2}, 2^1\}, \{a, b, c, d\}, \text{where } 3 \leq a \leq b \leq c \leq d, \gcd(a, b, c, d) = 1\)

The last is to prove that every point contributes at least \(n/3\) to Riemann-Hurwitz for \(\mathbb{C}(x, y)/\mathbb{C}(x)\).

Let \(S\) be the ramification type set, and \(a_1\) to be the number of 1s in \(S\). Since \(\gcd(S) = 1\), we have \(a_1 \leq n - 2\).

For each \(s\) in \(S\) we have \(\sum_{t \in S} (s - (s, t)) \geq s - 1\), since if at least two \(t\)'s aren't divisible by \(s\) then the sum is at least \(s/2 + s/2 = s\), and if at most one \(t\) isn't divisible by \(s\) then (since \(\gcd(S) = 1\)) some \(t\) is coprime to \(s\) while \(s\) divides all other \(t\)'s, so the sum equals \(s - 1\). Thus

\[
\sum_{s, t \in S} (s - (s, t)) \geq \sum_{s \in S} (s - 1) = n - \sharp S
\]

If \(\sum_{s, t \in S} (s - (s, t)) < n/2\), then \(\sharp S > n/2\) so \(a_1 > 0\).

\[
\sum_{s, t \in S} (s - (s, t)) \geq a_1 \sum_{s \in S} (s - 1) = a_1 (n - \sharp S) \geq \frac{a_1 (n - a_1)}{2}
\]

Since \(1 \leq a_1 \leq n - 2\), it follows that \(\sum_{s, t \in S} (s - (s, t)) \geq \frac{n - 1}{2}\), which is at least \(n/3\).

References


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Appendix A Research proposal Carney, Hortsch and Zieve showed that for any $f(x)$ in $\mathbb{Q}[x]$, all but finitely many rational numbers have at most 6 rational preimages under $f(x)$. In this thesis, I will prove an analogous result in which $\mathbb{Q}$ is replaced by the union of all quadratic number fields. More specifically, we prove that there is a $f \in \mathbb{Q}[x]$ such that $f$ induces all quadratic number fields.

Specifically, we prove that there is a $f \in \mathbb{Q}[x]$ such that $f$ induces all quadratic number fields. Moreover, $C_n$, and $D_n$ only occur for $x^n$ and $T_n(x)$ up to composing on both sides with degree one polynomials. So we may assume that the group is $A_n$ or $S_n$ and if it’s $A_n$ then $n$ must be odd since we know that the group contains a $n$ cycle.

In function field language, $\mathbb{Q}(x, y)$ has a subfield $L$ with $[\mathbb{Q}(x; y) : L] = 2$ and genus of $L$ is 0 or 1. Note that the automorphism $\sigma$ which switch $x$ and $y$ gives a subfield $\mathbb{Q}(x + y, xy)$, and $[\mathbb{Q}(x, y) : \mathbb{Q}(x + y, xy) = 2$.

Case 1:
Suppose $h(x) - h(y) \not\equiv 0 \mod{x - y}$. A theorem of Peter Müller says that if deg($f$) $\geq 32$, then the Galois group of $f(x) = t$ over $\mathbb{Q}(t)$ is either $C_n, D_n, A_n$ or $S_n$. Moreover, $C_n$ and $D_n$ only occur for $x^n$ and $T_n(x)$ up to composing on both sides with degree one polynomials. So we may assume that the group is $A_n$ or $S_n$ and if it’s $A_n$ then $n$ must be odd since we know that the group contains a $n$ cycle.

In function field language, $\mathbb{Q}(x, y)$ has a subfield $L$ with $[\mathbb{Q}(x; y) : L] = 2$ and genus of $L$ is 0 or 1. Note that the automorphism $\sigma$ which switch $x$ and $y$ gives a subfield $\mathbb{Q}(x + y, xy)$, and $[\mathbb{Q}(x, y) : \mathbb{Q}(x + y, xy) = 2$.

Case 1a: $L = \mathbb{Q}(x + y, xy)$, Theorem 1.2.1 of the Guralnick-Shareshian’s paper Symmetric and alternating groups as monodromy groups of Riemann surfaces determines all possible ramification types of degree-$n$ polynomials $f(x)$ with $Gal(f(x) = t, K(t) = S_n$ or $A_n$ for which $\mathbb{Q}(x + y, xy)$ has genus 0 or 1.

Case 1b: $L \neq \mathbb{Q}(x + y, xy)$. By Castelnuovo’s inequality, $\text{genus} (\mathbb{Q}(x, y)) \leq n$. By Riemann-Hurwitz
genus formula,
\[ 2 \text{genus}(Q(x, y)) - 2 \geq [Q(x, y) : Q(x + y, xy)](2 \text{genus}(Q(x + y, xy)) - 2), \]
which implies \( \text{genus}(Q(x + y, xy)) \leq (n + 1)/2 \).
We try to use this constraint and Guralnick-Shareshian’s method to determine \( f(x) \).

Case 2:
Suppose \( \frac{f(x) - f(y)}{x - y} \) is reducible. We write \( f = g \circ h \), such that \( H(x, y)|\frac{h(x) - h(y)}{x - y} \), and \( h \) is the minimal degree polynomial satisfying this property.

Case 2a:
If \( \frac{h(x) - h(y)}{x - y} \) is irreducible, it is reduced to the case 1.

Case 2b:
If \( \frac{h(x) - h(y)}{x - y} \) is reducible, write \( h = h_1 \circ h_2 \) with \( h_1 \) irreducible unless \( h_1 \) is \( x^n \) or \( T_n(x) \) up to composing on both sides with degree one polynomials. This case is reduced to the case 1. Therefore, we should deal with the case 1b first.

*Appendix B Translation* 在参考文献[22]中的Some unsolved problems on polynomials一章中，Schinzel提出研究形如 \( f(x) - g(y) \)的多项式的不约性的问題。在不同的假设条件下，至今已有很多非常优美的结论。比如Feit证明了

**Theorem 3.1.** (Feit) 若 \( f, g \in \mathbb{C}[x] \) 是非常数的多项式，且均不能写成两个更低阶多项式的复合，如果 \( f(x) - g(y) \) 是可约的，则要么 \( f \) 等于 \( g \) 复合上一个一阶多项式，或者 \( \deg(f) = \deg(g) \in \{7, 11, 13, 15, 21, 31\} \).

Fried证明了

**Theorem 3.2.** (Fried) 任意两个非常数的复系数多项式 \( f \) 和 \( g \)，如果 \( f(x) - g(y) \) 可约，则可将 \( f, g \) 写成 \( f = f_1 \circ f_2 \) 及 \( g = g_1 \circ g_2 \)。这里 \( f_1, f_2, g_1, g_2 \in \mathbb{C}[X] \)，使得 \( f_1(X) = g_1(Y) \) 是可约的且 \( f(X) - t \) 和 \( g(X) - t \) 在 \( \mathbb{C}(t) \) 上的分裂域是相同的。

若考虑 \( f, g \in \mathbb{C}(x) \)，则Fried定理有类似的推广，但仍然不知道Feit定理是不是有类似的推广。Guralnick-Shareshian的工作相当于再 \( f \in \mathbb{C}[x], g \in \mathbb{C}(x) \) 的条件下给出了推广（参见[1]）。而Guralnick-Shareshian的工作大部分是在决定所有的多项式 \( f \) 使得曲线 \( \{ (x, y) \in \mathbb{C}^2 | (f(X) - f(Y))/ (X - Y) = 0 \} \) 不可约。这个曲线在自同构 \( (X, Y) \mapsto (Y, X) \) 下的商空间的亏格为0。但是在[1]中，Guralnick-Shareshian的证明方法非常复杂，且篇幅较大，本文目的是对Guralnick-Shareshian的定理做一个更简洁的证明。首先我们从如下问题出发:

**Problem 1.** 若 \( f \in \mathbb{C}[x] \) 是不可分解的（即不能写成两个更低阶多项式的复合），\( g \in \mathbb{C}(x) \) 不可分解，若有 \( f(x) - g(y) \) 在 \( \mathbb{C}(y)[x] \) 上可约，则 \( f, g \) 满足什么条件？

Fried定理在有理函数上的推广证明了 \( f(x) - t \) 和 \( g(y) - t \) 在 \( \mathbb{C}(t) \) 上的分裂域是一样的，从而我们可以画出域扩张图如下:

Müller 在[8]中证明了

**Theorem 3.3.** (Müller) 任意 \( n \) 次复系数的不可分解多项式 \( f \) 的monodromy群 \( G \) 只能是循环群 \( C_n (n \) 为素数)或二面体群 \( D_n (n \) 为奇数)或 \( A_n (n \) 为奇数)或 \( S_n \), 或另外 \( 12 \) 个确定的群 \( (n \leq 31) \)。且当 \( G \) 为 \( C_n \) 时，\( f(x) = u(x) \circ x^n \circ v(x), \deg(u) = \deg(v) = 1 \); 当 \( G \) 为 \( D_n \) 时，\( f(x) = u(x) \circ T_n(x) \circ v(x), \deg(u) = \deg(v) = 1, T_n(x) \) 为切比雪夫多项式。
因此，当$n \geq 32$时，$G = Gal(\Omega/C(t)) = A_n, S_n, C_n, D_n$。

(1) 若$G = C_n,n$为素数，则由伽罗华理论知必有$\Omega = C(x) = C(y)$, 从而$x = \frac{aw+b}{cy+d}$。注意到$f(x) = t = g(y)$，我们有$f \circ h = g, h(x) = \frac{aw+b}{cy+d}$。

(2) 若$G = D_n,n$为奇素数，由$\Omega : C(t)] = 2n$知$\Omega : C(x)] = 2$, 而$\Omega : C(y)] = 2$或$\Omega : C(y)] = n$。若为前者，我们有$C(x) = C(y)$, 结论和(1)一样；若为后者，因为$f(x) = u(x) \circ T_n(x) \circ v(x), deg(u) = deg(v) = 1$，故不妨设$\Omega = C(w), t = w^n + w^{-n}, x = w + w^{-1}$, 此时$C(y) = C(w^n)$, 故$w^n = \frac{aw+b}{cy+d}, f(x) = T_n(x) = w^n + w^{-1} = h(y) + \frac{1}{h(y)}$。

所以我们只剩下讨论$G = A_n, S_n$的情况了。而Guralnick-Shareshian得工作就是研究：如果$G = A_n, S_n$, 这里$G$是$f \in \C[x]$的monodromy群，$\Omega$是$f(x) = t$的分裂域，$H$是$G$的非平凡极大子群，$H \neq A_n$, $\Omega^H$是$H$下的不动域，$\Omega^H$的$\ominus$为0。那么$H,f$要满足什么条件？Guralnick-Shareshian在[1]中证明了当$n > 8, G = S_n$时，$H$只能是$S_{n-2} \times S_2$或$S_{n-1}(G = A_n$是类似的)，而且计算出了所有的$f$的分歧类型（本文正是对$f$的分歧类型给出简短的计算）。

注意到当$G = S_n$时，问题1的假设等价于如下条件：

$$
\begin{align*}
\Omega &
\downarrow \\
\mathbb{C}(x) \cdot \Omega^H & \quad \downarrow \\
\mathbb{C}(x) & \quad f \\
\mathbb{C}(t) & \quad g \\
\end{align*}
$$

图5：Field extension(5)

这里要求$H$是$G$的非平凡极大子群，$H \neq A_n$, $\Omega^H$是$H$下的不动域，$\Omega^H$的$\ominus$为0。
这是因为$\Omega^H$的亏格为0等价于$\Omega^H$可以写成$\mathbb{C}(y)$的形式。由Lüroth 定理，$\mathbb{C}(t)$是$\mathbb{C}(y)/\mathbb{C}$的中间域推出存在有理函数$g$使得$y = g(y)$。同理，$H$是$G$的非平凡极大子群等价于$g$不可分解。而$f(x) - g(y)$在$\mathbb{C}(y)[x]$上可约等价于$[\mathbb{C}(x) : \mathbb{C}(x)] < [\mathbb{C}(x,y) : \mathbb{C}(y)]$，由Guralnick-Shareshian定理及伽罗华理论，$[\mathbb{C}(x,y) : \mathbb{C}(y)] = [S_{n-2} \times S_2 : S_{n-2}] = 2 < n = [\mathbb{C}(x) : \mathbb{C}(x)]$。

因此，除了$n \leq 10$和$G$为有限个特殊群的情况外，问题(1)的解答主要依赖于Guralnick-Shareshian的工作，即[1]。但是Guralnick-Shareshian的证明非常冗长，本文用更简洁的方法给出$f$的分歧类型的计算。

本文的证明思路是：

由Castelnuovo不等式可以推出$\mathbb{C}(x,y)$的亏格不大于$n$。对$\mathbb{C}(x,y)/\mathbb{C}(x)$运用Riemann-Hurwitz公式知

$$-2 = -2n + \sum_{i=1}^{r} (n - \sharp A_i)$$

这里$A_i$是$p_i$的分歧类型集，$p_i(i = 1, 2, \cdots, r)$是$f$的所有分歧点。

假设$p_i$是无穷远点，则$A_r = \{n\}$，上式变为

$$\sum_{i=1}^{r-1} (n - \sharp A_i) = n - 1 \quad (*)$$

对$\mathbb{C}(x,y)/\mathbb{C}(x)$用Riemann-Hurwitz公式，我们有

$$2g(\mathbb{C}(x,y)) - 2 = (n - 1)(0 - 2) + \sum_{i=1}^{r} \sum_{a,b \in A_i} (a - (a, b))$$

因为$genus(\mathbb{C}(x,y)) \leq n$，我们有

$$\sum_{i=1}^{r} \sum_{a,b \in A_i} (a - (a, b)) \leq 4n - 4$$

利用Do-Zieve定理的推广结果可以得到，对$f$的任意分歧点，若记$A$为其分歧类型集，$a_u$是$A$中$u$的个数，有估计

$$a_u \leq 4(u+1) \text{ 或者 } a_u \geq \frac{n}{u} - 4(u+1), u = 1, 2, 3 \quad (**)$$

利用(•)(**)分类讨论可以得到所有分歧点的分歧类型。

问题1的结果可以有很多有趣的应用。

Carney, Hortsch和Zieve证明了

**Theorem 3.4. (Carney-Hortsch-Zieve).** 对任意$f(x) \in \mathbb{Q}[x]$，除了有限个有理数外，任意有理数最多有6个有理数原像。

利用问题1的结果，我们很有希望将这个定理推广。更确切地说，我们想证明若有理系数多项式$f$将$E$映到$E$ (这里$E$是所有二次数域的并) 是否有类似的结果。

首先我们决定所有的$f$使得$f : E \to E$在无穷个点上是$(\leq 2)$对1的。Abramovich-Harris证明在这样的条件下可以推出$f(x) - f(y)$有一个不可约的因子$H(x,y)$使得曲线$H(x,y)=0$对
一个亏格为0的曲线有度数为1或2的映射。特别地，若\( \frac{f(x)-f(y)}{x-y} \)不可约时，曲线\( \frac{f(x)-f(y)}{x-y} = 0 \)对一个亏格为0的曲线有度数为1或2的映射。由Castelnuovo不等式知\( \frac{f(x)-f(y)}{x-y} = 0 \)的函数域\( \mathbb{Q}(x,y) \)的亏格不超过\( \deg(f) \)。然后利用本文的结果可以确定\( f \)的分歧点类型。当\( \frac{f(x)-f(y)}{x-y} \)可约时，利用不可约时的结果可以继续讨论。所以很有希望推广 Carney-Hortsch-Zieve的定理。
Abstract

We consider an original strategy to estimate the probability of systemic risk of an interconnected system described by a multi-parameter model of interacting agents. The systemic risk is the catastrophic event that corresponds to the simultaneous failure of the agents of the system. The strategy to estimate the probability of this rare event consists in simulating the system for one set of parameters of the model by an Importance Sampling (IS) technique. By using a Large Deviations Principle, we can estimate the probability of systemic risk for other sets of parameters that are close to the one we have just simulated, which allows to quantify the sensitivity of the probability of systemic risk with respect to the parameters of the model. Indeed we estimate the systemic risk for the other sets of parameters simply by multiplying the estimated probability of systemic risk for the reference parameters with the LDP ratio, which is the ratio of the probability of systemic risk of the target parameters versus the reference one as given by the Large Deviations Principle. We show that this strategy works well in some restricted region of parameters. With the help of the IS method, we will also show some by-products considering interconnected systems.

1 Introduction

Systemic risk is a risk that exists in interconnected systems, in which every component can fail individually. When systemic risk takes place, almost all the components fail simultaneously. That will result in a huge damage to the whole system. The 2008 financial crisis is just a vivid example in our life where the systemic risk almost happened. Fortunately, the probability of such a risk is quite small, meaning the chance for systemic risk to happen is quite rare. However, since the systemic risk is quite disastrous, it is of great importance for us to estimate the probability of this kind of risk. In our report, we mainly consider the models proposed in [1] and [2], and investigate the systemic risk events in these models, as well as the extended versions of these models where correlated driving Brownian motions are considered.

The typical method to estimate the probability of a specific event is the Monte Carlo (MC) method. However, for extremely rare events, say, events with probability of the
order of $10^{-4}$ or even less, the variance of the Monte Carlo method is too large for reasonable computational budgets. This problem exists when we consider MC to estimate the probability of the systemic risk event. Thus, we need to seek some variance reduction method to estimate the probability of systemic risk. In our report, we apply the Importance Sampling (IS) method, with the help of Cameron-Martion formula, a simple form of Girsanov transformation, to carry out the simulations.

The models used to describe the interconnected system have parameters and it is of interest to know the sensitivity of the probability of the systemic risk event with respect to these parameters (the evaluations of these sensitivities are of vital importance in risk management in mathematical finance for instance). One way would be to carry out IS simulations for different values of the parameters and to estimate the variations in the value of the probability, but it would require to carry out many sets of simulations. Furthermore, each IS simulation would be required to have high precision because the natural random fluctuations of the IS estimator would make it difficult to compute accurately the finite differences of the probability with respect to the change of the parameters. So this method would require much time to proceed. We want to address this problem by taking some powerful techniques into consideration.

The one we consider in this report is Large Deviations Principle (LDP). Large Deviations Principle gives an estimate of the probability of the systemic risk event, more precisely, it gives the exponential decay rate of the probability as a function of the number of components. However, this means that the estimate is only true up to a multiplicative constant or a slowly varying function that is difficult to assess. Our strategy in this project is to show that it is possible to combine IS simulations and LDP in an efficient way as follows:

- use IS to estimate the probability of the systemic risk event for reference values of the parameters.
- use LDP to evaluate the sensitivities, i.e., the relative changes of the probability of the systemic risk event as a function of the parameters.

During the analysis of our strategy, we have also succeeded in showing several 'by-results'. We have shown that in the case of correlated driving Brownian motions, the probability of systemic risk increases a lot when the correlation degree increases. We have also numerically confirmed that, while keeping the individual risk constant, the increase of the internal collaboration will cause the increase of the probability of systemic risk, as is mentioned in [2].

2 Theoretical Part

Our main target is to analysis the systemic risk of the model introduced in [2]. As a reference, we will apply our method to the model of [1] first. We will introduce the two models here.
The simple model is as follows.

\[ dY_i(t) = \frac{\alpha}{N} \sum_{j=1}^{N} (Y_j(t) - Y_i(t))dt + \sigma dW_i(t) \quad i = 1, ..., N \]  

(1)

where \( Y_i(t), i = 1, ..., N \) represent the log-monetary reserves of \( N \) banks possibly with lending and borrowing from each other, and \( W_i(t), i = 1, ..., N \) are \( N \) independent Brownian motions that drive the system. Here, the lending or borrowing rate between every two banks is assumed to be proportional to the difference of their log-monetary reserves, and \( \alpha \geq 0 \) represents the strength of the interconnection of the system, more precisely, the lending or borrowing rate of the system. The parameter \( \sigma > 0 \) represents the strength of the outer influence. For simplicity, we will assume the initial value of the \( N \) components are all 0, say, \( Y_i(0) = 0, i = 1, ..., N \), and consider the event \( \mathbf{Y}(t) = \frac{1}{N} \sum_{i=1}^{N} Y_i(t) \) reaches some specified value \( \eta < 0 \) within the given time period \([0, T]\) as the systemic risk event.

This model is simple because the equation is linear, which we can solve analytically, and the solution has Gaussian statistics that we can handle quite easily.

The complex model is a little bit more involved, where a quartic potential function is involved, and the equation is not linear any more. The model is as follows.

\[ dY_i(t) = -hU(Y_i(t))dt + \theta(\mathbf{Y}(t) - Y_i(t))dt + \sigma dW_i(t) \quad i = 1, ..., N \]  

(2)

here \( \mathbf{Y}(t) = \frac{1}{N} \sum_{i=1}^{N} Y_i(t) \) is the empirical average of all the components and \( U(y) = V'(y) \) is the restoring force. \( V \) is a potential, which we assume has two stable states. In this model, \( Y_i(t) \) can be considered as the representation of the state of the \( i \)th agent. The parameter \( h \) controls the level of intrinsic stabilization force, \( \sigma \) is the strength of the random destabilization influence, and \( \theta \) is the mean-reverting rate of each agent and characterizes the intensity of the interconnection of the system. We assume the potential \( V \) in this model has the quartic expression \( V(y) = \frac{1}{4}y^4 - \frac{1}{2}y^2 \). The two stable states is then \(-1\) and \(+1\). We choose \(-1\) as the normal state and \(+1\) as the failed state. The initial status of all the agents are assumed to be at \(-1\), so the presence of the potential \( V \) ensures that each component stays near the normal state. We consider the event that \( \mathbf{Y} \) switches from the normal state \(-1\) to the failed one \(+1\) (or \( \xi_b \) which will be detailed later) as the systemic risk event, and investigate its probability. More details can be found in [2].

We will change the independent Brownian motions assumption to correlated Brownian motions in the later sections, because the correlated Brownian motions setting can better describe the dependence between the outer random influences on different components, and at the same time, be of considerable practical interest [3]. Our way to introduce correlation in the driving Brownian motions is to change \( W_i(t) \) into \( \widetilde{W}_i(t) \), where

\[ \widetilde{W}_i(t) = \sqrt{\rho}W_0(t) + \sqrt{1 - \rho}W_i(t), \quad i = 1, ..., N \]

Here \( W_0(t) \) represents the common Brownian motion, and the parameter \( \rho \) controls the degree of correlation.


### 2.1 Monte Carlo Vs. Importance Sampling

**Monte Carlo Method**  
Monte Carlo (MC) method is a popular method for numerically estimating the probability of a given event. For the estimate of the probability of the systemic risk in (which we denote by $p$), the Monte Carlo estimator would be

$$
\hat{p}_n = \frac{1}{n} \sum_{k=1}^{n} \mathbb{I}_{\{\min_{0 \leq t \leq T} Y(t) \leq \eta\}}
$$

The estimator is unbiased:

$$
\mathbb{E}[\hat{p}_n] = \mathbb{E}[\mathbb{I}_{\{\min_{0 \leq t \leq T} Y(t) \leq \eta\}}] = p
$$

By the Law of Large Numbers (LLN), we also know:

$$
\lim_{n \to \infty} \hat{p}_n \to \mathbb{E}[\mathbb{I}_{\{\min_{0 \leq t \leq T} Y(t) \leq \eta\}}] = p
$$

in probability, so the MC estimator is consistent.

However, when we consider its variance, some problems happen:

$$
\text{Var}(\hat{p}_n) = \frac{1}{n} \text{Var}(\mathbb{I}_{\{\min_{0 \leq t \leq T} Y(t) \leq \eta\}}) = \frac{1}{n} p (1 - p)
$$

So the relative error would be:

$$
\text{Relative Error} = \sqrt{\frac{\text{Var}(\hat{p}_n)}{\mathbb{E}[\hat{p}_n]}} = \sqrt{\frac{1}{n} p (1 - p)}
$$

when $p$ is of the order $10^{-4}$, to get a reasonable relative error, say, of order $10^{-1}$, we should make $n$ the order of $10^6$, which is quite time-consuming and unwise. But, the probability of the systemic risk is, however, at that order. So we should consider some kind of variance reduction method instead of doing MC directly. We apply Importance Sampling (IS) method here.

**Importance Sampling Method**  
Importance Sampling method is a method using change of measure, with the aim to make the rare event not so rare, or say, more 'important' in the new probability measure. To give an intuitive example, assume $X$ is a random variable with probability density function (pdf) $p(x)$, and we want to estimate the expectation of $h(X)$, where $h$ is a measurable function (here we assume the expectation is finite). Then

$$
I \triangleq \mathbb{E}_p[h(X)] = \int h(x) p(x) dx = \int \frac{h(x) p(x)}{q(x)} q(x) dx = \mathbb{E}_q[\frac{h(X) p(X)}{q(X)}]
$$

The choice of the pdf $q(x)$ is decided by the user.
We can then choose the new estimator to be \( ˆI_n = \frac{1}{n} \sum_{k=1}^{n} h(X_k) \frac{p(X_k)}{q(X_k)} \), where \( X_k, k=1,...,n \) is sampled with pdf \( q \). This estimator is unbiased:

\[
E_q[ˆI_n] = E_q[h(X) \frac{p(X)}{q(X)}] = \int h(x) \frac{p(x)}{q(x)} q(x) dx = \int h(x) p(x) dx = E_p[h(X)] = I
\]

And the estimator is also consistent:

\[
ˆI_n = \frac{1}{n} \sum_{k=1}^{n} h(X_k) \frac{p(X_k)}{q(X_k)} \to E_q[h(X) \frac{p(X)}{q(X)}] = E_p[h(X)] = I \text{ in probability}
\]

The variance of this estimator is:

\[
Var(I_n) = \frac{1}{n} Var_q\left(h(X) \frac{p(X)}{q(X)}\right) = \frac{1}{n} \left(\frac{E_p[h(X)]^2}{q(X)} - \frac{E_p[h(X)]}{q(X)}\right)
\]

So by choosing a good pdf \( q \), we can efficiently reduce the variance. The choice of the pdf \( q \) is of critical importance here, and we will discuss the Importance Sampling we use in the following subsection. For more detailed discussion considering Monte Carlo or Importance Sampling, readers can refer to [4] or [5].

### 2.2 Our Importance Sampling

Because here we have to deal with Brownian motions and the event that some specified path reaches some default level within the given time region, we would like to consider the Cameron-Martin theorem. Here we will list some important results considering this theorem which will be used by us later. For more detailed discussion, readers can refer to [6] and [7].

**Proposition 2.1** (Exponential Martingales). Let \( \{W_t = W(t)\}_{t \geq 0} \) be a standard Brownian motion, defined on a probability space \( (\Omega,F,P) \), and \( \{F_t\}_{t \geq 0} \) the associated Brownian filtration. For any real number \( \vartheta \), the stochastic process \( \{Z_{\vartheta}(t)\}_{t \geq 0} \) is defined as follows:

\[
Z_{\vartheta}(t) = \exp \{\vartheta W(t) - \vartheta^2 t/2\}.
\]

Then the process \( \{Z_{\vartheta}(t)\}_{t \geq 0} \) is a positive martingale with respect to the Brownian filtration.

**Proposition 2.2** (Likelihood Ratios). Let \( (\Omega,F,P) \) be any probability space on which is defined a positive random variable \( Z \) with expectation \( E_Z=1 \). Then we can define a new probability measure \( Q \) as follows: For any event \( F \in F \), set

\[
Q(F) = E_P(Z \mathbb{1}_F).
\]

Furthermore, the expectation operators \( E_P \) and \( E_Q \) are related as follows: for any nonnegative random variable \( Y \),

\[
E_Q Y = E_P(Y Z)
\]
\[
E_P Y = E_Q(Y/Z)
\]
It is easy to check $Z_\theta(t)$ is of expectation one, so we can define probability measure $P_\theta$ with the likelihood ratio $Z_\theta(T)$ on the probability space $(\Omega, \mathcal{F}_T)$: that is for every $F \in \mathcal{F}_T$, $P_\theta(F) = E_0(Z_\theta(T)|\mathcal{F}_T)$, for some fixed $T$. Thanks to Proposition 1, the probability measure is well defined.

**Theorem 2.1** (Cameron-Martin). Under the probability $P_\theta$, the process $\{W(t)\}_{0 \leq t \leq T}$ has the same law as a Brownian motion with drift $\theta$; i.e., the stochastic process $\{W(t)\}_{0 \leq t \leq T}$ has the same law under $P_\theta$ as the process $\{W(t) + \theta t\}_{0 \leq t \leq T}$ has under the probability measure $P = P_0$.

### 2.3 Large Deviations Principle

To make the report as self-contained as possible, we will list some basic statements considering large deviation principle here. We refer the interested reader to [8] or [9].

Throughout this subsection, we assume $X$ is a topological space, and $\mathcal{B}$ is the completed Borel $\sigma$-field on it.

**Definition 2.1** (Rate Function). A rate function $J$ is a lower semicontinuous mapping $J: X \to [0, \infty]$ (such that for all $\alpha \in [0, \infty)$, the level set $\Psi_J(\alpha) \triangleq \{x : J(x) \leq \alpha\}$ is a closed subset of $X$). A good rate function is a rate function for which all the level sets $\Psi_J(\alpha)$ are compact subsets of $X$.

**Definition 2.2** (Large Deviation Principle). For a family of probability measures $\{\mu_\epsilon\}$ on $(X, \mathcal{B})$, $\{\mu_\epsilon\}$ satisfies the large deviation principle with a rate function $J$ if, for all $\Gamma \in \mathcal{B}$,

$$- \inf_{x \in \Gamma^\circ} J(x) \leq \liminf_{\epsilon \to 0} \epsilon \log \mu_\epsilon(\Gamma) \leq \limsup_{\epsilon \to 0} \epsilon \log \mu_\epsilon(\Gamma) \leq - \inf_{x \in \Gamma} J(x)$$

To show some intuition of the use of Large Deviation Principle (LDP), we can take the $\epsilon$ above as $1/N$, and the corresponding probability measure $\mu_\epsilon$ as the probability measure of the empirical average of all the components $Y$ in either model 1 on page 3 or model 2 on page 3 in the space of path under some regular conditions. If we can choose some event from the measurable sets of this space as the systemic risk event or some approximation of the systemic risk, and we can prove that $\mu_{1/N}$ satisfies LDP with some rate function $J$, then we can have a good estimate on the exponential scale of the corresponding probability. Furthermore, if the event $\Gamma$ we choose can satisfy $\inf_{x \in \Gamma^\circ} J(x) = \inf_{x \in \Gamma} J(x) \triangleq J_\Gamma$, then we can have $\lim_{N \to \infty} \frac{1}{N} \log \mu_{1/N}(\Gamma) = -J_\Gamma$, or equivalently, $\mu_{1/N}(\Gamma) \approx \exp\{-NJ_\Gamma\}$, up to a coefficient which changes slowly compared to the exponential term. This is why LDP is useful in the evaluation of the probability of systemic risk, and this is what is investigated intensively in [2].

To be more precise, we will state the important Large Deviations result for 2 on page 3 here. We will describe the space used, but the topology related is a little bit too complex and digressive for our report, so it will be not described here. For more detailed presentation and discussion, please refer to [10].

Here the element being investigated is the empirical measure of the components, which is $\mathcal{Y}_N(t) = \frac{1}{N} \sum_{i=1}^N \delta_{Y_i(t)}$. The space used is $\mathcal{E}_\nu = \{\phi \in C([0,T], M_\infty(\mathbb{R})) : \phi(0) = \nu\}$.
where \( \nu \in M_\infty(\mathbb{R}) \). \( M_\infty(\mathbb{R}) = \{ \mu \in M_1(\mathbb{R}) : \int \varphi(x) \mu(\mathrm{d}x) < \infty \} \), where \( \varphi(x) \) is chosen as 
\[
\varphi(x) = 1 + x^2 + \gamma x^4, \quad \text{for } 0 \leq \gamma \leq \frac{2}{7}.
\]
Here \( M_1(\mathbb{R}) \) is the space of probability measures on \( \mathbb{R} \).

The theorem is as follows.

**Theorem 2.2** (Dawson and Gartner, 1987). Given a finite horizon \([0, T]\), \( \nu \in M_\infty(\mathbb{R}) \) and \( A \subseteq \mathcal{E}' \), if \( \mathcal{Y}_N(0) = \frac{1}{N} \sum_{i=1}^{N} \delta_{Y_i(0)} \to \nu \) in \( M_\infty(\mathbb{R}) \) as \( N \to \infty \), then the law of \( \mathcal{Y}_N(t) = \frac{1}{N} \sum_{i=1}^{N} \delta_{Y_i(t)} \) satisfies the large deviation principle with the rate function \( J_h \): 
\[
- \inf_{\phi \in A^o} J_h(\phi) \leq \liminf_{N \to \infty} \frac{1}{N} \log P(\mathcal{Y}_N \in A) \leq \limsup_{N \to \infty} \frac{1}{N} \log P(\mathcal{Y}_N \in A) \leq - \inf_{\phi \in \overline{A}} J_h(\phi)
\]
(3)

where \( A^o \) and \( \overline{A} \) are the interior and closure of \( A \) in \( \mathcal{E}' \), respectively.

For the expression of the rate function \( J_h \) and the proof of this theorem, please refer to [10]. For more detail about the LDP result considering 2 on page 3, please refer to [2].

### 2.4 Our method

We will mainly discuss the case of model 1 on page 3 in the correlated Brownian motions setting here. For the other situations, the method is either similar or is just one specific case of the one considered here.

**Model:**
\[
dY_i(t) = a(\overline{Y}(t) - Y_i(t))dt + \sigma d\widetilde{W}_i(t) \quad i = 1, \ldots, N
\]
(4)

where \( \overline{Y}(t) = \frac{1}{N} \sum_{j=1}^{N} Y_j(t) \), the empirical average of all the components at time \( t \), and 
\( \widetilde{W}_i(t) = \sqrt{\rho} W_0(t) + \sqrt{1-\rho} W_i(t) \quad i = 1, \ldots, N \), the correlated Brownian motions.

We consider all the component starts from 0, say, \( Y_i(0) = 0 \) for all \( i = 1, \ldots, N \), and consider the event that the empirical average \( \overline{Y} \) reaches the default level \( y = \eta \) \( (\eta < 0) \) within the time interval \([0, T]\) as the systemic risk event. We want to use the Cameron-Martin theorem to make this event important.

From the Section 2.2, we know that the process \( \{Z_\theta(t)\}_{t \geq 0} \) is a martingale with respect to the natural filtration generated by \( \{W_i(t)\}_{0 \leq i \leq N} \), where \( Z_\theta(t) = \exp \{ \theta \sum_{i=0}^{N} W_i(t) - \frac{\theta^2}{2} t (N + 1) \} \); and for the specified parameter \( T \), consider the corresponding probability measure \( P_\theta \), which satisfies
\[
P_\theta(F) = \mathbb{E}_\theta[Z_\theta(T)^{\mathcal{Y}} F] \quad \forall F \in \mathcal{F}_T
\]

Then 
\[
P_\theta(\text{systemic risk}) = P_0(\overline{Y} \text{ reaches } \eta \text{ before time } T)
\]
\[
= \mathbb{E}_0[Z_\theta(T)^{-1} \mathbb{1}_{\{\overline{Y} \text{ reaches } \eta \text{ before time } T\}}]
\]
\[
= \mathbb{E}_0[\exp \{ -\theta \sum_{i=0}^{N} (W_i(T) - \theta T) - \frac{\theta^2}{2} T (N + 1) \} \mathbb{1}_{\{\overline{Y} \text{ reaches } \eta \text{ before time } T\}}]
\]
\[
= \mathbb{E}_0[\exp \{ -\theta \sum_{i=0}^{N} W_i(T) - \frac{\theta^2}{2} T (N + 1) \} \mathbb{1}_{\{\overline{Y} \text{ reaches } \eta \text{ before time } T\}}]
\]

The last equation above uses the Cameron-Martin Theorem, and the \( \widetilde{Y} \) satisfies
\[
d\widetilde{Y}_i(t) = (a(\overline{Y}(t) - \widetilde{Y}_i(t) + \sigma \vartheta (\sqrt{\rho} + \sqrt{1-\rho}))dt + \sigma d\widetilde{W}_i(t)
\]
60
here $\hat{W}_i(t) = \sqrt{T - \rho W_i(t)} + \sqrt{\rho} W_0(t)$. 

So that here we can use the estimator

$$\hat{I}_n = \frac{1}{n} \sum_{k=1}^{n} \exp \{ -\vartheta \sum_{i=0}^{N} W_i^k(T) - \frac{\vartheta^2}{2} T(N + 1) \} \eta \left\{ \min_{0 \leq t \leq T} \hat{Y}(t) \leq \eta \right\}$$ (5)

as the estimate of the probability of systemic risk. From the procedure shown above we can easily know that this estimator is unbiased. Its variance can be expressed as

$$\text{Var}(\hat{I}_n) = \frac{1}{n} \text{Var}_0 \left( \exp \left\{ -\vartheta \sum_{i=0}^{N} W_i(T) - \frac{\vartheta^2}{2} T(N + 1) \right\} \eta \left\{ \min_{0 \leq t \leq T} \hat{Y}(t) \leq \eta \right\} \right)$$ (6)

$$= \frac{1}{n} \left( \mathbb{E}_0 \exp \left\{ -2\vartheta \sum_{i=0}^{N} W_i(T) - \vartheta^2 T(N + 1) \right\} \eta \left\{ \min_{0 \leq t \leq T} \hat{Y}(t) \leq \eta \right\} \right) - P^2$$

Here $P$ is the probability of the systemic risk. The variance of the estimator can also be estimated during the simulation itself, as it can be expressed as expectations and the expectations can be estimated. The estimator of the variance is

$$\hat{\text{Var}}(\hat{I}_n) = \frac{1}{n} \left( \frac{1}{n} \sum_{k=1}^{n} \exp \left\{ -\vartheta \sum_{i=0}^{N} W_i^k(T) - \frac{\vartheta^2}{2} T(N + 1) \right\} \eta \left\{ \min_{0 \leq t \leq T} \hat{Y}(t) \leq \eta \right\} \right)$$ (7)

$$- \left( \frac{1}{n} \sum_{k=1}^{n} \exp \left\{ -\vartheta \sum_{i=0}^{N} W_i^k(T) - \frac{\vartheta^2}{2} T(N + 1) \right\} \eta \left\{ \min_{0 \leq t \leq T} \hat{Y}(t) \leq \eta \right\} \right)^2$$

So we can do numerical experiments for the estimator of the probability and the variance of this estimator at the same time, and figure out whether the simulation works well. The one thing to be done here is to decide the exact $\vartheta$ here. Due to intuition, we know $\vartheta$ should have the same sign of $\eta$, and more or less around $\frac{\eta}{\sqrt{\rho + \sqrt{1 - \rho} T}}$ (which is because if $\vartheta = \frac{\eta}{\sigma(\sqrt{\rho + \sqrt{1 - \rho} T})}$, then the effect of this drift is to pull the process to level $\eta$ if not taking the random perturbation into account, thus by the virtue of the Brownian motion, the probability of the systemic risk, say $\hat{Y} < \eta$ should be around 1/2). During our simulation for 1 on page 3, for which we can derive the explicit expression of the variance, we have tried 200 possible $\vartheta$’s between $-2$ and 0 with equal gap between the adjacent candidates, and choose the one with the least theoretical variance. We could have also used a dichotomy method to minimize the theoretical variance. For 2 on page 3, because of lacking analytical expression of the variance, we just tried 10 different $\vartheta$’s around the intuition value, and use the one with the least simulated variance.

For the complex model 2 on page 3 with correlated Brownian motions, the procedure is quite similar. For convenience of the reader, we will describe the model again here.

Model:

$$dY_i(t) = -hU(Y_i(t))dt + \theta(\hat{Y}(t) - Y_i(t))dt + \sigma d\hat{W}_i(t) \quad i = 1, ..., N$$ (8)
where $U(y) = y^3 - y$, $\mathbf{Y}(t)$ and $\mathbf{W}$ is defined as in 4 on page 7.

The expression for the estimator of the probability and the corresponding variance is exactly the same as mentioned above, except that $\mathbf{Y}$ satisfies the following

$$d\mathbf{Y}_t = -hU(\mathbf{Y}_t)dt + (\theta(\mathbf{Y}(t) - \bar{\mathbf{Y}}(t)) + \sigma d(\sqrt{\rho} + \sqrt{1-\rho}))dt + \sigma d\mathbf{W}_t$$

and the systemic risk event we choose to be $\{ \max_{0 \leq t \leq T} \mathbf{Y}(t) \geq \xi_b\}$, where $\xi_b$ is the positive stable status of the distribution

$$u^\xi(y) = \frac{1}{Z_{\xi} \sqrt{2\pi \sigma^2}} \exp\left\{ -\frac{(y-\xi)^2}{2\sigma^2} - h\frac{\sigma^2}{\sigma^2 + \theta} V(y) \right\}$$

where $V(y) = \frac{1}{4} y^4 - \frac{1}{2} y^2$. That is to say, $\xi_b$ is the positive solution of the equation

$$\xi = \int y u^\xi(y) dy$$

We view $\xi_b$ as the failed status, and it has the expression

$$\xi_b = \sqrt{1 - \frac{3\sigma^2}{2\theta}} (1 + h \frac{6\sigma^2}{2\theta} \left( \frac{1 - 2(\sigma^2/2\theta)}{1 - 3(\sigma^2/2\theta)} \right) + O(h^2)$$

for $h$ small.

Here if we take $\rho = 0$, then the models we considered here become the independent Brownian motions cases, and the estimates also work for the corresponding models.

### 3 Results and Discussion

In this section, we show the main simulation results of our project, as well as the discussion along with it. The catalogue is as follows.

- Simulation result Vs. Theoretical value 4 on page 7
- Importance Sampling & Large Deviations Principle 8 on the preceding page
- The effect of the correlated Brownian Motion 8 on the previous page
- The tradeoff between $h$ and $\theta$ 8 on the preceding page

#### 3.1 Simulation result Vs. Theoretical value

In this subsection, I will show my simulation works well in the model 4 on page 7. The parameter I use is as follows.

$$N = 10, \alpha = 100, \sigma = 1, \rho = 0.01, T = 1, \eta = -0.7$$
The parameter for IS simulation is as follows.

\[ n = 1000, \ \vartheta = -1.66 \]

Thanks to this model, the theoretical value can be calculated. By adding 4 on page 7 among \( i = 1, \ldots, N \) and taking the average, we have

\[
d\overline{Y}(t) = \frac{\sigma}{N} d\left( \sum_{i=1}^{N} \tilde{W}_i(t) \right)
\]

with the initial value \( \overline{Y}(0) = 0 \). So

\[
\overline{Y}(t) = \frac{\sigma}{N} \sum_{i=1}^{N} \tilde{W}_i(t) = \frac{\sigma}{N} \sqrt{1 - \rho} \sum_{i=1}^{N} W_i(t) + \sigma \sqrt{\rho} W_0(t)
\]

Then

\[
P(\min_{0 \leq t \leq T} \overline{Y}_t \leq \eta) = P(\min_{0 \leq t \leq T} (\frac{\sigma}{N} \sqrt{1 - \rho} \sum_{i=1}^{N} W_i(t) + \sigma \sqrt{\rho} W_0(t)) \leq \eta)
\]

\[
= P(\min_{0 \leq t \leq T} W_t \leq \frac{\eta}{\sigma} \sqrt{\frac{N}{1 - \rho + N \rho}})
\]

\[
= P(|W_T| \geq -\frac{\eta}{\sigma} \sqrt{\frac{N}{1 - \rho + N \rho}})
\]

\[
= 2\Phi\left(\frac{\eta \sqrt{N}}{\sigma \sqrt{T}} \frac{1}{\sqrt{1 - \rho + N \rho}}\right)
\]

(10)

Here the third equation is because \( \min_{0 \leq t \leq T} W_t \) has the same law as \(-|W_T|\), which can be proved easily by the reflection principle.

With the given parameter values, we can calculate the theoretical value of the probability of the systemic risk, and get \( p = P(\min_{0 \leq t \leq T} \overline{Y}_t \leq \eta) = 0.0340 \).

For our estimator 5 on page 8, by Central Limit Theorem, as \( n \to \infty \),

\[
\hat{I}_n \to N(\mathbb{E}[I_1], \frac{1}{n} Var(I_1))
\]

where \( I_1 = \exp \{-\vartheta \sum_{i=0}^{N} W_i(T) - \frac{\sigma^2}{2} T (N+1)\} \mathbb{1}_{\{|\min_{0 \leq t \leq T} \overline{Y}_t| \leq \eta\}} \)

Here we know \( \mathbb{E}[I_1] = p \) and \( \frac{1}{n} Var(I_1) = Var(\hat{I}_n) \), which can be approximated by \( \hat{Var}(\hat{I}_n) \) as shown in 7 on page 8.

So we should have \( P(p \in (\hat{I}_n - 1.96\hat{Var}(\hat{I}_n), \hat{I}_n + 1.96\hat{Var}(\hat{I}_n))) \approx 0.95 \).

The following two figures are the 100 simulation results of \( \hat{I}_n \) and \( n\hat{Var}(\hat{I}_n) \) (estimate for \( Var(I_1) \))
1 and 2 on the following page are boxplots of the estimated probability $\hat{I}_n$ and the estimated variance $n\hat{\text{Var}}(I_n) = \hat{\text{Var}}(I_1)$. In either figure, each column plots the results of 20 simulated samples, with the red line in the middle representing the median of the corresponding set of samples, the edges of the box representing the 25th and 75th percentiles. The whiskers show the most extreme sample points considered to be not outliers, and the outliers are plotted individually. From the sample sets above, which is in total 100 samples, when we consider the 95% confidence interval, the number of times that the true value lies within the confidence interval we constructed, which is, the number of times that $p \in (\hat{I}_n - 1.96\hat{\text{Var}}(I_n), \hat{I}_n + 1.96\hat{\text{Var}}(I_n))$ holds, is 95, which exactly corresponds to the theoretical expectation.

The result here shows our method works well for simple model 4 on page 7, and we can then move on to our main target.

3.2 Importance Sampling & Large Deviations Principle

This part is the main part of our report, and we will only discuss the uncorrelated case here (2 on page 3 with $\rho = 0$), for the LDP result in [2] is only considering uncorrelated
Brownian Motions. We will mainly show that the LDP combined with IS sampling technique together can give out really nice predictions for the probability of systemic risk in the regime of small change of parameters.

Since the LDP result in [2] only works for small $h$, we will only consider small $h$ here. There is still something to be mentioned here. In the following tables, I will first give out a set of reference parameters, say $(h_0, N_0, \theta_0, \sigma_0, T_0)$, and get the IS simulation result $\hat{p}_{IS}(h_0, N_0, \theta_0, \sigma_0, T_0)$. Then for small change of parameters, we will compare our prediction based on LDP result, which is

$$\hat{p}_{prediction}(h, N, \theta, \sigma, T) = \frac{p_{LDP}(h, N, \theta, \sigma, T)}{p_{LDP}(h_0, N_0, \theta_0, \sigma_0, T_0)} \hat{p}_{IS}(h_0, N_0, \theta_0, \sigma_0, T_0)$$  \hspace{1cm} (11)

with the IS simulation result $\hat{p}_{IS}(h, N, \theta, \sigma, T)$. Here $p_{LDP}$ has a simple explicit expression as described below.
3.2.1 Tables

Each IS simulation result here is derived with \( n = 8000 \) simulations, with \( \theta \) in the IS simulation to be 0.12. We assume all the components start from \(-1\), and take the event 'the mean of all the components reach the failed state, say \( \xi_b \)' as the systemic risk event.

The LDP result is taken from [2], which is

\[
p_{LDP} = \exp \left( -N \frac{2\xi_b^2}{\sigma^2 T} \right) \tag{12}
\]

where \( \xi_b = \sqrt{1 - 3 \frac{\sigma^2}{2\theta}} \left( 1 + h \frac{6}{\sigma^2} \left( \frac{\sigma^2}{2\theta} \right) \left( \frac{1-2\sigma^2/2\theta}{1-3(\sigma^2/2\theta)} \right) \right) + O(h^2). \)

<table>
<thead>
<tr>
<th>( N )</th>
<th>( \theta )</th>
<th>( \sigma )</th>
<th>( T )</th>
<th>( \hat{p}_{\text{prediction}} )</th>
<th>( \hat{p}_{\text{IS}} )</th>
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<td>100</td>
<td>10</td>
<td>1</td>
<td>15</td>
<td>6.25 \times 10^{-7}</td>
<td>6.25 \times 10^{-7}</td>
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<tr>
<td>100</td>
<td>11</td>
<td>1</td>
<td>15</td>
<td>5.22 \times 10^{-7}</td>
<td>5.56 \times 10^{-7}</td>
</tr>
</tbody>
</table>

Table 1: Comparison between IS estimates of the probability of systemic risk with LDP predictions. Here \( h = 0.02 \) and we look for changes of parameters that induce smaller probabilities.

<table>
<thead>
<tr>
<th>( N )</th>
<th>( \theta )</th>
<th>( \sigma )</th>
<th>( T )</th>
<th>( \hat{p}_{\text{prediction}} )</th>
<th>( \hat{p}_{\text{IS}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
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<td>15</td>
<td>6.28 \times 10^{-7}</td>
<td>6.28 \times 10^{-7}</td>
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<tr>
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<td>10</td>
<td>1</td>
<td>16</td>
<td>1.28 \times 10^{-6}</td>
<td>1.36 \times 10^{-6}</td>
</tr>
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<td>90</td>
<td>10</td>
<td>1</td>
<td>15</td>
<td>1.95 \times 10^{-6}</td>
<td>2.34 \times 10^{-6}</td>
</tr>
<tr>
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</tr>
<tr>
<td>100</td>
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<td>1</td>
<td>15</td>
<td>7.83 \times 10^{-7}</td>
<td>6.99 \times 10^{-7}</td>
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</tbody>
</table>

Table 2: Comparison between IS estimates of the probability of systemic risk with LDP predictions. Here \( h = 0.02 \) and we look for changes of parameters that induce larger probabilities.
Table 3: Comparison between IS estimates of the probability of systemic risk with LDP predictions. Here \( h = 0.1 \) and we look for changes of parameters that induce smaller probabilities.

<table>
<thead>
<tr>
<th>( N )</th>
<th>( \theta )</th>
<th>( \sigma )</th>
<th>( T )</th>
<th>( \hat{p}_{\text{prediction}} )</th>
<th>( \hat{p}_{\text{IS}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>10</td>
<td>1</td>
<td>15</td>
<td>( 5.19 \times 10^{-7} )</td>
<td>( 5.19 \times 10^{-7} )</td>
</tr>
<tr>
<td>100</td>
<td>10</td>
<td>1</td>
<td>14</td>
<td>( 2.30 \times 10^{-7} )</td>
<td>( 2.09 \times 10^{-7} )</td>
</tr>
<tr>
<td>110</td>
<td>10</td>
<td>1</td>
<td>15</td>
<td>( 1.66 \times 10^{-6} )</td>
<td>( 1.55 \times 10^{-6} )</td>
</tr>
<tr>
<td>100</td>
<td>10</td>
<td>0.9</td>
<td>15</td>
<td>( 2.27 \times 10^{-8} )</td>
<td>( 1.86 \times 10^{-8} )</td>
</tr>
<tr>
<td>100</td>
<td>11</td>
<td>1</td>
<td>15</td>
<td>( 4.35 \times 10^{-7} )</td>
<td>( 4.58 \times 10^{-7} )</td>
</tr>
</tbody>
</table>

Table 4: Comparison between IS estimates of the probability of systemic risk with LDP predictions. Here \( h = 0.1 \) and we look for changes of parameters that induce larger probabilities.

<table>
<thead>
<tr>
<th>( N )</th>
<th>( \theta )</th>
<th>( \sigma )</th>
<th>( T )</th>
<th>( \hat{p}_{\text{prediction}} )</th>
<th>( \hat{p}_{\text{IS}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>10</td>
<td>1</td>
<td>15</td>
<td>( 5.19 \times 10^{-7} )</td>
<td>( 5.19 \times 10^{-7} )</td>
</tr>
<tr>
<td>100</td>
<td>10</td>
<td>1</td>
<td>16</td>
<td>( 1.06 \times 10^{-6} )</td>
<td>( 1.15 \times 10^{-6} )</td>
</tr>
<tr>
<td>100</td>
<td>10</td>
<td>1</td>
<td>15</td>
<td>( 1.62 \times 10^{-6} )</td>
<td>( 1.91 \times 10^{-6} )</td>
</tr>
<tr>
<td>100</td>
<td>10</td>
<td>1.1</td>
<td>15</td>
<td>( 5.25 \times 10^{-6} )</td>
<td>( 6.90 \times 10^{-6} )</td>
</tr>
<tr>
<td>100</td>
<td>9</td>
<td>1</td>
<td>15</td>
<td>( 6.43 \times 10^{-7} )</td>
<td>( 5.64 \times 10^{-7} )</td>
</tr>
</tbody>
</table>

\[ \text{3.2.2 Discussion} \]

From Table 1 to Table 4, we can see that in the case that parameters other than \( h \) change in the scale of about 10\%, the \( \hat{p}_{\text{prediction}} \), using 11 on page 12 gives a quite good estimate for \( \hat{p}_{\text{IS}} \), which is in turn, a representative of the true probability of the systemic risk. The error of the estimate \( \hat{p}_{\text{prediction}} \) compared to \( \hat{p}_{\text{IS}} \) is in general no larger than 15\%.

Amongst all the parameters considered above, the parameter \( \sigma \) and \( \theta \) is quite sensitive, but within the scale of 10\% change, the prediction result \( \hat{p}_{\text{prediction}} \) can be still viewed as a considerably nice estimate of the true probability.

<table>
<thead>
<tr>
<th>( h )</th>
<th>( \hat{p}_{\text{prediction}} )</th>
<th>( \hat{p}_{\text{IS}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.02</td>
<td>( 6.28 \times 10^{-7} )</td>
<td>( 6.28 \times 10^{-7} )</td>
</tr>
<tr>
<td>0.05</td>
<td>( 6.21 \times 10^{-7} )</td>
<td>( 6.29 \times 10^{-7} )</td>
</tr>
<tr>
<td>0.075</td>
<td>( 6.16 \times 10^{-7} )</td>
<td>( 6.21 \times 10^{-7} )</td>
</tr>
<tr>
<td>0.1</td>
<td>( 6.10 \times 10^{-7} )</td>
<td>( 5.19 \times 10^{-7} )</td>
</tr>
<tr>
<td>0.15</td>
<td>( 5.99 \times 10^{-7} )</td>
<td>( 3.38 \times 10^{-7} )</td>
</tr>
<tr>
<td>0.2</td>
<td>( 5.89 \times 10^{-7} )</td>
<td>( 1.92 \times 10^{-7} )</td>
</tr>
</tbody>
</table>

Table 5: Comparison between IS estimates of the probability of systemic risk with LDP predictions. Here \( N = 100, \theta = 10, \sigma = 1, T = 15 \), and we investigate the effect of change of the parameter \( h \).
From Table 5, we know the effect of parameter \( h \) in the prediction of the probability of systemic risk \( \hat{p}_{\text{prediction}} \). For \( h < 0.1 \), the prediction \( \hat{p}_{\text{prediction}} \) is quite close to \( \hat{p}_{\text{IS}} \), which is a representative of the true probability. However, for \( h \geq 0.1 \), we can see that the prediction \( \hat{p}_{\text{prediction}} \) is quite far away from the true probability. One possible reason is that the LDP theory we applied here on page 13 only works for small \( h \), so for comparatively large \( h \), the LDP result does not work well. On the other hand, the scale of change of parameter \( h \) is also quite large (far more than 10\%, even if we consider 0.075 as \( h_0 \)), so it is reasonable that the prediction based on reference parameters is not that good.

After all, we know from all these tables above that for small \( h \) and small changes of parameters, the strategy we use, which is to use LDP result and the reference IS result to do prediction for the probability of systemic risk of close parameters, works quite well. This result will be of essential importance, because therefore, we will save a lot of time in estimating the probability of systemic risk for a set of parameters that do not differ so much, which only costs one round of IS simulations and nearly cost-free prediction procedures by 11 on page 12.

### 3.3 The effect of correlated Brownian Motions

In most articles investigating systemic risk, the independent Brownian motions assumption is taken. However, in real life, each component can not be totally independent. In finance, there should be some common influence from the whole market, or some local market affecting a portion of the components. Professor Francois Delarue also mentioned that by introducing the common Brownian motion, the analytic properties of the systemic risk problem, say, the existence and uniqueness of the more complex stochastic differential equations can be achieved [3].

So it is of interest to investigate the situation of correlated Brownian motions, and here we will do several numerical simulations with respect to different levels of correlated Brownian motions and compare the corresponding probability of systemic risk.

The model we adopt here is 8 on page 8 with the parameters taken as follows:

\[
h = 0.1, \quad N = 100, \quad \theta = 10, \quad \sigma = 1, \quad T = 15
\]

The parameters for simulation is taken as follows:

\[
\vartheta = 0.12, \quad n = 2000, \quad \text{default level} = 1
\]

Here default level means at which level the empirical average of the components reaches we consider as the systemic risk taking place. The value of the default level here is a little bit different from the ones in the numerical simulations in the early parts of this report; this is because here for different correlated situations (different \( \rho \)), the failed state of the system may varies, so it would be good to compare the probability of systemic risk in one common default setting.

Here \( \overline{\text{Var}}_{\text{IS}} \) is \( \overline{\text{Var}}(\overline{I}_1) = n\overline{\text{Var}}(\overline{I}_n) \), where the latter is derived by 7 on page 8. The *’s that appear are to show that the results in the last two rows are derived by Monte Carlo method, not IS method, for the probabilities of systemic risk in these two situations are so large that IS method is not necessary.
From the table above, we can find out obviously that, with the increasing of the parameter $\rho$, the probability of the systemic risk increases considerably. This phenomenon is also easy to understand. With the increase of $\rho$, the influence of the common random perturbation increases in the system, then if the common 'noise' directs to the default level, it is more probable that almost the whole system fails and the systemic risk happens.

### 3.4 The tradeoff between $h$ and $\theta$

In this section, we numerically confirm one statement from [2], which is '...,when the number of agents $N$ is large and when $h$ is small. In this regime of parameters we find that systemic risk increases with cooperation...' In other words, that is to say, under the model 2 on page 3, when $N$ is large and $h$ is small, to keep individual risk unchanged\(^1\), increasing cooperation will cause an increasing in the probability of systemic risk. We carried out a set of simulations to test this statement.

We calculate the individual risk with respect to distribution function (2.2) given in [2], which is

$$u_\xi(y) = \frac{1}{Z_\xi} \exp \left\{ - \frac{(y - \xi)^2}{2 \frac{\sigma^2}{\theta}} - h \frac{2 \sigma^2}{\theta} V(y) \right\}$$

(13)

where

$$V(y) = \frac{3}{4} y^4 - \frac{1}{2} y^2$$

and $\xi$ is calculated by

$$\xi = \sqrt{1 - 3 \frac{\sigma^2}{2\theta} (1 + h \frac{6 \sigma^2 (\sigma^2/2\theta)^2}{2 \theta} + O(h^2))}$$

as before.

We take the variance of the above distribution as a representative of the individual risk, and modify the parameters $h$ and $\theta$ simultaneously to keep the individual risk stationary. The default level we consider here is $\xi$, as we did in Section 2.

Parameters we use:

$$N = 100, \; \sigma = 1, \; T = 15$$

Parameters for simulation:

$$n = 2000, \; \vartheta = 0.12$$

\(^1\) Although this condition is not mentioned in the statement cited, it is implied in the context of [2].
Table 7: Tradeoff between $h$ and $\theta$

<table>
<thead>
<tr>
<th>$h$</th>
<th>$\theta$</th>
<th>$\hat{p}_{IS}$</th>
<th>$\hat{\text{Var}}_{IS}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>20</td>
<td>$3.34 \times 10^{-7}$</td>
<td>$9.45 \times 10^{-13}$</td>
</tr>
<tr>
<td>0.05</td>
<td>19.9</td>
<td>$3.00 \times 10^{-7}$</td>
<td>$1.08 \times 10^{-12}$</td>
</tr>
<tr>
<td>0.1</td>
<td>19.8</td>
<td>$2.19 \times 10^{-7}$</td>
<td>$1.45 \times 10^{-5}$</td>
</tr>
<tr>
<td>0.2</td>
<td>19.6</td>
<td>$7.14 \times 10^{-8}$</td>
<td>$2.32 \times 10^{-13}$</td>
</tr>
<tr>
<td>0.5*</td>
<td>19.1</td>
<td>$1.54 \times 10^{-11}$</td>
<td>$3.99 \times 10^{-19}$</td>
</tr>
</tbody>
</table>

In the table above, we always keep the variance of the distribution 13 on the previous page at the value $2.50 \times 10^{-2}$. The last line with * represents that the sample size $n$ in that simulation is different than the rest, which is $n = 10^4$.

From the table above, we can easily figure out that while $h$ increases and $\theta$ decreases, the systemic risk decreases considerably. If we view it in the opposite direction, then we can get the statement we want, which is, as we keep the individual risk unchanged, increasing collaboration (increasing $\theta$) will cause an increase in the probability of systemic risk.

4 Summary and Conclusion

The aim of this report is to study an original strategy to estimate the probability of systemic risk of the model proposed in [2], which is a strategy combining an Importance Sampling method with the help of Cameron-Martin theorem and Large Deviations Principle proved in [2]. The model used here is a system of bistable diffusion processes that interact through their empirical mean. The random processes generating this system can be either independent or correlated. The system has several parameters: the parameter $N$ is the number of interacting agents, the parameter $h$ is a measure of intrinsic stability of each agent, the parameter $\theta$ is a measure of collaboration between agents, and the parameter $\sigma$ is the strength of the external driving noise. Our strategy is as follows. For sets of parameters close to each other, pick up one set as the reference set and use the Importance Sampling method to get an estimate of the probability of systemic risk for this set of parameters. For the other sets of parameters, use Large Deviations Principle to get the ratio between the probability of systemic risk of the corresponding set of parameters with respect to that of the reference set, and multiply this ratio by the simulated probability of systemic risk of the reference set. The formula has been written in Section 3.2. We have successfully shown that for small $h$ and small changes in parameters, this method works quite well, and saves a lot of computational times and therefore allows for a sensitivity analysis of the systemic risk (sensitivity with respect to the parameters of the model). Besides that, we have shown that, when the driving Brownian motions are correlated, more correlation will enhance the probability of systemic risk. We have also numerically confirmed the statement that, when individual risk is kept to a fixed value, in the regime of $h$ small and $N$ large, the probability of systemic risk increases when cooperation increases, which was predicted by Large Deviations Theory in [2].
Acknowledgement

Many thanks to Ecole Normale Superieure for helping me with my living and studying there. Thanks a lot to my university, Tsinghua University, for providing me with this opportunity to do my undergraduate thesis at ENS. Most importantly, thanks to my advisor Josselin Garnier for his instruction and help for me, I am really grateful to them.

References


偏微分方程在期权定价中的应用

王艺博
指导教师：江宁 副教授

摘要

在近代金融工程学中，数学方法有着重要的应用。通过将金融理论与数学推导的整合，很多以前只能凭主观预测而做出的决定，现在可以通过准确的数学理论模型来给出更加令人信服的结论。Black-Scholes 模型就是一个偏微分方程与金融理论结合得出的漂亮的结论，它给出了期权定价与金融市场变量的关系。

本篇论文，主要介绍了纽约大学柯朗数学所 Robert V. Kohn 教授在其研究生 PDE for Finance 课上的讲义[1]的一部分，重点内容为 Black-Scholes 微分方程与经典热方程的联系和其在各种期权定价中的应用。为了使论文内容更加完整并且更具有可读性，我还参考了经典的金融学圣经多伦多大学 John Hull 教授所著的《Option Futures and Other Derivatives》和北京大学的《偏微分方程》教材。

关键词：偏微分方程; 金融理论; Black-Scholes 模型; 期权定价
第 1 章 引言

1.1 研究背景

1.1.1 衍生品证券的发展

在过去的三十年中，衍生品证券在金融学中变得越来越重要。人们不仅交易股票，债券和大众商品，而且更多的买卖期货期权这类金融衍生品，他们的价格依据股票，债券和大众商品的价格。

期权产生于 18 世纪末。刚开始的时候，由于不完整的制度，期权交易受到抑制。在 19 世纪 20 年代，看涨看跌期权的交易商都是职业交易员。他们只是当价格对他们有利的时候才报价，这样的交易缺乏普遍性而且很难交易，所以那时的交易的流动性收到限制。

1973 年 4 月 26 日，芝加哥期权交易所开业。期权的买卖更加标准化和一致化。随着期权的交易越来越普遍，怎样给期权定价成了一个很大也很严肃的问题。在 1900 年，法国金融经济学家 Bachelier 发表了第一篇关于期权定价的文章[2]。在这之后，众多学者提出了很多经典公式和数量模型，但是由于种种局限都没有被广泛使用。

最早完整的期权定价模型是由 Fisher Black 和 Myron Scholes 在 1973 年提出的[3]。几乎在同一时间，Merton 找到了同样的公式给期权定价还有一些其他关于期权定价有用的结论。现在 Black-Scholes-Merton 模型被广泛应用在金融领域。Scholes 和 Merton 因为他们在期权定价方面做出的贡献获得了诺贝尔奖，当时 Black 已经去世了。

1.1.2 偏微分方程的发展

偏微分方程是由 Euler, Poisson 等人在 17 世纪兴起。19 世纪，偏微分方程随着数学在物理中的应用而快速发展。法国数学家傅里叶在他的热传导系统中提出偏微分方程。现在随着现代金融的快速发展，数学方法在金融中有着广泛的应用。尤其是偏微分方程在金融，如期权定价领域。
第 2 章 预备知识

2.1 一些基本的金融概念

看涨期权

看涨期权给予期权持有人在某一预先设定的时间以某一预先设定的价格购买一项资产的权利。

看跌期权

看跌期权给予期权持有人在某一预先设定的时间以某一预先设定的价格售出一项资产的权利。

到期时间

期权产品所指定的特定时间被称为到期时间。

执行价格

期权产品中所说的确定价格被称为执行价格。

美式期权

美式期权是指在到期日之前的任何时间，期权持有人可以行权的一种期权。

欧式期权

欧式期权是指期权持有人只有在预先设定的到期日才可以行权的一种期权。

波动率

股票波动率是用于度量股票所提供收益不确定性的量，一般介于 15%~60%之间。

障碍期权

\[5\]

\(\) 定义引自 [5]
障碍期权取决于标的资产价格在一段时间内是否达到某个特定水平。
下跌-敲出看涨期权是一种普通看涨期权，但当资产价格下跌到一障碍水平 $X$ 时，期权自动消失；
下跌-敲入看涨期权是一种普通看涨期权，但只有当资产价格下跌到一定水平 $X$ 时，期权才会生效；
上升-敲出看涨期权是一种普通看涨期权，但当资产价格到一障碍水平 $X$ 时，期权自动消失；
上升-敲入看涨期权是一种普通看涨期权，但只有当资产价格上升到一定水平 $X$ 时，期权才会生效。

2.2 关于偏微分方程和热方程的一些基本概念

定义 2.2.1
偏微分方程是与一个未知的多元函数及它的偏导数有关的方程。

定义 2.2.2
我们用符号 $\Delta$ 代表拉普拉斯算子，定义 $\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$，如果我们有一个偏微分方程，且满足 $\Delta u = 0$，则我们称这是拉普拉斯方程。

定义 2.2.3
如果我们有一个偏微分方程，满足 $u_t - a^2 \Delta u = f$，其中 $u = u(x,t)$，$x = (x_1,x_2,\cdots,x_n) \in \Omega \subset R^n$，$u_t = \frac{\partial u}{\partial t}$，$t$ 和 $a$ 是两个大于零的常数，$f = f(x,t)$ 是一已知函数，则我们称这个方程为热方程。

要解一个偏微分方程，仅知道方程的形式是不够的，我们需要一些其他条件。热方程中，我们有一些如下简单的条件：
初值条件：
$$u(x,0) = \varphi(x), x \in \Omega, \varphi(x) \text{ 是一个已知函数}$$

\(^{\circledast}\) 定义引自[8]
边值条件:
\[ u(x, t) = g(x, t), \quad x \in \partial \Omega, \quad t \geq 0 \]
运用常微分方程和傅里叶变换的知识，我们可以求解如下热方程:
\[
\begin{aligned}
\frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} &= f(x, t), \quad (x, t) \in R \times R_+ \\
u(x, 0) &= \varphi(x), \quad x \in R
\end{aligned}
\quad (2-2-1)
\]
下面我们将会介绍如何求解。先介绍傅里叶变换的概念和性质。

定义 2.2.4
令 \( f(x) \in L^1(R) \), 则无穷积分
\[
\hat{f} (\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{f}(x) e^{-i\lambda x} dx
\]
存在，它被称作 \( f(x) \) 的傅里叶变换，记 \((f(x))^\wedge = \hat{f}(\lambda)\)

定理 2.2.5
令 \( f(x) \in L^1(R) \cap C^1(R) \), 则对任意 \( x \in R \), 有
\[
\lim_{N \to +\infty} \frac{1}{\sqrt{2\pi}} \int_{-N}^{+N} \hat{f}(\lambda) e^{i\lambda x} d\lambda = f(x)
\]
这被称作反演公式，上面积分左侧被称作傅里叶逆变换，记为 \( \hat{f}(\lambda)^\vee \)

性质 2.2.6
(1) 若 \( f_i(x) \in L^1(R), \ a_i \in C(i = 1, 2) \), 则
\[
(a_1 f_1(x) + a_2 f_2(x))^\wedge = a_1 \hat{f}_1(\lambda) + a_2 \hat{f}_2(\lambda)
\]
(2) 若 \( f(x), f'(x) \in L^1(R) \cap C(R) \), 则
\[
(f'(x))^\wedge = i\lambda \hat{f}(\lambda)
\]
(3) 若 \( f(x), xf(x) \in L^1(R) \), 则
\[
(xf(x))^\wedge = i \frac{d}{d\lambda} \hat{f}(\lambda)
\]
(4) 对任意常数 $k$, 若 $f(x) \in L^1(\mathbb{R})$, 则
\[(f(x - k))^ = e^{-ik\lambda}\hat{f}(\lambda)\]

(5) 若 $f(x) \in L^1(\mathbb{R})$, 则
\[(f(kx))^ = \frac{1}{|k|}\hat{f}\left(\frac{\lambda}{k}\right), \quad k \neq 0\] 是一个常数

(6) 若 $f(x) \in L^1(\mathbb{R})$, 则
\[(f(x))^v = \hat{f}(-\lambda)\]

(7) 若 $f(x), g(x) \in L^1(\mathbb{R})$, 定义 $f \ast g(x) = \int_{-\infty}^{+\infty} f(x - t)g(t)dt$，则
\[f \ast g(x) \in L^1(\mathbb{R})\] 并且 $(f \ast g(x))^v = \sqrt{2\pi}\hat{f}(\lambda)\hat{g}(\lambda)$

好了，现在我们有了傅里叶变换这个工具，回到初始问题，对方程（2.2.1）

关于 $x$ 运用傅里叶变换，再应用性质（1）和（2），可以得到：
\[
\begin{aligned}
\frac{d\hat{u}}{dt} + a^2\lambda^2\hat{u} &= \hat{f}(\lambda, t) \\
\hat{u}(\lambda, 0) &= \hat{\phi}(\lambda)
\end{aligned}
\]
解这个常微分方程初值问题，得：
\[
\hat{u}(\lambda, t) = \hat{\phi}e^{-a^2\lambda^2t} + \int_0^t \hat{f}(\lambda, \tau)e^{-a^2\lambda^2(t-\tau)}d\tau
\]
做傅里叶逆变换，得：
\[
u(x, t) = (\hat{u}(\lambda, t))^v = (\hat{\phi}e^{-a^2\lambda^2t})^v + \int_0^t (\hat{f}(\lambda, \tau)e^{-a^2\lambda^2(t-\tau)})^v d\tau
\]
由性质（5），得：
\[e^{-a^2\lambda^2t} = (g(x, t))^, \quad \text{其中} \quad g(x, t) = \frac{1}{a\sqrt{2\pi}}e^{-\frac{x^2}{4a^2t}}\]
所以，
\[(\hat{\phi}e^{-a^2\lambda^2t})^v = (\hat{\phi}\hat{g})^v = \frac{1}{\sqrt{2\pi}}\varphi \ast g = \frac{1}{2a\sqrt{\pi}t} \int_{-\infty}^{+\infty} \varphi(\epsilon)e^{-\frac{(x-\epsilon)^2}{4a^2t}}d\epsilon
\]
类似的，
\[
(f(\lambda, \tau) e^{-a^2 \lambda^2 (t-\tau)})^\vee = \frac{1}{2a\sqrt{\pi(t-\tau)}} \int_{-\infty}^{+\infty} f(\epsilon, \tau) e^{-\frac{(x-\epsilon)^2}{4a^2(t-\tau)}} d\epsilon
\]

最终，我们得到：

\[
u(x, t) = \int_{-\infty}^{+\infty} K(x - \epsilon, t) \varphi(\epsilon) d\epsilon + \int_{0}^{t} d\tau \int_{-\infty}^{+\infty} K(x - \epsilon, t - \tau) f(\epsilon, \tau) d\epsilon
\]

这里

\[
K(x, t) = \begin{cases} 
\frac{1}{2a\sqrt{\pi t}} e^{-\frac{x^2}{4a^2t}}, & t \geq 0 \\
0, & t \leq 0
\end{cases}
\]

我们称 \(K(x, t)\) 为 Poisson 核。

接下来，我们给出热方程边值初值问题的解

\[
\begin{align*}
\begin{cases}
u_t - a^2 \nu_{xx} = f(x, t) & (x, t) \in (0, l) \times (0, T) \\
u(x, 0) = \varphi(x) & x \in [0, l] \\
u(0, t) = g_1(t), u(l, t) = g_2(t) & t \in [0, T]
\end{cases}
\end{align*}
\]

(2-2-2)

省去推导过程，我们直接给出解的表述形式

\[
u(x, t) = \int_{0}^{t} \varphi(\epsilon) \frac{2}{l} \sum_{n=1}^{\infty} \sin \frac{n\pi}{l} \epsilon \sin \frac{n\pi}{l} x e^{-\frac{(n\pi a)^2}{4l^2}(t-\tau)} d\epsilon
\]

\[
+ \int_{0}^{t} d\tau \int_{0}^{t} f(\epsilon, \tau) \frac{2}{l} \sum_{n=1}^{\infty} \sin \frac{n\pi}{l} \epsilon \sin \frac{n\pi}{l} x e^{-\frac{(n\pi a)^2}{4l^2}(t-\tau)} d\epsilon
\]

记 \(G(x, t; \epsilon, \tau) = \frac{2}{l} \sum_{n=1}^{\infty} \sin \frac{n\pi}{l} \epsilon \sin \frac{n\pi}{l} x e^{-\frac{(n\pi a)^2}{4l^2}(t-\tau)} H(t-\tau), \ t > \tau\)

上式可以化简为

\[
u(x, t) = \int_{0}^{t} G(x, t; \epsilon, 0) \varphi(\epsilon) d\epsilon + \int_{0}^{t} d\tau \int_{0}^{t} G(x, t; \epsilon, \tau) f(\epsilon, \tau) d\epsilon
\]

2.3 维纳过程和伊藤引理

定义 2.3.1

\footnote{定义引自[5]}
如果一个随机过程，只有标的变量当前值与未来的预测有关，变量的历史与未来的预测无关，则称之为马尔科夫过程。

定义 2.3.2

维纳过程一种期望为 0，方差率为 1 的特殊的马尔科夫过程，用 $dz$ 表示。

定义 2.3.3

由 $dz$ 给出广义维纳过程：

$$dx = adt + b dz$$

其中 $a$ 和 $b$ 是常数

定义 2.3.4

我们进一步定义伊藤过程。伊藤过程是一个更为广义的维纳过程，其中 $a,b$ 均为变量 $x$ 和时间 $t$ 的函数。伊藤过程的表达式为：

$$dx = a(x,t)dt + b(x,t)dz$$

引理 2.3.5（伊藤引理）

假设变量 $x$ 服从伊藤过程：

$$dx = a(x,t)dt + b(x,t)dz$$

(2-3-1)

其中 $dz$ 是维纳过程，$a,b$ 均为变量 $x$ 和时间 $t$ 的函数。伊藤引理 $x$ 和 $t$ 的函数 $G$ 遵循以下过程：

$$dG = \left( \frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2 \right) dt + \frac{\partial G}{\partial x} b dz$$

其中 $dz$ 是和 (2-3-1) 中一样的维纳过程。

第 3 章 偏微分方程在期权定价中的应用

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3.1 Black–Scholes 微分方程

我们假设股票价格满足以下随机过程

\[ dS = \mu S dt + \sigma S dz \]  (3-1-1)

其中 \( \mu \) 是股票的预期收益率，\( \sigma \) 是股票价格的波动率。事实上，这个模型是期权定价二叉树模型在时间间隔趋近于 0 时候的极限形式。

假设 \( V \) 是一份看涨期权的价格。\( V \) 是股票价格 \( S \) 和时间 \( t \) 的函数。由伊藤引理可得到:

\[ dV = \left( \frac{\partial V}{\partial S} \mu S + \frac{\partial V}{\partial t} + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial V}{\partial S} \sigma S dz \]  (3-1-2)

(3-1-1) 和 (3-1-2) 两式的离散形式为:

\[ \Delta S = \mu S \Delta t + \sigma S \Delta z \]  (3-1-3)

和

\[ \Delta V = \left( \frac{\partial V}{\partial S} \mu S + \frac{\partial V}{\partial t} + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \sigma^2 S^2 \right) \Delta t + \frac{\partial V}{\partial S} \sigma S \Delta z \]  (3-1-4)

方程 (3-1-3) 和 (3-1-4) 中的 \( \Delta z \) 是一样的。这样我们可以构造一个由股票和标的该股票的期权组成的投资组合，以使维纳过程抵消。

这样的投资组合是:

\(-1\): 期权

\(+ \frac{\partial V}{\partial S}\): 股票

这时投资组合的持有人手里持有一短头寸的期权和 \( \frac{\partial V}{\partial S} \) 长头寸的股票。定义 \( \Pi \) 为该投资组合的价值。由定义:

\[ \Pi = -V + \frac{\partial V}{\partial S} S \]  (3-1-5)

在 \( \Delta t \) 时间内投资组合价值的改变量 \( \Delta \Pi \) 为:

\[ \Delta \Pi = -\Delta V + \frac{\partial V}{\partial S} \Delta S \]  (3-1-6)

把 (3-1-3) 和 (3-1-4) 带入方程 (3-1-6) 可推导出:

\[ \Delta \Pi = -\left( \frac{\partial V}{\partial t} + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \sigma^2 S^2 \right) \Delta t \]  (3-1-7)
我们假设，这个投资组合的收益等于同时期同样回报率的无风险证券的收益。
即有：

$$\Delta \Pi = r\Pi \Delta t$$  \hspace{1cm} (3-1-8)

其中 $r$ 是无风险利率。把 (3-1-5) 和 (3-1-7) 带入 (3-1-8) 可得到：

$$\left( \frac{\partial V}{\partial t} + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \sigma^2 S^2 \right) \Delta t = r \left( V - \frac{\partial V}{\partial S} S \right) \Delta t$$

所以，

$$\frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 = rV$$  \hspace{1cm} (3-1-9)

这就是 Black–Scholes 方程的微分形式。
若我们考虑欧式看涨期权，则其边界条件为：

$$f = \max(S - K, 0), \text{ 当 } t = T$$

其中 $K$ 代表该看涨期权的执行价格。

### 3.2 转变 Black–Scholes 微分方程为标准热方程的形式

设 Black–Scholes 微分方程和其边值问题如下：

$$\frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \sigma^2 S^2 = rV, \hspace{1cm} 0 \leq S, 0 \leq t \leq T$$

$$V(S, T) = f(S) = \max(S - K, 0), V(0, t) = 0$$

下面我们进行一次变量代换：

$$S = e^x, \hspace{1cm} t = T - \frac{2\tau}{\sigma^2}$$

$$V(x, \tau) = v(\ln S, \frac{\sigma^2}{2} (T - t))$$

用 $v$ 对 $x$ 和 $\tau$ 的偏导数表示 $V$ 对 $S$ 和 $t$ 的偏导数如下：

$$\frac{\partial V}{\partial t} = -\frac{\sigma^2}{2} \frac{\partial v}{\partial \tau}$$

$$\frac{\partial V}{\partial S} = \frac{1}{S} \frac{\partial v}{\partial x}$$

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\[
\frac{\partial^2 V}{\partial S^2} = -\frac{1}{S^2} \frac{\partial v}{\partial x} + \frac{1}{S^2} \frac{\partial^2 v}{\partial x^2}
\]

把上述式子的右面带入到 Black–Scholes 微分方程中并进行简单的化简，我们可以得到如下等式:

\[
\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + \left(2r - \frac{1}{\sigma^2} \right) \frac{\partial v}{\partial x} - \frac{1}{2} \frac{\sigma^2}{\sigma^2} v
\]

令 \( \kappa = \frac{2r}{\sigma^2}, t = \tau \)，则 Black–Scholes 边值问题变为:

\[
\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + (\kappa - 1) \frac{\partial v}{\partial x} - \kappa v, \quad -\infty < x < \infty, \quad 0 \leq t \leq \frac{\sigma^2}{2} T \quad (3-2-1)
\]

\( v(x,0) = V(e^x, T) = f(e^x), \quad -\infty < x < \infty \)

为了得到标准热方程的形式，再进行一次变量代换，以消去上面 (3-2-1) 等式右侧的最后两项。设:

\[
v(x, t) = e^{ax + \beta t} u(x, t) = \phi u
\]

\( \alpha \) 和 \( \beta \) 的值我们之后再取。计算 \( v \) 对 \( t \) 和 \( x \) 的偏导数，得到:

\[
\frac{\partial v}{\partial t} = \beta \phi u + \phi \frac{\partial u}{\partial t}
\]

\[
\frac{\partial v}{\partial x} = \alpha \phi u + \phi \frac{\partial u}{\partial x}
\]

\[
\frac{\partial^2 v}{\partial x^2} = \alpha^2 \phi u + 2 \alpha \phi \frac{\partial u}{\partial x} + \phi \frac{\partial^2 u}{\partial x^2}
\]

把上述三式带入 (3-2-1) 等式中，并且令:

\[
\alpha = -\frac{1}{2} (\kappa - 1) = -\frac{\sigma^2 - 2r}{2\sigma^2}
\]

\[
\beta = -\frac{1}{4} (\kappa + 1)^2 = -\left( \frac{\sigma^2 + 2r}{2\sigma^2} \right)^2
\]

我们得到:

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad 0 \leq t \leq \frac{\sigma^2}{2} T
\]

\( u(x,0) = e^{-ax} v(x,0) = e^{-ax} f(e^x) = e^{-ax} \max(e^x - K, 0), -\infty < x < \infty \)

由 2.2 中关于热方程的知识，我们得到上述热方程的解为:

\[
u(x, t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} u(\epsilon, 0) e^{-\frac{(x-\epsilon)^2}{4t}} d\epsilon
\]

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3.3 柯尔莫戈洛夫方程

如果我们把一个随机微分方程和一个初值条件合并，就可以得到一个扩散过程。我们用两种方法定义空间和时间的随机函数：

（1）考虑某个收益的预期值
（2）考虑在某一时刻某一状态出现的概率

Black–Scholes 偏微分方程是 (1) 的一个例子。这里用到最主要的数学知识是向后柯尔莫戈洛夫方程。

（2）与（1）不同，事实上（2）是（1）的一个对偶形式。用概率密度偏微分方程可以解另一个方程：向前柯尔莫戈洛夫方程。

推导向前向后柯尔莫戈洛夫方程和我们运用的主要的工具是伊藤积分和积分
\[ \int_a^b f dz \] 的期望是零。

下面我们首先讨论向后柯尔莫戈洛夫方程。

3.3.1 向后柯尔莫戈洛夫方程

假设 \( y(t) \) 是如下微分方程的解
\[ dy = f(y,s)ds + g(y,s)dz \]
令
\[ u(x,t) = E_{y(t)=x} [ \Phi(y(T))] \]
是在 \( y(t) = x \) 的条件下，某个收益函数 \( \Phi \) 在到期时间 \( T \) 的期望值。
则 \( u \) 满足下面方程
\[ u_t + f(x,t)u_x + \frac{1}{2}g^2(x,t)u_{xx} = 0 \] 对 \( t < T \) 成立，且 \( u(x,t) = \Phi(x) \)

证明：

对任意函数 \( \phi(y,t) \)，我们由伊藤引理知道有
\[ d(\phi(y(s),s)) = \left( \phi_x + f\phi_y + \frac{1}{2}g^2\phi_{yy} \right) dt + g\phi_y dz \]
选择 \( \phi = u \)，对上式左右两边积分，得到：

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\[ u(y(T), T) - u(y(t), t) = \int_t^T \left( u_t + f u_y + \frac{1}{2} g^2 u_{yy} \right) ds + \int_t^T g u_y dz \]

对两边取期望，并且带入已知的偏微分方程，我们可得:
\[ E_{y(t)} [\Phi(y(T))] - u(x, t) = 0 \]

证毕。

我们还有向量形式的向后柯尔莫戈洛夫方程

设 \( y \) 满足下面微分方程
\[ dy_i = f_i(y, s) \, ds + \sum_j g_{ij}(y, s) \, dz_j \]

其中 \( z_j \) 是相互独立的布朗运动，则
\[ u(x, t) = E_{y(t)} [\Phi(y(T))] \]

满足下列方程
\[ u_t + Lu = 0 \quad \text{对} \quad t < T \quad \text{成立，并且} \quad u(x, T) = \Phi(x) \]

\( L \) 定义为下列微分算子
\[ Lu(x, t) = \sum_i f_i \frac{\partial u}{\partial x_i} + \frac{1}{2} \sum_{i,j,k} g_{ik} g_{jk} \frac{\partial^2 u}{\partial x_i \partial x_j} \]

3.3.2 Feynman–Kac 公式

如果我们考虑一个有折扣的终期收益
\[ u(x, t) = E_{y(t)} \left[ e^{- \int_t^T b(y(s), s) \, ds} \Phi(y(T)) \right] \]

我们依旧假定
\[ dy = f(y, s) \, dt + g(y, s) \, dz \]

则 \( u \) 满足下列微分方程
\[ u_t + f(x, t) u_x + \frac{1}{2} g^2(x, t) u_{xx} - b(x, t) u = 0 \quad \text{对} \quad 0 < t < T, \quad \text{且} \quad u(x, T) = \Phi(x) \quad (3-3-1) \]
（3-3-1）式有一些简单的应用

（1）如果 \( y \) 满足对数正态分布，\( b \) 是利率，则（3-3-1）是 Black-Scholes 偏微分方程的形式。

（2）若 \( b \) 是当前利率，\( \Phi = 1 \)，我们可以得到到期时间为 \( T \) 的零息债券在时间 \( t \) 的值。

证明：

伊藤引理的多维形式如下：

\[
d[z_1(s)z_2(s)] = z_1dz_2 + z_2dz_1 + dz_1dz_2
\]

我们应用上式，令

\[
z_1(s) = e^{-\int_t^s b(y(r),r)dr}
\]

令

\[
z_2(s) = \phi(y(s),s)
\]

我们得到

\[
d\left(e^{-\int_t^s b(y(r),r)dr}u(y(s),s)\right) = z_1gu_ydz
\]

上式右面的期望值是零

\[
E_{y(t) = x}[z_1(T)z_2(T)] = z_1(t)z_2(t) = u(x, t)
\]

证毕。

以上我们讨论从 \( t \) 到 \( T \) （一个固定的时间）随机积分的两个例子。为了解这些随机微分方程，我们需要在 \( x \in \mathbb{R}^n \) 空间上面积分。事实上，很多情形下我们感兴趣不是从 \( t \) 到 \( T \) 的积分，而是从 \( t \) 到 \( y \) 第一次离开某一个固定区域的时间。

3.3.3 向后柯尔莫戈洛夫的边值问题

令 \( D \) 是 \( \mathbb{R}^n \) 中的一个固定区域。设 \( y \) 是一个扩散过程，并且是下述方程的解

\[
dy = f(y, s)ds + g(y, s)dz \quad \text{对} \ s > t , \ \text{并且} \ y(t) = x
\]

其中 \( x \in D \)。令

\[
\tau(x) = y(s) \ \text{第一次离开区域} \ D \ \text{的时间，若该时刻在} \ T \ \text{之前，否则} \ \tau(x) = T
\]

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这是停时的一个例子，用最简单的语言解释停时：在$t$时刻是否停止只取决于$t$时刻之前的信息。
基本的结论是：
\[ u(x,t) = E_{y(t)=x}[\int_t^{\tau(x)} \Psi(y(s),s)ds + \Phi(y(\tau(x),\tau(x)))] \]
满足下列方程
\[ u_t + Lu + \Psi = 0, \quad x \in D \]
边值条件
\[ u(x,t) = \Phi(x,t), \quad x \in \partial D \]
终值条件
\[ u(x,T) = \Phi(x,T), \quad x \in D \]
证明：
这个证明几乎与我们之前的例子一样。唯一不同的地方是我们最后一步积分的时候，积分上限是停时而不是终止时间$T$。
这里有一个小问题。在我们最后一步积分之前，需要确定任何停时都满足$E[\tau] < \infty$。然后应用 Dynkin 定理可以得到$E[\int_t^\tau f dz] = 0$。因为有例子表明当$E[\tau] = \infty$时，$E[\int_t^\tau f dz] \neq 0$。
这里还有一个容易误导人地方我需要解释一下。我们用符号$\Phi$表示边值问题的函数和终值问题的函数。但是$\Phi$应该被认为是两个不同的函数。一个定义在空间边界$\partial D \times [0,T]$，另一个定义在时间边界$D \times \{T\}$。通常$\Phi$的两个部分中的一个为0，而另一部分非0。比如，在障碍期权中，$\Phi$在敲出价格的时候是0，在到期时间等于收益函数。
3.3.4 向前柯尔莫戈洛夫方程
向后柯尔莫戈洛夫方程的解是一个马尔科夫过程，所以我们可以定义它的转移概率：
对任意满足 \( t < s_1 < s \) 的 \( s_1 \)

令 \( p(\cdot; s, x, t) \) 表示 \( s \) 时刻的概率密度函数，在初始时刻 \( t \) 从 \( x \) 出发的条件下。那么 \( p(\cdot; s, x, t) \) 是一个高斯随机变量的密度函数，其均值为 \( x \)，方差为 \( s - t \)。

如果随机微分方程 \( dy = f(y)dt + g(y)dz \) 中的 \( f \) 和 \( g \) 只与 \( y \) 有关，那么

\[
    p(z, s; x, t) = p(z, s - t; x, 0)
\]

如果随机微分方程 \( dy = f(y)dt + g(y)dz \) 中的 \( f \) 和 \( g \) 只与 \( t \) 有关，那么

\[
    p(z, s; x, t) = p(z - x, s; 0, t)
\]

转移概率最重要的性质是，他是柯尔莫戈洛夫向前方程的解

\[
    -p_s - \sum_i \frac{\partial}{\partial z_i} (f_i(z, s)p) + \frac{1}{2} \sum_{i,j,k} \frac{\partial^2}{\partial z_i \partial z_j} (g_{ik}(z, s)g_{jk}(z, s)p) = 0 \quad \text{当} \ s > t
\]

方程的初值条件是

\[
    p = \delta_x(z) \quad \text{当} \ s = t
\]

我们把柯尔莫戈洛夫向前方程记做如下形式

\[
    -p_s + L^*p = 0
\]

其中

\[
    L^*p = - \sum_i \frac{\partial}{\partial z_i} (f_i p) + \sum_{i,j,k} \frac{\partial^2}{\partial z_i \partial z_j} (g_{ik}(z, s)g_{jk}(z, s)p)
\]

柯尔莫戈洛夫向前方程通过解一个初值问题，描述了一个概率分布。而柯尔莫戈洛夫向后方程通过解一个终值问题，描述了预期最终收益。事实上，期权定价中二叉树方法就是类似的应用。其中在每一个节点的股票价格是由前面每一个节点决定的，而每一个节点的期权价格是由后面每一个节点的期权价格决定的。

证明柯尔莫戈洛夫向前方程

首先我们注意到

\[
    E_{y(t)=x}[\Phi(y(T))] = \int \Phi(z)p(z, T; x, t) \, dz \quad (3-3-2)
\]
我们已经知道通过解柯尔莫戈洛夫向后方程可以得到上式左边，这个关系式唯一的决定
\( p(z,T;x,t) \)。

下面我们把注意力转向 \( L \) 和 \( L^* \)。简单的说，\( L^* \) 是 \( L \) 在 \( L^2 \) 内积空间中的对偶
回顾线性代数中若 \( A \) 是内积空间中的一个线性算子，那么 \( A \) 的对偶 \( A^* \) 被定义为

\[ \langle Ax, y \rangle = \langle x, A^* y \rangle \]

在这个问题中，我们用类似的方法，首先定义包含所有平方可积的函数的内积空间

\[ \langle v, w \rangle = \int_{\mathbb{R}^n} v(x)w(x)dx \]

我们断定

\[ \langle L v, w \rangle = \langle v, L^* w \rangle \]  \hspace{1cm} (3-3-3)

接下来，回顾推导柯尔莫戈洛夫向后方程的过程

\[ E_{y(t)} \equiv x[\phi(y(T),T)] - \phi(x,t) = E_{y(t)} = x[\int_t^T (\phi_s + L\phi)(y(s),s)ds] \]  \hspace{1cm} (3-3-4)

对任何函数 \( \phi(y,s) \) 都成立。之前我们总是选择函数 \( \phi \) 使得方程右边等于零。事实上，这个式子对于任何函数 \( \phi(y,s) \) 都成立。

联合（3-3-2）和（3-3-4），我们可以得到

\[ \int_{\mathbb{R}^n} \int \Phi(z,T)p(z,T;x,t)dz - \phi(x,t) = \int_t^T \int_{\mathbb{R}^n} (\phi_s + L\phi)(z,s)p(z,s;x,t)dzds \]  \hspace{1cm} (3-3-5)

对（3-3-5）方程右边应用（3-3-3），得到

\[ \int_t^T \int_{\mathbb{R}^n} (-\phi p_s + \phi L^* p)dzds + \int_t^T \int_{\mathbb{R}^n} \phi(z,s)p(z,s;x,t)dzds \]  \hspace{1cm} (3-3-6)

方程（3-3-5）的左面只与初始时间 \( t \) 和终止时间 \( T \) 有关，我们得到

\[ -p_s + L^* p = 0 \]

因此由（3-3-5）和（3-3-6）可以得到

\[ \int_{\mathbb{R}^n} \phi(z,t)p(z,t;x,t)dz = \phi(x,t) \]
对任何 $\phi$ 成立，证毕。

为了抓住其本质，我们用柯尔莫戈洛夫向前方程给出下面关系的一个新的证明：

$$ u $$ 是柯尔莫戈洛夫向后方程的解可以推出 

$$ \frac{d}{ds} E[u(y(s), s)] = 0 $$

事实上，若 $\rho(z,s)$ 是 $s$ 时刻位置的密度分布函数，则

$$ \frac{d}{ds} E[u(y(s), s)] = \frac{d}{ds} \int u(z,s) \rho(z,s) dz 
= \int u_s \rho + u \rho_d dz = \int u_s \rho + (Lu)\rho dz = 0 $$

上式最后一个等号用到我们假设 $u$ 是柯尔莫戈洛夫向后方程的解。

### 3.4 热方程的初边值问题

我们下面讨论热方程的初边值问题，这些结果在障碍期权中有很多的应用。

首先考虑半空间上的初边值问题：

$u_t = u_{xx}$ 对 $t > 0$ 和 $x > x_0$，且 $u = g$ 当 $t = 0$ 和 $u = \phi$ 当 $x = x_0$

如果 $u$ 是连续的，则一定会有 $g(0) = \phi(0)$。

由线性性质，我们可以把这个问题分解为两个问题：$u = v + w$

其中 $v$ 满足：

$$ v_t = v_{xx} \text{ 对 } t > 0 \text{ 和 } x > x_0, \text{ 且 } v = g \text{ 当 } t = 0 \text{ 和 } v = 0 \text{ 当 } x = x_0 \quad (3-4-1) $$

换句话说，$v$ 是具有同样初值但是边值为 0 的同一个偏微分方程的解。

$w$ 满足：

$$ w_t = w_{xx} \text{ 对 } t > 0 \text{ 和 } x > x_0, \text{ 且 } w = 0 \text{ 当 } t = 0 \text{ 和 } w = \phi \text{ 当 } x = x_0 \quad (3-4-2) $$

换句话说，$w$ 是具有同样边值但是初值为 0 的同一个偏微分方程的解。

接下来，我们把精力放在 $v$ 这个方程上（不失一般性，取 $x_0 = 0$）

为了得到这个方程的解，我们考虑把初值函数 $g$ 定义于全空间，方法是在 $x < 0$ 时，定义 $-g(-x)$，也就是说绕着原点做一个奇反射

$$ \tilde{g}(x) = \begin{cases} g(x) & x > 0 \\ -g(-x) & x < 0 \end{cases} $$
令 $v(x,t)$ 是全空间初值问题的解，其初值函数为 $g(x)$，我们有如下两个断言：

（1）当 $t > 0$ 时，$v$ 是 $x$ 和 $t$ 的光滑函数

（2）对任何 $t$，$v(x,t)$ 是 $x$ 的奇函数，即 $v(x,t) = -v(-x,t)$

由 $v(x,t)$ 的奇函数性质，有 $v(0,t) = -v(0,t)$，所以 $v(0,t) = 0$ 对任何 $t > 0$ 成立。

这样 $v(x,t) = v(x,t)$ （限制在 $x > 0$ 上）是方程（3-4-1）的解。由热方程的解，我们可以写出 $v(x,t)$ 的解

$$v(x,t) = \int_0^\infty k(x-y,t)g(y)dy$$

$$+ \int_{-\infty}^0 k(x-y,t)(-g(-y))dy$$

$$= \int_0^\infty [k(x-y,t) - k(x+y,t)]g(y)dy$$

其中 $k(x,t)$ 是热方程的基础解

$$k(z,t) = \frac{1}{\sqrt{4\pi t}}e^{-z^2/4t}$$

我们换一个写法

$$v(x,t) = \int_0^\infty G(x,y,t)g(y)dy$$

其中

$$G(x,y,t) = k(x-y,t) - k(x+y,t)$$

函数 $G$ 被称为半空间问题的“格林函数”。由我们对向前柯尔莫戈洛夫方程的讨论，可以把 $G(x,y,t)$ 看做是一个布朗运动粒子在 $0$ 时刻从 $y$ 出发，在第一次到达边界之前，$t$ 时刻到达 $x$ 出的概率。

下面我们考虑 $w$，他是初值问题为 $0$，边值问题为 $\phi(t)$ 的半空间热方程的解。$w$ 的解由下列方程给出

$$w(x,t) = \int_0^t \frac{\partial G}{\partial y}(x,0,t-s)\phi(s)ds$$

（3-4-3）
其中 \(G(x, y, t)\) 是由上面定义的格林函数。代入 \(G(x, y, t)\)，我们可得

\[
w(x, t) = \int_0^t \frac{x}{(t-s)\sqrt{4\pi(t-s)}} e^{-x^2/4(t-s)} \phi(s)ds
\]

得到 \(w(x, t)\) 的推导比较复杂，这里不做详细推导。

3.5 单侧障碍期权的定价

在前期知识中，我们介绍过障碍期权的定义和四种障碍期权各自的要求。比如一个下降敲出看涨期权，执行价格为 \(K\)，到期时间为 \(T\)，障碍为 \(X\)，获得的收益为 \((s - K)_+\) 当股票价格保持在 \(X\) 之上。然而当股票价格在到期时间 \(T\) 之前跌倒 \(X\) 之下，则收益为 0。为了得出障碍期权的定价公式，我们考虑一个下降敲出看涨期权，其中 \(X < K\)。这个看涨期权的值为

\[
V(s, t) = C(s, t) - \left(\frac{s}{X}\right)^{1-k} C\left(\frac{X^2}{s}, t\right)
\]

其中 \(k = \frac{2}{\sigma^2}, C(s, t)\) 是一般执行价格为 \(K\)，到期时间为 \(T\) 的欧式看涨期权执行的价值。回顾我们把 Black–Scholes 微分方程转变成标准热方程时候用过的变量代换：

\[
s = e^x, \quad \tau = \frac{1}{2}\sigma^2(T - t), \quad V(s, t) = e^{\alpha x + \beta \tau} u(x, \tau)
\]

其中 \(\alpha = \frac{1-k}{2}, \beta = -\frac{(k+1)^2}{4}\)

这样 Black–Scholes 微分方程变为

\[
u_t = u_{xx}, \quad \text{其中} x > \log X
\]

初值条件为 \(u_0(x) = e^{-\alpha x}(e^x - K)_+\)

我们需要找到一个 \(u\) 满足 \(u_t = u_{xx}\)，对所有的 \(x\) 成立，并且满足

\[
u(x', t) = -u(x, t) \quad \text{其中} x' \text{ 是} x \text{ 关于} \log X \text{ 的对称点}
\]

所以一定有 \(u(\log X, t) = 0\)。
经过简单的计算可知

\[ u(2\log X - x, t) = -u(x, t) \] 对于所有 \( x \) 成立

我们接下来的任务（1）是扩展初值问题到全空间，（2）在 \( t > 0 \) 的条件下，解这个线性热方程。

（2）当 \( x > \log X \) 的时候，\( u_0(x) = e^{-\alpha x}(e^x - K)_+ \)，又由假设 \( X < K \)，所以 \( u_0(x) = 0 \)，当 \( x \leq \log X \) 的时候。

所以扩展后的初值为 \( f_0(x) = u_0(x) - u_0(2\log X - x) \)

利用我们偏微分方程的线性性质，障碍期权的价值 \( V(s, t) \) 可以分为两部分。

\[ f_1(x, t) \text{ 为初值为 } u_0(x) \text{ 的全空间热方程的解} \]

\[ f_2(x, t) \text{ 为初值为 } u_0(2\log X - x) \text{ 的全空间热方程的解} \]

事实上，\( f_1(x, t) \) 就是一般欧式期权的价值 \( C(s, t) \)，\( f_2(x, t) = f_1(2\log X - x, t) \)

\[ e^{\alpha x + \beta t} f_2(x, t) = e^{\alpha x + \beta t} f_1(2\log X - x, t) = e^{\alpha x + \beta t} e^{-(\alpha|2\log X - x| + \beta t)} C(e^{2\log X - x}, t) \]

\[ = X^{-2\alpha S^2\alpha} C \left( \frac{X^2}{S}, t \right) = \left( \frac{S}{X} \right)^{1-k} C \left( \frac{X^2}{S}, t \right) \]

这样我们得出了下降敲出看涨期权的价格。一份下降敲出看涨期权与下降敲入看涨期权组成的投资组合与一份标准欧式期权是等价的，所以我们也得出了下降敲入看涨期权的价格。

3.6 双侧障碍期权定价

双侧障碍期权既有上限又有下限，若假定标的资产价格服从对数正态分布，那么其定价公式可以转化成双边值热方程问题。所以我们感兴趣的是如何求解位置变量在一个区间的热方程问题。简单起见，我们把位置区间设定为 \( 0 < x < 1 \)。

这样，我们的目标是求解：

\[ u_t = u_{xx}, \quad t > 0, \quad 0 < x < 1 \]
初值条件为：

\[ u = g, \text{ 当 } t = 0 \]

边值条件为：

\[ u = \phi_0 \text{ 当 } x = 0, \quad u = \phi_1 \text{ 当 } x = 1 \]

我们先考虑 \( \phi_0 = \phi_1 = 0 \) 的情形。

我们知道如下关于傅里叶展开的事实：

若定义在 \((0,1)\) 的函数 \( f(x) \) 且满足 \( f(0) = f(1) = 0 \)，可以展开成如下傅里叶级数的形式：

\[ f(x) = \sum_{k=1}^{\infty} a_n \sin(n\pi x) \quad (3-6-1) \]

其中 \( a_n \) 可以由函数 \( f \) 表示出来

\[ a_n = 2 \int_{0}^{1} f(x) \sin(n\pi x) dx \quad (3-6-2) \]

由 (3-6-1)，我们可以把 \( u \) 表示成如下形式

\[ u(x, t) = \sum_{k=1}^{\infty} a_n(t) \sin(n\pi x) \]

把上式带入方程 \( u_t = u_{xx} \)，可得

\[ \frac{da_n}{dt} = -n^2 \pi^2 a_n \]

因此 \( a_n(t) = a_n(0)e^{-n^2\pi^2t} \)

为了满足满足初值条件，由 (3-6-2)

\[ a_n(0) = 2 \int_{0}^{1} g(x) \sin(n\pi x) dx \]

所以，综上

\[ u(x, t) = \sum_{k=1}^{\infty} a_n(0) e^{-n^2\pi^2t} \sin(n\pi x) \]

我们可以把 \( u(x, t) \) 表示成下述形式

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\[ u(x, t) = \int_{0}^{1} G(x, y, t)g(y)dy \]

\[ G(x, y, t) = 2 \sum_{k=1}^{\infty} e^{-n^2\pi^2t} \sin(n\pi x) \sin(n\pi y) \]  

(3-6-3)

下面我们回到原始初值边值问题

\[ u_t = u_{xx}, \quad t > 0, \quad 0 < x < 1 \]

初值条件为:

\[ u = g, \quad \text{当} t = 0 \]

边值条件为:

\[ u = \phi_0 \quad \text{当} x = 0, \quad u = \phi_1 \quad \text{当} x = 1 \]

可以通过作函数替换 \( u(x, t) = v(x, t) + x\phi_1 + (1-x)\phi_0 \) 得到 \( v \) 的齐次边值问题。然后可以得到 \( u \) 的解。直接给出结果:

\[ u(x, t) = \int_{0}^{t} \frac{\partial G}{\partial y}(x, 0, t-s)\phi_0(s)ds - \int_{0}^{t} \frac{\partial G}{\partial y}(x, 1, t-s)\phi_1(s)ds \]

其中 \( G \) 由 (3-6-3) 给出。事实上, \( \frac{\partial G}{\partial y}(x, 0, t-s) \) 是随机游动的点, 0 时刻从 \( x \) 出发, 在 \( t \) 时刻先到 0 这个边界的概率。类似的 \( -\frac{\partial G}{\partial y}(x, 1, t-s) \) 是随机游动的点, 0 时刻从 \( x \) 出发, \( t \) 时刻先到 1 这个边界的概率。

3.7 最大值和唯一性定理

我们已经得出热方程在全空间，半空间和有限区间上的解了，并且知道他们分别对应普通期权，单侧障碍期权和双侧障碍期权的定价。但是我们还不知道我们得出的解是不是唯一的解。这一个小节，我们来证明解的唯一性。由线性，等价于证明 \( u = 0 \) 是初值和边值问题为 0 的热方程的唯一解。

闭区域最大值定理
$D$ 是一个有界区域。假设 $f_t - \Delta f \leq 0$ 对所有的 $x \in D$ 和 $0 < t < T$ 都成立。那么 $f$ 在闭区域 $D \times [0, T]$ 上的最大值，只可能在以下两种可能取到，(1)在初值边界 $t = 0$，(2)在空间边值边界 $x \in \partial D$。

证明：
若 $f_t - \Delta f < 0$，我们的任务是证明 $f$ 的最大值不在内部取到，并且不在 $t = T$ 的时候取到。由微积分的知识可知，内部极大值点一阶导数为 $0$，且 $\frac{\partial^2 f}{\partial x_i^2} \leq 0$ 对每一个 $i$ 成立，这样就有 $f_t - \Delta f \geq 0$，与假设 $f_t - \Delta f < 0$ 矛盾。若在 $t = T$ 的时候取到最大值，依旧有 $\frac{\partial^2 f}{\partial x_i^2} \leq 0$ 成立，且有 $f_t \geq 0$，这推出 $f_t - \Delta f \geq 0$，依旧与假设矛盾。

若我们只能确定 $f_t - \Delta f \leq 0$，那么就不能应用之前的证明过程了。但是通过简单处理，我们把之前的证明过程应用到 $f_\epsilon(x, t) = f(x, t) - \epsilon t$ 中，其中 $\epsilon$ 是任意大于 $0$ 的数。再令 $\epsilon$ 趋近于 $0$，即得证。

把最大值定理应用到函数 $-f$ 上，可以得到如下最小值定理

闭区域最小值定理

$D$ 是一个有界区域。假设 $f_t - \Delta f \geq 0$ 对所有的 $x \in D$ 和 $0 < t < T$ 都成立。那么 $f$ 在闭区域 $D \times [0, T]$ 上的最小值，只可能在以下两种可能取到，(1)在初值边界 $t = 0$，(2)在空间边值边界 $x \in \partial D$。

由最大值定理和最小值定理，可以很容易得到唯一性定理

闭区域唯一性定理

若 $f_t - \Delta f = 0$ 对所有的 $x \in D$ 和 $0 < t$ 都成立，并且假设 $f(x, 0) = 0$ 对 $x \in D$ 成立，$f(x, t) = 0$ 对 $x \in \partial D$ 成立。则有 $f(x, t) = 0$ 对所有的 $x \in D$, $t > 0$ 成立。
事实上，由最大值和最小值定理，可得 $f$ 的最大值和最小值都是 0。所以 $f$ 在该闭区域内恒为 0。

以上我们证明了热方程初值问题在闭区域的解的唯一性，若要证明热方程初值问题在全空间的唯一性，则还要多一些工作。

全空间唯一性定理

若 $f_t - \Delta f = 0$ 对所有的 $x \in \mathbb{R}^n$ 和 $0 < t$ 都成立，并且假设 $f(x, 0) = 0$ 对 $x \in \mathbb{R}^n$ 成立，$|f(x, t)| \leq Me^{c|x|^2}$ 对某个 $M$ 和 $c$ 成立。则 $f(x, t) = 0$ 对 $x \in \mathbb{R}^n$, $t > 0$ 成立。

我们只要证明存在某个 $t_0 > 0$，使得 $f = 0$ 对 $0 < t \leq t_0$ 成立，然后应用这个结论 $k$ 次，可得 $f = 0$ 对 $t \leq kt_0$ 成立，再让 $k$ 趋近于正无穷即可。事实上，我们只要证明 $f \leq 0$，然后把这个结论应用到 $-f$ 上就可得 $f = 0$。

证明：

令 $g(x, t) = f(x, t) - \frac{\delta}{(t_1-t)^{\frac{3}{2}}} e^{\frac{|x|^2}{2(t_1-t)}}$，$\delta$ 和 $t_1$ 为某个常数，他们的选择由后面的推导得出。我们构造一个闭区间 $D \times [0, t_0]$，$D$ 是一个大球，而 $t_0 < t_1$。

直接计算可得 $g_t - \Delta g = 0$。令 $D$ 是一个半径为 $r$ 的大球，由最大值定理可知 $g$ 在 $D \times [0, t_0]$ 上的最大值在边界取得，在初值边界，有：

$$g(x, 0) < f(x, 0) = 0$$

在空间边界，有 $|x| = r$，所以

$$g(x, t) = f(x, t) - \frac{\delta}{(t_1-t)^{\frac{3}{2}}} e^{\frac{|x|^2}{2(t_1-t)}} \leq Me^{c|x|^2} - \frac{\delta}{(t_1-t)^{\frac{3}{2}}} e^{\frac{|x|^2}{2(t_1-t)}} \leq Me^{cr^2} - \frac{\delta}{r^2} e^{\frac{r^2}{2t_1}}$$

我们选择 $t_1$ 使得 $\frac{1}{4t_1} > c$，则 $r$ 足够大的时候有：

$$g(x, t) \leq 0$$

在 $|x| = r$ 的时候

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由最大值定理我们可得在整个空间上有 $g(x, t) \leq 0$, 对足够大的 $r$ 的成立。所以我们得到:

$$f(x, t) \leq \frac{\delta}{(t_1 - t)^\frac{n}{2}} e^{\frac{|x|^2}{4(t_1 - t)}}$$

对所有 $x \in \mathbb{R}^n$, $t < t_1$ 成立，限制 $t < t_0$, $t_0$ 是某个小于 $t_1$ 的数，然后让 $\delta$ 趋近于 0，可得 $f \leq 0$. 证毕。

### 第 4 章 结论

本文主要根据纽约大学大学柯朗所 Robert V. Kohn 教授的讲义 PDF for Finance，介绍了偏微分方程在期权定价中的应用。其中核心内容是 Black–Scholes 微分方程:

$$\frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \sigma^2 S^2 = rV$$

的推导和应用。

在我们假设股票服从对数正态分布的前提下，运用伊藤引理和无风险对冲理论，我们可以得到上述方程，对上述方程的求解在金融背景下有很多方法，其中一种就是将其转化为标准热方程的形式，然后直接利用热方程解的表达式得解。

在给障碍期权定价的时候，我们会遇到障碍条件，对应的问题正是热方程里面边界值问题，所以说我们在已知热方程问题理论的前提下，很容易给各类期权定价。

当然，Black–Scholes 微分方程的成立是要满足很多前提的，但是这些前提条件往往在现实中很难满足，所以 Black–Scholes 微分方程是理想化的方程。不过在多种因素混合的条件下，由 Black–Scholes 方程得出的结果和真实结果相差很小，所以 Black–Scholes 方程还是受到了金融工程师广泛的应用。
随着人们对金融市场规律的把握越来越准确，数学方法在金融中越来越多的应用，很有可能在不久的将来会产生新的，更加准确的期权定价公式，他们可能需要更加实际的假设条件，一定会受到更广泛的应用。

由此可见，数学的强大不仅在于数学家孜孜不倦的追求更深奥的数学理论，更在于数学理论在各个领域的应用会让我们的生活变得更加美好！

参考文献

[1]. Robert V. Kohn, Courant Institute of Mathematical Sciences, PDE for Finance Notes, 2011.


附录 A 外文资料的调研阅读报告

布朗运动和偏微分方程

1. 引言

早在柯尔莫戈洛夫方程出现的时候，概率论和分析学就有着很紧密的联系。我们通过几个例子来阐述两者之间的联系。

很多椭圆和抛物线偏微分方程的解可以表示为随机函数的期望，这种表示方法方便我们获得解的特性，并且反过来，也方便我们通过相关的偏微分方程问题来确定随机过程不同函数的分布。

我们主要讨论布朗运动和一维热方程的联系。我们给出热方程解的存在性和唯一性的概率证明和解释。反过来，对如何从热方程的解计算得出布朗运动的边界概率做了介绍。

2 一维热方程
考虑 (t, x) 平面中 x 轴上的一个绝热、无限长的棒，用 \( f(x) \) 来表示橡胶在 \( t = 0 \) 时刻的温度函数，若 \( u(t, x) \) 表示 \( t \geq 0 \) 时刻橡胶在 x 处的温度，则 u 满足热方程

\[
\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2}
\]  \hspace{1cm} (2-1)

满足初值条件 \( u(0, x) = f(x) \)。

我们用概率方法处理 (2-1) 式的起点是观察到一维布朗运动的转移概率密度

\[
p(t; x, y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}}
\]  \hspace{1cm} (2-2)

满足偏微分方程

\[
\frac{\partial p}{\partial t} = \frac{1}{2} \frac{\partial^2 p}{\partial x^2}
\]  \hspace{1cm} (2-3)

假设 \( f \) 是一个 Borel 可测函数，并且满足

\[
\int_{-\infty}^{\infty} e^{-ax^2} |f(x)| \, dx < \infty \quad \text{对某个 } a > 0 \text{ 成立}
\]  \hspace{1cm} (2-4)

定义

\[
u(t, x) = E^x f(W_t) = \int_{-\infty}^{\infty} f(y)p(t; x, y) \, dy \quad , \quad 0 < t < \frac{1}{2a} , \quad x \in R
\]  \hspace{1cm} (2-5)

无穷次可微，且满足热方程 (2-1)。

若 \( f \) 有界连续，把 (2-5) 式子写成 \( u(t, x) = E^0 f(x + W^t) \) 的形式，由有界收敛定理可以得到

\[
f(x) = \lim_{t \to 0} u(t, y), \quad \forall x \in R
\]  \hspace{1cm} (2-6)

A Tychonoff 唯一性定理

我们称 \( p(t; x, y) \) 为寻找函数 u 满足 (2-1) 和特定初值条件这个问题的基本解。

定理 2.1

假设函数 \( u \in C^{1,2} \) 在带状区域 \( (0, T] \times R \) 中满足 (2-1)，且满足条件
对某个常数 $K$ 和 $a$ 成立，则在 $(0, T) \times R$ 中 $u = 0$

推论

若 $u_1$ 和 $u_2$ 满足 (2-1) 和 (2-8)，且

$$
\lim_{y \to x} u_1(t, y) = \lim_{y \to x} u_2(t, y) 
$$

对 $u_1 - u_2$ 应用定理 2.1，可得 $u_1 = u_2$

B 热方程的非负解

若初始温度 $f$ 是非负的，由 (2-4) 可以得出，整个棒的温度在任何时刻都是非负的。

定理 2.2

令 $v(t, x)$ 是定义在带状区域 $(0, T) \times R$ 的非负函数，其中 $0 < T < \infty$. 以下四个条件是等价的:

(1) 存在某个非减函数 $F: R \to R$，使得

$$
v(t, x) = \int_{-\infty}^{\infty} p(T - t; x, y) dF(y); \quad 0 < t < T, \quad x \in R
$$

(2) $v$ 是定义在 $(0, T) \times R$ 上的 $C^{1, 2}$ 函数，且满足热方程

$$
\frac{\partial v}{\partial t} + \frac{1}{2} \frac{\partial^2 v}{\partial x^2} = 0
$$

(3) 对布朗运动 ${W_s, \ 0 \leq s < \infty}$ 和任何固定的 $t \in (0, T)$，过程 ${v(t + s, W_s), 0 \leq s < T - t}$ 是一个鞅

(4) 对布朗运动 ${W_s, \ 0 \leq s < \infty}$，我们有

$$
v(t, x) = E^x v(t + s, W_s); \quad 0 < t \leq t + s < T, \quad x \in R
$$
推论

令 \( u(t, x) \) 是定义在带状区域 \((0, T] \times R\) 的非负函数，其中 \(0 < T < \infty\)。以下四个条件是等价的:

1. 存在某个非减函数 \( F: R \rightarrow R \)，使得
   \[
   u(t, x) = \int_{-\infty}^{\infty} p(T - t; x, y) dF(y); \quad 0 < t < T, \; x \in R
   \]
2. \( u \) 是定义在 \((0, T] \times R\) 上的 \(C^1 \times 2\) 函数，且满足热方程 (2-1)
3. 对布朗运动 \( \{W_s, 0 \leq s < \infty\} \) 和任何固定的 \( t \in (0, T) \)，过程 \( \{u(t - s, W_s), 0 \leq s < T - t\} \) 是一个鞅
4. 对布朗运动 \( \{W_s, 0 \leq s < \infty\} \)，我们有
   \[
   u(t, x) = E^x u(t - s, W_s); \quad 0 \leq s < t < T, \; x \in R
   \]

C 混合初值边界值问题

我们简要的讨论半无限棒上面温度的概率以及在原点被吸收的布朗运动概念的关系。

假设 \( f: (0, \infty) \rightarrow R \) 是一个 Borel 可测函数，满足
\[
\int_0^{\infty} e^{-ax^2} |f(x)| dx < \infty, \quad \text{对某个 } a > 0 \text{ 成立}
\]

定义 \( u_1(t, x) = E^x[f(W_t)1_{[T_0 > t]}]; \quad 0 < t < \frac{1}{2a}, \; x > 0 \)

由反射定理我们有

\[
P^x[W_t \in dy, T_0 > t] = [p(t; x, y) - p(t; x, -y)] dy, \quad \text{对某个 } t > 0, \; x, y > 0
\]

所以

\[
u_1(t, x) = \int_0^{\infty} f(y)p(t; x, y)dy - \int_{-\infty}^0 f(-y)p(t; x, y)dy
\]

\( u_1 \) 无穷次可微，满足热方程 (2-1)，在 \( f \) 的所有连续点上满足 (2-6)，且

\[
\lim_{s \rightarrow t, x \downarrow 0} u_1(s, x) = 0; \quad 0 < t < \frac{1}{2a}
\]
我们把 $u_1(t,x)$ 看作半无限棒被放在 $x$ 正半轴的温度。

现在假设整个棒的初始温度为 0，但是 $t$ 时刻 $x=0$ 处的温度函数为 $g(t)$，
其中 $g:(0,\frac{1}{2}\alpha) \to \mathbb{R}$ 是有界连续函数。则函数 $g$ 的 Abel 变换

$$u_2(t,x) = E^x\left[g(t-T_0)1_{[r_0 \leq t]}\right] = \int_0^t g(t-\tau)h(\tau,x)\,d\tau = \int_0^t g(s)h(t-s)\,ds$$

方程（2-1）的解，其中 $0 < t < \frac{1}{2a}$, $x > 0$, $h(t,x) = -\frac{\partial}{\partial x}p(t;x,0)$

我们把上式写成下面的形式

$$u_2(t,x) = E^0\left[g(t-T_x)1_{[r_x \leq t]}\right]; \quad 0 < t < \frac{1}{2a}, \quad x > 0$$

由有界收敛定理我们得到

$$\lim_{s \to t, x \to 0} u_2(s,x) = g(t): \quad 0 < t < \frac{1}{2a}$$

$$\lim_{t \to 0, y \to x} u_2(t,y) = 0: \quad 0 < x < \infty$$

我们把 $u_1$ 和 $u_2$ 相加就能够得到初始条件是 $f$ 且边值 $x = 0$ 处温度 $g(t)$
的方程的解。

参考文献

[1]. Ioannis Karatzas, Steven E. Shreve. Brownian Motion and Stochastic Calculus
2nd ed. Chapter 4 Springer.
1 Introduction

It is known that every compact, connected Riemann Surface of genus larger than 1 has universal covering \( \mathbb{H} \), which gives a canonical projective structure (given by a coordinate covering for \( \mathbb{C} \), with transitions functions all Möbius functions, \( i.e. \), of form \( z \mapsto \frac{az+b}{cz+d} \), where \( ad - bc = 1 \)).

Our first goal is to define projective connections, and to construct relations between these and projective structures, by which we can show that all the projective structures form an affine space \( B \) for \( H^0(\mathbb{C}, K_C^2) \), where \( K_C \) is the canonical line bundle over \( \mathbb{C} \), and \( K_C^2 := K_C \otimes K_C \), called the line bundle of quadratic differentials over \( \mathbb{C} \). Then we will give an embedding of \( B \) into \( H^1(\mathbb{C}, PSL(2, \mathbb{C})) \), and show that \( H^1(\mathbb{C}, PSL(2, \mathbb{C})) \) can be identified with \( \text{Hom}(\pi_1(\mathbb{C}), PSL(2, \mathbb{C}))/PSL(2, \mathbb{C}) \) (the equivalence relation is given by conjugation of \( PSL(2, \mathbb{C}) \) on \( \text{Hom}(\pi_1(\mathbb{C}), PSL(2, \mathbb{C})) \)). Using this embedding, we can give a geometric explanation of projective structures.

Above results are mostly based on [1].

Denote \( S := C \times C, \Delta : C \hookrightarrow S, x \mapsto (x, x) \) the diagonal embedding, and let \( L := K_S^2(2\Delta) \), a line bundle over \( S \). Another main result is that \( B \) can be canonically identified with those trivialisation of \( L|_{3\Delta} \) which can be restricted to the canonical trivialisation of \( L|_{2\Delta} \) (these trivialisations form an affine space \( \Omega \subseteq H^0(S, L|_{3\Delta}) \)). This is proved by [2].

2 Projective Connections

In this section, \( C \) represents a compact, connected Riemann Surface (not necessarily of genus larger than 1 if not additionally mentioned).

2.1 Schwarzian Derivative

**Definition 2.1.** Suppose \( D \subseteq \mathbb{C} \) open, \( f : D \rightarrow \mathbb{C} \) locally biholomorphic, we define the *Schwarzian Derivative* of \( f \) to be a holomorphic function \( S(f) \) on \( D \), such that

\[
S(f)(z) := \frac{2f''(z)f'''(z) - 3(f''(z))^2}{2f'(z)^2}
\]
The Schwarzian Derivative is characterized by the following two propositions.

**Proposition 2.2.** For \( f : D \rightarrow \mathbb{C} \) as before, \( S(f) = 0 \) is equivalent to: \( f \) is locally a Möbius function.

*Proof.* If locally \( f(z) = \frac{az + b}{cz + d} \), for some \( a, b, c, d \in \mathbb{C} \) with \( ad - bc = 1 \), then it can be checked by direct calculation that \( S(f) = 0 \).

Suppose now that \( S(f) = 0 \). Since it suffices to prove that \( f \) is a Möbius transformation locally (by uniqueness of analytic functions, see ([4], Chapter 4, Section 3)), we may assume that the values of \( f' \) is contained in a simply connected area in \( \mathbb{C} - 0 \), so that we can well define \( \sqrt{f'} \) over \( D \). Define

\[
g := \frac{f}{\sqrt{f'}}
\]

calculate

\[
g' = \frac{f'\sqrt{f'} - f f''}{2f'\sqrt{f'}} = \sqrt{f'} - \frac{f f''}{2f'\sqrt{f'}}
\]

and

\[
g'' = \frac{f''}{2\sqrt{f'}} - \frac{\left( f'f'' + f f''' \right)f'\sqrt{f'} - f f''^2}{2(f')^3} = \frac{1}{2(f')^3} \left( \frac{3}{2} f(f'')^2 \sqrt{f'} - f f''' f' \sqrt{f'} \right) = \frac{f\sqrt{f'}}{4(f')^3} \left( 3(f'')^2 - 2f' f''' \right) = 0
\]

This implies that \( g(z) = az + b \), for some \( a, b \in \mathbb{C} \), so we get

\[
f^2 = (az + b)^2 f'
\]
solve the ODE, we get

\[
f = \frac{az + b}{cz + d}
\]
for some \( a, b, c, d \in \mathbb{C}, ad - bc = 1 \). \( \square \)

**Proposition 2.3.** Let \( f, g \) be locally biholomorphic functions defined on \( D_1, D_2 \) respectively. Assume that \( g(D_2) \subseteq D_1 \), then we have the chain rule

\[
S(f \circ g)(z) = S(f)(g(z))g'(z)^2 + S(g)(z)
\]

*Proof.* Denote \( w = g(z) \), we calculate

\[
(f \circ g)'(z) = f'(w)g'(z)
\]

\[
(f \circ g)''(z) = f''(w)g'(z)^2 + f'(w)g''(z)
\]

\[
(f \circ g)'''(z) = f'''(w)g'(z)^3 + 3f''(w)g'(z)g''(z) + f'(w)g'''(z)
\]
\[ S(f \circ g)(z) = \frac{(f \circ g)'''(z)}{(f \circ g)'(z)} \]

\[
= \left( \frac{f'''(w)g'(z)^2}{f'(w)} + \frac{g'''(z)}{g'(z)} \right) - \frac{3}{2} \left( \frac{f''(w)g'(z)}{f'(z)} \right) + \frac{g''(z)}{g'(z)} \right)^2
\]

\[
= S(f)(g(z))g'(z)^2 + S(g)(z).
\]

\[ S(f \circ g)dz \otimes dz = g^*(S(f)dz \otimes dz) + S(g)dz \otimes dz \]

which may be more natural, and will be used later.

### 2.2 Projective Connection

Next we are going to define the projective connections on \( C \).

Suppose \( \mathcal{U} = (U_\alpha, z_\alpha) \) is a coordinate covering for \( C \). For \( \alpha, \beta \) with \( U_{\alpha\beta} := U_\alpha \cap U_\beta \) nonempty, define \( f_{\alpha\beta} := z_\alpha z_\beta^{-1} \), which is biholomorphic from \( z_\beta(U_{\alpha\beta}) \) to \( z_\alpha(U_{\alpha\beta}) \). It is given by \( U_{\alpha\beta} \mapsto K_{\alpha\beta} := (f'_{\alpha\beta})^{-1} \) a 1-cocycle in \( \hat{\mathcal{C}}^1(\mathcal{U}, \mathcal{O}^\times) \), which induces the canonical line bundle \( K_C \in \hat{H}^1(\mathcal{U}, \mathcal{O}^\times) \).

Define line bundle of quadratic differentials by \( K_C^2 := K_C \otimes K_C \), and construct a 1-cocycle \( \sigma \in \hat{\mathcal{C}}^1(\mathcal{U}, K_C^2) \) by

\[ \sigma_{\alpha\beta}(p) := (Sf_{\alpha\beta})(z_\beta(p))dz_\beta \otimes dz_\beta, p \in U_{\alpha\beta} \]

We can check that \( \sigma \) is indeed a cocycle.

In fact, for \( U_{\alpha\beta} \neq \emptyset \), we have \( f_{\alpha\gamma} = f_{\alpha\beta}f_{\beta\gamma} \), acting by \( S \), and using the chain rule we have proved, we can get

\[ Sf_{\alpha\gamma}dz_\gamma \otimes dz_\gamma = (Sf_{\alpha\beta})dz_\beta \otimes dz_\beta + (Sf_{\beta\gamma})dz_\gamma \otimes dz_\gamma \]

which is by definition

\[ \sigma_{\alpha\gamma} = \sigma_{\alpha\beta} + \sigma_{\beta\gamma} \]

hence \( \sigma \) is a 1-cocycle.

Denote \( \delta : \hat{\mathcal{C}}^0(\mathcal{U}, K_C^2) \rightarrow \hat{\mathcal{C}}^1(\mathcal{U}, K_C^2) \) to be the coboundary map of cochains.

**Definition 2.5.** A projective connection on \( C \) is a 0-cochain in \( \hat{\mathcal{C}}^0(\mathcal{U}, K_C^2) \), denoted by \( h = (h_\alpha dz_\alpha \otimes dz_\alpha) \), \( h_\alpha \) is a holomorphic function on \( U_\alpha \), such that \( \delta(h) = \sigma, i.e., h_\beta dz_\beta \otimes dz_\beta - h_\alpha dz_\alpha \otimes dz_\alpha = \sigma_{\alpha\beta}, \) for all \( \alpha, \beta \) with \( U_{\alpha\beta} \neq \emptyset \).

Obviously, if there exists one projective connection \( h \) on \( C \), then \( h + H^0(C, K_C^2) \) is the space of all the projective connections.
2.3 The First Main Correspondence

**Definition 2.6.** A projective structure on $\mathbb{C}$ is given by a coordinate covering for $\mathbb{C}$, of which the transition functions are all Möbius functions (we call it a projective coordinate covering for $\mathbb{C}$). Two projective structures are the same if and only if the union of the corresponding atlas still give a projective structure.

**Lemma 2.7.** Let $D$ be an open subset of $\mathbb{C}$, $h : D \rightarrow \mathbb{C}$ holomorphic. Then for all $p \in D$, there exists $V$ open, $p \in V \subseteq D$, $f : V \rightarrow \mathbb{C}$ biholomorphic, and $S(f) = h|_V$.

**Proof.** First consider the ODE

$$g'' = -\frac{1}{2}gh$$

It has a nonvanishing solution $g$ on a neighborhood $V'$ of $p$. Consider the ODE

$$f'' = f'g^2$$

It has a nonvanishing solution $f$ on an open subset $V''$, $p \in V'' \subseteq V'$ with additional property that $f'(p) \neq 0$. Then we can find an open subset $V$, $p \in V \subseteq V''$, $f|_V$ is biholomorphic.

Let us check that $S(f) = h|_V$. Find $W$ open, $p \in W \subseteq V$, such that $\sqrt{f'}$ is well defined on $W$, we can assume that $f/\sqrt{f'} = g$, then $(f/\sqrt{f'})'' = -\frac{1}{2}fh/\sqrt{f'}$. Combine with the calculation in the proof of Proposition 2.2, we have

$$\frac{f}{\sqrt{f'}}(h - S(f)) = 0 \implies h = S(f)$$

on $W$, thus, by uniqueness of analytic functions, we have $h|_V = S(f)$.

Suppose now we have a projective connection $\mathfrak{h} = (h_\alpha dz_\alpha \otimes dz_\alpha)$ on $\mathbb{C}$, we want to construct a projective structure $\Phi(\mathfrak{h})$ from $\mathfrak{h}$. We can regard $h_\alpha$ as a holomorphic function on $z_\alpha(U_\alpha)$, for all $p \in U_\alpha$. By the above lemma, we can find $V$ open, $p \in V \subseteq U_\alpha$, such that there exist $w : z_\alpha \rightarrow \mathbb{C}$ biholomorphic, $S(w) = h_\alpha|_{z_\alpha(V)}$. Such $V$'s give a new covering for $\mathbb{C}$, which is finer than $\mathcal{U}$. In fact, we can use this new coordinate covering instead of $\mathcal{U}$. Thus, without loss of generality, we may assume that there exists $w_\alpha : z_\alpha(U_\alpha) \rightarrow \mathbb{C}$ biholomorphic, with $S(w_\alpha) = h_\alpha|_{U_\alpha}$ for all $\alpha$. ($U_\alpha, w_\alpha \circ z_\alpha$) is a new coordinate covering for $\mathbb{C}$, and we claim that this gives a projective structure on $\mathbb{C}$.

In fact, for all $\alpha, \beta$ with $U_{\alpha\beta} \neq \emptyset$, denote $f = w_\alpha \circ z_\alpha \circ (w_\beta \circ z_\beta)^{-1}$, then $f \circ w_\beta = w_\alpha \circ z_\alpha \circ z_\beta^{-1} = w_\alpha \circ f_{\alpha\beta}$. For $p \in U_{\alpha\beta}$,

$$(f \circ w_\beta)(z_\beta(p)) = (w_\alpha \circ f_{\alpha\beta})(z_\beta(p))$$

hence

$$(Sf)((w_\alpha \circ z_\beta)(p))(w_\beta(p)) + h_\beta(z_\beta(p)) = h_\alpha((f_{\alpha\beta} \circ z_\beta)(p))(f_{\alpha\beta}(p)) + (Sf_{\alpha\beta})(z_\beta(p))$$

write by quadratic differentials, we have

$$S(f)dw_\beta \otimes dw_\beta + h_\beta dz_\beta \otimes dz_\beta = h_\alpha dz_\alpha \otimes dz_\alpha + S(f_{\alpha\beta})dz_\beta \otimes dz_\beta$$

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hence
\[ S(f)dw_\beta \otimes dw_\beta = \sigma_{\alpha\beta} + h_\alpha dz_\alpha \otimes dz_\alpha - h_\beta dz_\beta \otimes dz_\beta = 0 \]
hence \( Sf = 0 \), which implies that \( f \) is Möbius.

Thus, \((U_\alpha, w_\alpha \circ z_\alpha)\) gives a projective structure on \( C \), denoted by \( \Phi(h) \).

**Theorem 2.8.** The map \( \Phi : \{ \text{projective connections} \} \to \{ \text{projective structures} \} \) given by the above discussion is a well defined bijective map.

**Proof.** Note that in the above discussion, no matter how fine the new coordinate covering is, we can always get same projective structure, as long as \( Sw_\alpha \) coincide with our 0-cochain \( h_\beta \). Thus, \( \Phi(h) \) is well defined.

Suppose \((U_\alpha, z_\alpha)\) is a projective coordinate covering for \( C \), define \( h_\alpha(p) = 0, p \in U_\alpha \)

\( h = (h_\alpha dz_\alpha \otimes dz_\alpha) \) gives a 0-cochain, with \( \delta(h) = \sigma \) (note that \( \sigma_{\alpha\beta}(z_\beta(p)) = S(f_{\alpha\beta}) = 0 \)). This produces a projective connection from a projective structure, which can be shown to be the inverse of \( \Phi \).

Denote \( \mathcal{B} \) be the space of all the projective structures on \( C \), by the above theorem, we see that \( \mathcal{B} \) is an affine space for \( H^0(C, K_C^2) \) provided \( \mathcal{B} \) itself is nonempty.

### 2.4 Existence of Projective Structures

If \( \text{genus}(C) = 0 \), we must have \( C = \mathbb{P}^1 \), thus there is a standard projective structure on \( C \) (in fact, this is the only one, we will return to this later). If \( \text{genus}(C) = 1 \), \( C \) is isomorphic to a complex torus, and one can show that \( C \) admits affine structures (given by a coordinate covering for \( C \) with transition functions all linear transformations), which must be also projective structures. Details can be found in [1].

**Theorem 2.9.** There exists a projective connection on \( C \).

**Proof.** Since the question for the case when \( \text{genus}(C) \leq 1 \) has already been dealt with, we now assume that \( \text{genus}(C) \geq 2 \).

We only need to show that \( \sigma \) is exact, i.e., \([\sigma] = 0 \) in \( H^1(U, K_C^2) \). By Serre’s duality theorem (see ([3], Chapter 3, Section 7)), \( H^1(U, K_C^2) \cong H^0(U, K_C^{-1}) \). Since \( \deg(K_C^{-1}) = -\deg(K_C) = 2 - 2\text{genus}(C) < 0 \), \( K_C^{-1} \) has no global sections over \( C \), thus \( H^0(C, K_C^{-1}) = 0 \), hence \( H^1(C, K_C^2) = 0 \), which suffices to show that \([\sigma] = 0 \).

**Remark 2.10.** For the case when \( \text{genus}(C) \geq 2 \), \( \mathcal{B} \) is an affine space for \( H^0(C, K_C^2) \). Using Riemann-Roch theorem, we can compute

\[ \dim(\mathcal{B}) = \dim(H^0(C, K_C^2)) = \chi(C, K_C^2) = \deg(K_C^2) + 1 - \text{genus}(C) = 3\text{genus}(C) - 3 \]
We want to give a projective structure on $C$ explicitly. Suppose now that $\text{genus}(C) \geq 2$, and $\pi : \mathbb{H} \to C$ is the universal covering of $C$. We can embed $\mathbb{H}$ into $\mathbb{P}$ naturally, (identity $\mathbb{P}$ with $\mathbb{C} \cup \{\infty\}$, and regard $\mathbb{H}$ as a subset of $\mathbb{C}$).

For $p \in C$, choose arbitrarily a preimage point $q$ of $p$ under $\pi$, and an open subset $U$ of $\mathbb{H}$, with $q \in U$, such that $\pi|_U : U \to \pi(U)$ is biholomorphic. $(\pi(U), (\pi|_U)^{-1})$ gives a chart around $p$ in $C$, and such charts form an coordinate covering for $C$. We claim that this gives a projective coordinate covering for $C$.

Suppose there is another preimage point $q'$ of $p$ under $\pi$, and an open subset $U'$ of $\mathbb{H}$, with $q' \in U'$, such that $\pi|_{U'} : U' \to \pi(U')$ is also biholomorphic. Without loss of generality, we can assume that $\pi(U) = \pi(U')$. Find a path from $q$ to $q'$ in $\mathbb{H}$, denote by $\gamma$. Then $\pi(\gamma)$ is a loop in $C$ which is based on $p$, denote $[\pi(\gamma)]$ be the element in $\pi_1(C, p)$ induced by $\pi(\gamma)$. Recall that $\pi_1(C, p)$ can be identified with the covering transformation group of $\pi : \mathbb{H} \to C$, and we know that the holomorphic automorphisms of $\mathbb{H}$ form the group $PSL(2, \mathbb{R}) \subseteq PSL(2, \mathbb{C})$. Thus, $[\pi(\gamma)]$ induces a holomorphic automorphism of $\mathbb{H}$, denoted by $\sigma_\gamma$, and we have $\sigma_\gamma \in PSL(2, \mathbb{R}) \subseteq PSL(2, \mathbb{C})$. $\sigma_\gamma$ maps $U$ biholomorphic to $U'$, thus, the two coordinate functions on $\pi(U) = \pi(U')$, $(\pi|_U)^{-1}$ and $(\pi|_{U'})^{-1}$, differ by $\sigma_\gamma \in PSL(2, \mathbb{R}) \subseteq PSL(2, \mathbb{C})$, which proves our claim.

We constructed a projective structure on $C$ from the canonical complex structure of $\mathbb{H}$, such a projective structure on $C$ is called the canonical projective structure on $C$.

**Remark 2.11.** For the case when $\text{genus}(C) = 0$ and $\text{genus}(C) = 1$, we can also construct canonical projective structure on $C$ by similar process. We indeed gave another proof for the existence of the projective structures on a compact, connected Riemann surface in a explicitly way.

## 3 Čech Cohomology with values in a Nonnecessarily Commutative Group

This section is a preparation for the next one. We always denote $M$ to be a topological surface and $G$ be a group with discrete topology.

### 3.1 Čech Cohomology with values in a Nonnecessarily Commutative Group

Let $\mathcal{U}$ be an open covering for $M$, associate to every $U_{\alpha \beta} \neq \emptyset$ an element of $G$, say $\varphi_{\alpha \beta}$, satisfies that

1. $\varphi_{\alpha \beta} \varphi_{\beta \gamma} = \varphi_{\alpha \gamma}$, when $U_{\alpha \beta \gamma} \neq \emptyset$,
2. $\varphi_{\alpha \beta} = \varphi_{\beta \alpha}^{-1}$, when $U_{\alpha \beta} \neq \emptyset$

Such a $(\varphi_{\alpha \beta})$ is called a 1-cocycle with values in $G$. All 1-cocycles with values in $G$ form a set, denoted by $Z^1(\mathcal{U}, G)$

Suppose $(\varphi_{\alpha \beta})$ and $(\psi_{\alpha \beta})$ are two 1-cocycles, and for all $\alpha$, there exist $\lambda_\alpha \in G$, such that $\psi_{\alpha \beta} = \lambda_\alpha^{-1} \varphi_{\alpha \beta} \lambda_\beta$, when $U_{\alpha \beta} \neq \emptyset$, then $(\varphi_{\alpha \beta})$ and $(\psi_{\alpha \beta})$ are said to be equivalent. We
defined an equivalence relation on \( Z^1(U, G) \), and denote \( H^1(U, G) \) be the set of equivalence classes.

Suppose \( U = (U_\alpha) \) and \( V = (V_\beta) \) are two open coverings for \( M \), and \( r : V \to U \) is a refinement, i.e., \( r(V_\beta) \) is one of the \( U_\alpha \)'s with \( V_\beta \subseteq r(V_\beta) \), and suppose we have a 1-cocycle \( \varphi \) in \( Z^1(U, G) \). If \( V_{\beta_1} \cap r(V_{\beta_2}) \neq \emptyset \), then obviously \( r(V_{\beta_1}) \cap r(V_{\beta_2}) \neq \emptyset \), thus we can associated to \( V_{\beta_1} \) the element in \( G \) which has been associated to \( r(V_{\beta_1}) \cap r(V_{\beta_2}) \neq \emptyset \) via \( \varphi \), we then get a 1-cocycle in \( Z^1(V, G) \).

Above process gives a natural map from \( Z^1(U, G) \) to \( Z^1(V, G) \). It is easy to see that equivalent elements in \( Z^1(U, G) \) map to equivalent elements in \( Z^1(V, G) \), thus we have a natural map from \( H^1(U, G) \) to \( H^1(V, G) \).

The open coverings for \( M \) form a directed set, and \( (H^1(U, G))_{U_0} \) form a directed system, (for directed set and directed system, inverse system, see([6], Page 3-6)). We define

\[
H^1(M, G) := \lim_{\to} H^1(U, G)
\]

Remark 3.1. When \( G \) is abelian, we just defined the Čech cohomology with values in \( G \) in degree 1. But when \( G \) is non-abelian, we can’t use the usual method to define cohomology with values in \( G \) in degree more than 1.

### 3.2 A New Approach to the Fundamental Group

It is known that the fundamental group of \( M \) is the set of equivalent classes of loops based on a point, and here two loops are equivalent if and only if they can be continuously deformed to each other. We first give another approach to this.

Let \( U = (U_\alpha) \) be an open covering for \( M \). A sequence \( (U_{\alpha_0}, U_{\alpha_1}, \ldots, U_{\alpha_n}) \) with \( U_{\alpha_0} = U_\alpha \) and \( U_{\alpha_i} \neq \emptyset \), \( 0 \leq i \leq n - 1 \), is called a closed chain for \( U \). Choose and fix \( U_0 \in U \). We consider the set of all the closed chains for \( U \) with beginning and ending terms both \( U_0 \), denoted by \( A \). We can define composition rule in \( A \) by the following

\[
(U_{\alpha_0}, U_{\alpha_1}, \ldots, U_{\alpha_n}) \cdot (U_{\beta_0}, U_{\beta_1}, \ldots, U_{\beta_m}) := (U_{\alpha_0}, U_{\alpha_1}, \ldots, U_{\alpha_n}, U_{\beta_0}, U_{\beta_1}, \ldots, U_{\beta_m})
\]

We say \( (U_{\alpha_0}, U_{\alpha_1}, \ldots, U_{\alpha_i}, U_{\alpha_{i+1}}, \ldots, U_{\alpha_n}) \) and \( (U_{\alpha_0}, U_{\alpha_1}, \ldots, U_{\alpha_i}, U_{\beta}, U_{\beta_1}, U_{\beta_1}, \ldots, U_{\beta_m}) \) can jumped to each other, if and only if \( U_{\alpha_i} \neq \emptyset \). Two close chains in \( A \) are said to be equivalent if and only if they can be related to each other via finitely many jumpings. The set of equivalent classes of \( A \) is defined to be \( \pi_1(U, U_0) \), which has a natural group structure inherit from the composition rule in \( A \).

Let \( U, V \) be two open coverings for \( M \) and \( r : V \to U \) be a refinement, then a closed chain for \( V \) can be contained in a closed chain for \( U \) via \( r \), thus we have a natural map \( \pi_1(V, V_0) \to \pi_1(U, U_0) \) for \( r(V_0) = U_0 \).

Fix a point \( p \in M \), consider pairs \( (U, U_0) \) with \( p \in U_0 \in U \). For two such pairs \( (V, V_0) \) and \( (U, U_0) \), if \( V \) is a refinement of \( U \), and \( U_0 \) is the one containing \( V_0 \), then we say \( (V, V_0) \) is a refinement of \( (U, U_0) \). \( (U, U_0) \)'s form a directed set, and as we have derived, \( (\pi_1(U, U_0))_{(U, U_0)} \) form an inverse system. Define

\[
\pi_1(M, p) := \lim_{\to} \pi_1(U, U_0)
\]
Theorem 3.2. The group we defined above coincide with the fundamental group of \( M \) based on \( p \).

One can see this is natural, but we will not give the details here. See [1].

3.3 A Nice Correspondence

Let \( \mathcal{U} = (U_\alpha) \) be an open covering for \( M \), and \( U_0 \in \mathcal{U} \) fixed. Next we will construct a relation between \( H^1(\mathcal{U}, G) \) and \( \pi_1(\mathcal{U}, U_0) \).

Suppose \( \varphi = (\varphi_{\alpha\beta}) \in Z^1(\mathcal{U}, G) \), \( (U_{a_0}, U_{a_1}, \ldots, U_{a_n}) \) is a closed chain in \( A \), then we can associate to \( (U_{a_0}, U_{a_1}, \ldots, U_{a_n}) \) with \( \varphi_{a_0a_1}\varphi_{a_1a_2} \cdots \varphi_{a_n-1a_n} \in G \). If \( (U_{a_0}, U_{a_1}, \ldots, U_{a_n}) \) can jump to \( (U_{a_0}, U_{a_1}, \ldots, U_{a_i}, U_{\beta}, U_{a_{i+1}}, \ldots, U_{a_n}) \), then \( U_{a_i} \beta_{a_{i+1}} \neq \emptyset \), thus \( \varphi_{a_i\beta_\beta_{a_{i+1}}} = \varphi_{a_i\alpha_{i+1}} \), this implies that \( (U_{a_0}, U_{a_1}, \ldots, U_{a_n}) \) and \( (U_{a_0}, U_{a_1}, \ldots, U_{a_i}, U_{\beta}, U_{a_{i+1}}, \ldots, U_{a_n}) \) are associated with the same element in \( G \). Thus, the element which is associated to \( (U_{a_0}, U_{a_1}, \ldots, U_{a_n}) \) only depend on the equivalent class of \( (U_{a_0}, U_{a_1}, \ldots, U_{a_n}) \). We showed that \( \varphi \) induces a map from \( \pi_1(\mathcal{U}, U_0) \) to \( G \), and which is obviously a group homomorphism.

Denote

\[
\Psi : Z^1(\mathcal{U}, G) \longrightarrow Hom(\pi_1(\mathcal{U}, U_0), G)
\]

Suppose \( \varphi = (\varphi_{\alpha\beta}) \) and \( \psi = (\psi_{\alpha\beta}) \) are two equivalent 1-cocycles in \( Z^1(\mathcal{U}, G) \), i.e., there exists \( \lambda_\alpha \in G \) associated to \( U_\alpha \), such that \( \psi_{\alpha\beta} = \lambda_\alpha^{-1} \varphi_{\alpha\beta} \lambda_\beta \). For a closed chain \( (U_{a_0}, U_{a_1}, \ldots, U_{a_n}) \in A \), compute

\[
\Psi(\psi)((U_{a_0}, U_{a_1}, \ldots, U_{a_n})) = \psi_{a_0a_1} \cdots \psi_{a_{n-1}a_n} = (\lambda_{a_0}^{-1} \varphi_{a_0a_1} \lambda_{a_1}) \cdots (\lambda_{a_{n-1}}^{-1} \varphi_{a_{n-1}a_n} \lambda_{a_n}) = \lambda_{a_0}^{-1} (\varphi_{a_0a_1} \cdots \varphi_{a_{n-1}a_n}) \lambda_{a_n} = \lambda_{a_0}^{-1} (\varphi_{a_0a_1} \cdots \varphi_{a_{n-1}a_n}) \lambda_0
\]

where \( \lambda_0 \) is the element associated to \( U_0 \).

We can define the conjugate action of \( G \) on \( Hom(\pi_1(\mathcal{U}, U_0), G) \) by the following

\[
(g \cdot X)(\cdot) := g^{-1}X(\cdot)g
\]

where \( g \in G \) and \( X \in Hom(\pi_1(\mathcal{U}, U_0), G) \). Denote \( Hom(\pi_1(\mathcal{U}, U_0), G)/G \) to be the set of conjugate classes.

By the previous computation, two equivalent 1-cocycles in \( Z^1(\mathcal{U}, G) \) map to conjugated elements of \( Hom(\pi_1(\mathcal{U}, U_0), G) \), thus, \( \Phi \) induces a natural map from \( H^1(\mathcal{U}, G) \) to \( Hom(\pi_1(\mathcal{U}, U_0), G)/G \), still denoted by \( \Phi \).

Theorem 3.3. We defined an bijection

\[
\Phi : H^1(\mathcal{U}, U_0) \longrightarrow Hom(\pi_1(\mathcal{U}, U_0), G)/G
\]

For the proof, one can see [1]

Corollary 3.4. There is a canonical bijection between \( Hom(\pi_1(M, p), G)/G \) and \( H^1(M, G) \).

Proof. In the previous theorem, take direct limit on both the two sides, we get the correspondence immediately. \qed

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4 Projective Coordinate and Geometric Realization

4.1 Projective coordinate

First we consider the general case. Let $M$ be a topological surface, denote $\mathcal{B}$ to be the space of projective structures on $M$. For a projective coordinate covering $U = (U_\alpha, z_\alpha) \in \mathcal{B}$, associated to $U_{\alpha\beta} \neq \emptyset$ the transition function $f_{\alpha\beta} := z_\alpha z_\beta^{-1} \in PSL(2, \mathbb{C})$, and this gives a 1-cocycle on $M$ with values in $PSL(2, \mathbb{C})$. We may use a different representation $\mathcal{V} = (V_\beta, w_\beta)$ for the same projective structure. After a refinement, we can assume $\mathcal{V} = (U_\alpha, w_\alpha)$. Set $\lambda_\alpha := z_\alpha w_\alpha^{-1} \in PSL(2, \mathbb{C})$, then $w_\alpha w_\beta^{-1} = \lambda_\alpha^{-1} f_{\alpha\beta} \lambda_\beta$ when $U_{\alpha\beta} \neq \emptyset$. Thus, different representations give equivalent 1-cocycles in $Z^1(M, PSL(2, \mathbb{C}))$. We constructed a natural map from $\mathcal{B}$ to $H^1(M, PSL(2, \mathbb{C}))$.

Remark 4.1. The image of a projective structure under the map is called the projective coordinate of that projective structure. The map constructed above is neither one-to-one nor onto, but things are better when $M = \mathbb{C}$ is a compact, connected Riemann surface.

Theorem 4.2. The natural map $\mathcal{B} \rightarrow H^1(\mathbb{C}, PSL(2, \mathbb{C}))$ is one-to-one, i.e., the projective structure on a compact, connected Riemann surface $\mathbb{C}$ is uniquely determined by its projective coordinate.

Proof of the theorem is complicated, one can see [1].

4.2 Geometric Realization of Projective structure

Suppose $\mathbb{C}$ is a compact, connected Riemann surface, $\pi: \tilde{\mathbb{C}} \rightarrow \mathbb{C}$ is the universal covering of $\mathbb{C}$.

Theorem 4.3. A projective structure on $\mathbb{C}$ naturally induces a projective structure on $\tilde{\mathbb{C}}$.

Proof. Suppose $\mathcal{U} = (U_\alpha, z_\alpha)$ is a projective coordinate covering for $\mathbb{C}$. We may assume that $\mathcal{U}$ is fine enough, so that every $U_\alpha$ is well covered by $\pi$, i.e., the preimage of $U_\alpha$, $\pi^{-1}(U_\alpha)$ is a disjoint union of certain pieces, and each piece is biholomorphic to $U_\alpha$ via $\pi$. We claim that the collection of these pieces gives a projective covering for $\tilde{\mathbb{C}}$. In fact, suppose we have $V_\alpha, V_\beta$, with $\pi: V_\alpha \rightarrow U_\alpha$ and $\pi: V_\beta \rightarrow U_\beta$ biholomorphic, and $V_{\alpha\beta} := V_\alpha \cap V_\beta \neq \emptyset$. Then the transition function between $V_\alpha, V_\beta$ is $(z_\alpha \circ \pi) \circ (z_\beta \circ \pi)^{-1} = z_\alpha z_\beta^{-1} \in PSL(2, \mathbb{C})$. Thus, we induce a projective structure on $\tilde{\mathbb{C}}$ from the projective structure $\mathcal{U} = (U_\alpha, z_\alpha)$ on $\mathbb{C}$.

It is easy to see that another representation of the same projective structure induces the same projective structure on $\tilde{\mathbb{C}}$, and we finish the proof.

Remark 4.4. Also by checking locally, we can prove that different projective structures on $\mathbb{C}$ induce different projective structures on $\tilde{\mathbb{C}}$.

Note that $\tilde{\mathbb{C}}$ is simply connected, by Corollary 3.4, $H^1(\tilde{\mathbb{C}}, PSL(2, \mathbb{C}))$ is trivial, thus every projective structure on $\tilde{\mathbb{C}}$ is associated with a trivial projective coordinate.
Remark 4.5. 1. When $\text{genus}(C) = 0$, we have $C \cong \mathbb{P}^1$, combining with Theorem 4.2, we can deduce that there is only one projective structure on $\mathbb{P}^1$, which must be the standard one.

2. When $\text{genus}(C) = 1$, we have $\tilde{C} \cong \mathbb{C}$; when $\text{genus}(C) = 2$, we have $\tilde{C} \cong \mathbb{H}$. In these two cases, $\tilde{C}$ is noncompact, thus, we can not use Theorem 4.2 to claim that there is only one projective structure on $\tilde{C}$.

Suppose now we have a projective coordinate covering $\mathcal{U} = (U_\alpha, z_\alpha)$ for $C$, which induces a projective coordinate covering $\tilde{\mathcal{U}}$ for $\tilde{C}$. Since the projective coordinate associated to $\tilde{U}$ is trivial, we can find a refinement $\mathcal{V}$ of $\tilde{\mathcal{U}}$, such that the 1-cocycle in $Z^1(\mathcal{V}, PSL(2, \mathbb{C}))$ induced by the projective coordinate covering $\tilde{\mathcal{U}}$ is equivalent to the trivial one in $Z^1(\mathcal{V}, PSL(2, \mathbb{C}))$. Thus, we can add certain coordinate functions to $\mathcal{V}$, giving the same projective structure as $\tilde{\mathcal{U}}$, for which the transition functions are all equal to $1 \in PSL(2, \mathbb{C})$. Thus, the coordinate functions on $\mathcal{V}$ can be glued together to get a locally biholomorphic map $\tilde{\rho}$ from $\tilde{C}$ to $D \subseteq \mathbb{P}^1$. Since $\rho$ is biholomorphic, $D$ must be an open, connected subset of $\mathbb{P}^1$.

Remark 4.6. We have to allow the coordinate functions on $\mathcal{V}$ having values in $\mathbb{P}^1$, because we need to do some Möbius transformations to make the transition functions equal to 1.

It is possible that another map $\rho' : \tilde{C} \rightarrow D' \subseteq \mathbb{P}^1$ is also given in this way. Since $\rho, \rho'$ are locally biholomorphic, for a point $p \in \tilde{C}$, there exists $U_p$, $p \in U_p \subseteq \tilde{C}$, such that $\rho|_{U_p}$ and $\rho'|_{U_p}$ are coordinate functions. Then $\rho'|_{U_p} \circ \rho|_{U_p}^{-1} \in PSL(2, \mathbb{C})$, thus, there exists $g_p \in PSL(2, \mathbb{C})$, such that

$$\rho'|_{U_p} = g_p \circ \rho|_{U_p}$$

the $g_p$’s should coincide on the intersection of $U_p$’s, thus there exists a $g \in PSL(2, \mathbb{C})$, such that $\rho' = g \circ \rho$. We have the following

Theorem 4.7. A projective structure on $C$ determines a locally biholomorphic map $\rho : \tilde{C} \rightarrow D \subseteq \mathbb{P}^1$ unique up to a Möbius transformation.

As we know, $\pi_1(C)$ can be identified with the covering transformation group of $\pi : \tilde{C} \rightarrow C$. Suppose $T : \tilde{C} \rightarrow \tilde{C}$ is a covering transformation, then $T$ preserves the projective structures, i.e., $T$ can be written as a Möbius transformation when we fix the projective coordinate locally. For $p \in \tilde{C}$, there exists $T_p \in PSL(2, \mathbb{C})$, $\rho \circ T(g) = T_p \circ \rho(g)$ for $g$ around $p$. $T_p$ depends on $p$ continuously, with $PSL(2, \mathbb{C})$ endowed with discrete topology. Thus, there exists $T_0$, such that $\rho \circ T(p) = T_0 \circ \rho(p)$, for all $p$. We associated to $T \in \pi_1(C)$ an element $T_0 \in PSL(2, \mathbb{C})$, this is indeed a group homomorphism $\tilde{\rho} : \pi_1(C) \rightarrow PSL(2, \mathbb{C})$, and we have

$$\rho \circ T(p) = \tilde{\rho}(T) \rho(p)$$

for all $T \in \pi_1(C)$, for all $p \in \tilde{C}$.

Suppose $\rho' = g \rho$, then

$$\rho' \circ T(p) = g \rho \circ T(p) = g \tilde{\rho}(T) \rho(p) = g \tilde{\rho}(T) g^{-1} \rho'(p)$$

thus

$$\tilde{\rho}' = g \tilde{\rho} g^{-1} : T \mapsto g \tilde{\rho}(T) g^{-1}$$
This tells us that we should regard $\tilde{\rho}$ as an element in $H^1(\pi_1(C), PSL(2, \mathbb{C}))/PSL(2, \mathbb{C})$, and in fact, this element coincides with the one we obtained before.

**Definition 4.8.** The pair $(\rho, \tilde{\rho})$ with $\rho : \tilde{C} \rightarrow D \subseteq \mathbb{P}^1$ and $\tilde{\rho} : \pi_1(C) \rightarrow PSL(2, \mathbb{C})$ obtained above is called a geometric realization of the projective structure on $C$.

We have another way to understand projective structures.

Fix a projective structure on $C$. For $p \in C$, consider the chart $(z_p, U_p)$, which gives the fixed projective structure locally. Every two such charts differ by a unique Möbius transformation, and if the Möbius transformation is trivial, we define the two charts to be the same. Thus, the space defined by $E := \{(p, z_p, U_p)|(z_p, U_p) \text{ is a chart for } p \text{ which gives the fixed projective structure}\}$ is a principal $PSL(2, \mathbb{C})$ bundle over $C$, where $PSL(2, \mathbb{C})$ is endowed with discrete topology. (For principal bundle, see ([5], Page 40-43))

$E$ is in fact a covering of $C$. Fix a point $p \in C$, and $(p, z_p, U_p) \in E$. An element $\gamma$ in $\pi_1(C, p)$ can be lifted to be a homotopy class of pathes in $E$ with beginning point $p$. We denote the ending point (which does not depend on the choice of the loop in the class $\gamma$) by $(p, z'_p, U'_p) = (p, z_p, U_p) \cdot g$, where $g \in PSL(2, \mathbb{C})$.

We constructed a map from $\pi_1(C, p)$ to $PSL(2, \mathbb{C})$ by sending $\gamma$ to $g$. If we fix a different beginning point in $E$, then we can get another map from $\pi_1(C, p)$ to $PSL(2, \mathbb{C})$, which can be easily shown to be conjugated to the one we obtained before.

We get the map from $B$ to $Hom(\pi_1(C), PSL(2, \mathbb{C}))/PSL(2, \mathbb{C})$ once again.

### 5 Trivialisation of $\mathcal{L}$ on an infinitesimal neighborhood of $\Delta$

#### 5.1 Some Basic Notations

Let $C$ be a compact connected Riemann Surface, denote $S := C \times C$, denote $p_1, p_2$ be the two projections from $S$ to $C$, recall that $K_C$ is the canonical line bundle of $C$, $K_S := p_1^*K_C \otimes p_2^*K_C$ is the canonical line bundle of $S$. Denote $\Delta : C \hookrightarrow S, x \mapsto (x, x)$ be the diagonal embedding, and $\Delta$ can be regarded as a prime Weil divisor of $S$. Define $\mathcal{L} := K_S(2\Delta) := K_S \otimes O_S(\Delta)^2$.

**Remark 5.1.** The sections of $\mathcal{L}$ are 2-forms which admit poles of 2-order along the diagonal $\Delta$. Explicitly say, for a point $p \in C$, give a local chart $(U, z)$ for $p$, with $z(p) = 0$, then the section around $(p, p) \in S$ can be represented as $f(z_1, z_2)dz_1 \wedge dz_2$, where $f : z(U) \times z(U) \rightarrow \mathbb{C}$ is a holomorphic function.

Denote $n\Delta$ the $(n - 1)$-th infinitesimal neighborhood of $\Delta$ in $S$, see ([3], Chapter 2, Section 9). We have

$$(n - 1)\Delta \hookrightarrow n\Delta \hookrightarrow S$$

for $n \geq 2$ and the two maps are both closed immersions of analytic spaces. We use the convention that every line bundle on a closed subspace can be regarded as a line bundle on the whole space via extension by 0. A nowherevanishing global section of $\mathcal{L}|_{n\Delta}$ is called a trivialisation of $\mathcal{L}|_{n\Delta}$ or a trivialisation of $\mathcal{L}$ over $n\Delta$. A line bundle is called trivialisable if it admits at least one trivialisation.
Let’s make it more clear what is $L|_{\Delta}$ like. Around $(p, p) \in \Delta \subseteq S$, a section of $L$ can be represented as $f(z_1, z_2) \frac{dz_1 - dz_2}{(z_1 - z_2)^2}$, where $f : z(U) \times z(U) \to \mathbb{C}$ is a holomorphic function. If we have another coordinate chart for $p$ in $C$, say $\zeta$, write

$$z(\zeta) = a_0 + a_1 \zeta + a_2 \zeta^2 + \ldots, a_1 \neq 0$$

then in the $\zeta$-chart,

$$s = \frac{f(z(\zeta_1), z(\zeta_2))dz(\zeta_1)dz(\zeta_2)}{(z(\zeta_1) - z(\zeta_2))^2} = \frac{f(a_1 + 2a_2 \zeta_1 + \ldots)(a_1 + 2a_2 \zeta_2 + \ldots)}{(a_1(\zeta_1 - \zeta_2) + a_2(\zeta_1^2 - \zeta_2^2) + \ldots)^2} d\zeta_1 \wedge d\zeta_2$$

$$= \frac{fd\zeta_1 \wedge d\zeta_2}{(\zeta_1 - \zeta_2)^2} \cdot \frac{f(a_1 + 2a_2 \zeta_1 + \ldots)(a_1 + 2a_2 \zeta_2 + \ldots)}{(a_1 + a_2(\zeta_1 + \zeta_2) + \ldots)^2}$$

Note that $\frac{(a_1 + 2a_2 \zeta_1 + \ldots)(a_1 + 2a_2 \zeta_2 + \ldots)}{(a_1(\zeta_1 - \zeta_2) + a_2(\zeta_1^2 - \zeta_2^2) + \ldots)^2} = 1$ when $\zeta_1 = \zeta_2$, thus $s = \frac{f(z(\zeta_1), z(\zeta_2))d\zeta_1 \wedge d\zeta_2}$ when restrict to $\Delta \subseteq S$.

Now take $f = 1$, we just proved $\frac{dz_1 \wedge dz_2}{(z_1 - z_2)^2}$ and $\frac{d\zeta_1 \wedge d\zeta_2}{(\zeta_1 - \zeta_2)^2}$ represent the same section around $(p, p)$ in $L|_{\Delta}$, thus all terms of form $\frac{dz_1 \wedge dz_2}{(z_1 - z_2)^2}$ glue to a global section of $L|_{\Delta}$, which is nowherevanishing on $\Delta$, thus we get a canonical trivialisation of $L|_{\Delta}$.

Let $\iota$ be the involution of $S$ defined by $(x, y) \mapsto (y, x)$, of which $\Delta$ is the fixed points set. We can lift $\iota$ to the line bundle $L$ and $L|_{n\Delta}$, still denoted by $\iota$.

### 5.2 The Second Main Correspondence

**Theorem 5.2.**

1. $L|_{n\Delta}$ is trivialisable, for all $n \geq 1$.
2. $L|_{2\Delta}$ admits a trivialisation, which is invariant under the action of $\iota$ and restrict to the canonical trivialisation of $L|_{\Delta}$. Such a trivialisation is unique.

**Proof.** We will not give the proof of 1 and the former part of 2, see [2]. Let us prove the uniqueness in 2.

We have short exact sequence

$$0 \to K_S(\Delta)/K_S \to L|_{2\Delta} \to L|_{\Delta} \to 0$$

If we have a global section $s \in H^0(S, L|_{\Delta})$, which is invariant under the action of $\iota$ and vanishes under a restriction to $\Delta$. Then we can find a chart $(U, z)$, and write $s$ locally as $\frac{f(z_1, z_2)}{z_1 - z_2} dz_1 \wedge dz_2$, with $f$ a holomorphic function on $z(U) \times z(U)$.

Since $\frac{f(z_1, z_2)}{z_1 - z_2} dz_1 \wedge dz_2$ is invariant under the action of $\iota$, we know that $s$ must be odd, thus $f(z, z) = 0$, thus $\frac{f(z_1, z_2)}{z_1 - z_2}$ is holomorphic on $z(U) \times z(U)$, thus $s = 0$ as a global section of $L|_{2\Delta}$. We proved the uniqueness. \[\square\]

**Theorem 5.3.** Denote $\Omega$ to be the space of the trivialisations of $L|_{3\Delta}$ which restrict to the unique canonical trivialisation of $L|_{2\Delta}$, then $\Omega$ is an affine space for $H^0(C, K^2_C)$ and there exists a canonical identification between $B$ and $\Omega$ as affine spaces.
Proof. Denote $\rho : \mathcal{L}|_{3\Delta} \longrightarrow \mathcal{L}|_{2\Delta}$ to be the canonical restriction.
Sections of $\ker(\rho)$ are those 2-forms without poles along the diagonal $\Delta$ and being ignored the part which contains factor $z_1 - z_2$, thus $\ker(\rho)$ is the restriction of $K_S$ to $\Delta$, which can be identified with $K_C^2$.

In an other word, denote $\mathcal{J}_\Delta$ to be the ideal sheaf which gives $\Delta \subseteq S$, then

\[ \mathcal{L}|_{3\Delta} = K_S(2\Delta)/\mathcal{J}_\Delta^3 K_S(2\Delta) = \mathcal{J}_\Delta^{-2} K_S/\mathcal{J}_\Delta K_S \]

\[ \mathcal{L}|_{2\Delta} = \mathcal{J}_\Delta^{-2} K_S/K_S \]

thus

\[ \ker(\rho) \cong K_S/\mathcal{J}_\Delta K_S = K_S|_\Delta \cong K_C^2 \]

Now, we have a short exact sequence:

\[ 0 \longrightarrow K_C^2 \longrightarrow \mathcal{L}|_{3\Delta} \longrightarrow \mathcal{L}|_{2\Delta} \longrightarrow 0 \]

which induces a long exact sequence

\[ 0 \longrightarrow H^0(C, K_C^2) \longrightarrow H^0(S, \mathcal{L}|_{3\Delta}) \longrightarrow H^0(S, \mathcal{L}|_{2\Delta}) \longrightarrow H^1(C, K_C^2) \longrightarrow \ldots \]

$\Omega \subseteq H^0(S, \mathcal{L}|_{3\Delta})$ is the set of preimages of the canonical trivialisation of $\mathcal{L}|_{2\Delta}$, thus, $\Omega$ is an affine space for $H^0(C, K_C^2)$.

Next, let's construct a bijection $F : \mathcal{B} \longrightarrow \Omega$, which preserves the affine structures.

Suppose now we are given a projective coordinate covering $U = (U_\alpha, z_\alpha)$. Consider around $(p, p) \in S$, suppose there are two charts $z_\zeta, \zeta$ for $p \in C$, differ by a M"{o}bius transformation, we claim that $\frac{dz_1 \wedge dz_2}{(z_1 - z_2)^2}$ and $\frac{dz_1 \wedge dz_2}{(\xi_1 - \xi_2)^2}$ give the same sections of $\mathcal{L}$.

In fact, suppose $z = \frac{a\zeta + b}{c\zeta + d}$, $ad - bc = 1$, then

\[ \frac{dz_1 \wedge dz_2}{(z_1 - z_2)^2} = \frac{\frac{dz_1}{(c\zeta + d)^2} \wedge \frac{dz_2}{(c\zeta + d)^2}}{\frac{dz_1}{(c\zeta + d)^2} \wedge \frac{dz_2}{(c\zeta + d)^2}} = \frac{d\zeta_1 \wedge d\zeta_2}{(\zeta_1 - \zeta_2)^2} \]

Thus, we can glue the sections of $\mathcal{L}$ which are near the diagonal $\Delta$ and of form $\frac{dz_1 \wedge dz_2}{(z_1 - z_2)^2}$ together to get a section of $\mathcal{L}$ over a neighborhood (in the ordinary topology) of $\Delta$, and after restricting to $3\Delta$, we get a trivialisation of $\mathcal{L}|_{3\Delta}$. This gives our map $F$.

We claim that

\[ F(\mathfrak{h} + \gamma) = F(\mathfrak{h}) + \frac{\gamma}{6} \]

for $\mathfrak{h} \in \mathcal{B}$ and $\gamma \in H^0(C, K_C^2)$, which shows that the map $F$ is a bijection and preserves the affine structures of $\mathcal{B}$ and $\Omega$.

Now fix our $\mathfrak{h} \in \mathcal{B}$ and $\gamma \in H^0(C, K_C^2)$, there exists a projective coordinate covering $U = (U_\alpha, z_\alpha)$ for $C$, which gives $\mathfrak{h}$, and $\lambda = (\lambda_\alpha dz_\alpha \otimes dz_\alpha)$ with $\lambda_\alpha$ on $U_\alpha$, which gives the 0-cocycle $\gamma$. And we may assume that $U$ is fine enough, so that there exists $w_\alpha : z_\alpha(U_\alpha) \longrightarrow \mathbb{C}$ biholomorphic with $(Sw_\alpha)(z_\alpha(p)) = \lambda_\alpha(z_\alpha(p)), p \in U_\alpha$. According to the proof of Theorem 2.8, $(U_\alpha, w_\alpha \circ z_\alpha)$ is a projective coordinate covering for $C$, which gives $\mathfrak{h} + \gamma \in \mathcal{B}$. We care about the difference of $F(\mathfrak{h} + \gamma), F(\mathfrak{h}) \in H^0(S, \mathcal{L}|_{3\Delta})$. 

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Let’s compute around \((p, p) \in S\). Suppose \((U, z)\) is a chart for \(p \in C\) \(((U, z)\) is one of the \((U_\alpha, z_\alpha)\)'s), with \(z(p) = 0\). We have \(\lambda\) on \(U\), and \(\lambda dz \otimes dz\) locally represents \(\gamma\). We can find \(w\) on \(z(U)\) with \((Sw)(z(p)) = \lambda(z(p)), p \in U\). Since we can always compose \(w\) with a Möbius transformation and this does not change \(Sw\). We can write

\[w(z) = z + a_1 z^2 + a_2 z^3 + \ldots\]

Compute around \((p, p) \in S\)

\[
F(h + \gamma) - F(h) = \frac{dw(z_1) \wedge dw(z_2)}{(w(z_1) - w(z_2))^2} - \frac{dz_1 \wedge dz_2}{(z_1 - z_2)^2} \\
= \frac{(1 + 2a_1z_1 + 3a_2z_1^2 + \ldots)(1 + 2a_1z_2 + 3a_2z_2^2 + \ldots)dz_1 \wedge dz_2}{(z_1 - z_2)^2} - \frac{dz_1 \wedge dz_2}{(z_1 - z_2)^2} \\
= \frac{(1 + 2a_1z_1 + \ldots)(1 + 2a_1z_2 + \ldots) - (1 + a_1(z_1 + z_2) + \ldots)^2 dz_1 \wedge dz_2}{(1 + a_1(z_1 + z_2) + a_2(z_1^2 + z_1z_2 + z_2^2) + \ldots)^2 (z_1 - z_2)^2} \\
= \frac{(a_2 - a_1^2)(z_1 - z_2)^2 + (z_1 - z_2)^2O(\sqrt{z_1^2 + z_2^2}) dz_1 \wedge dz_2}{(1 + a_1(z_1 + z_2) + a_2(z_1^2 + z_1z_2 + z_2^2) + \ldots)^2 (z_1 - z_2)^2}
\]

Thus \(F(h + \gamma) - F(h) = (a_2 - a_1^2)dz \otimes dz\) as \(z_1, z_2 \rightarrow 0\).

On the other hand, we can compute \(\gamma\) around \(p\)

\[
\lambda(z) = (Sw)(z) = \frac{2w'w'' - 3(w'')^2}{2(w')^2}(z) = \frac{2(1 + 2a_1z + \ldots)(6a_2 + \ldots) - 3(2a_1 + \ldots)^2}{2(1 + 2a_1z + \ldots)^2}
\]

thus \(\lambda(0) = 6(a_2 - a_1^2)\).

We see immediately that \(F(h + \gamma) - F(h) = \frac{\gamma}{\lambda}\). Thus, \(F\) is a bijection from \(B\) to \(\Omega\), and preserves the affine structures.

\[\square\]

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First I would like to thank my supervisor, Professor Eduard Looijenga, for introducing the beautiful topic to me. And I learned a lot, not only mathematics, but also some useful skills and good characteristics, from Professor Looijenga, which benefits for my life.

I thank the department of mathematical science, for providing a lot of sources for my study.

I have to thank my beloved family, for their love and support, which are always around me during my life.

I also thank my friends, for helping me learning a lot of mathematics through our communication.

References


毕业感言

李嘉伦*

本科四年，倒是有半年在巴黎度过，好在仍拿到了清华的学位。既然毕业了，就写写自己的经历感受。

刚进大学时，面对校园里的活动，看花了眼。于是参加不少的社团。有人说：社团就是学长学妹呢。可我是学弟，没有非分之想，在社团是好好干活的，尽心尽力，并且积极参与组会，活动。但渐渐的，开始的新奇，热情消退后，发觉自己不适合。活动时常是与小孩子接触，可，书读得越多自己却越缺乏亲和力。或是有点冷漠，在小孩子面前愣头楞脑。但第一年在社团的经历，至少令我知道不擅长什么，少了些无谓的幻想。

第二年开始，要好好学习，可大二要求的课程较少，我便去听一些高年级的课程。其中一门是周坚老师的微分几何，周老师的风格大家是知道的。课上似懂非懂，但不觉厌倦。有所不懂，也不吱声，担心问题太傻，或者问题太多，而不知道问什么。但这课很给劲，一上来就“欧高黎嘉陈”，“究天人之际，通古今之变”，有种“玄而又玄，众妙之门”的感觉。当时有一节课是在周五晚上，上完课后，在六教楼下的寒风中，内热外冷，有种飘飘然的感觉。和同学显摆时，无一不佩服的样子。在以后的学习中，发现自己其实并不懂，只稍稍领教思想，得从头开始理解定义、细节。但周坚老师的课把我领到一个新的世界，令自己对几何产生浓厚的兴趣，数学还可以是这样子的。

后两年，主要讲讲巴黎的学习生活。经过一轮申请的选拔后，身边的同学都是爱数学的。日常生活中数学成了最大的话题。散步时聊，吃饭时聊。吃饭的时候，中国人抱团坐在一起，吃着法国菜，说着中国话。面对说着中文的一群中国人，外国人不会加入我们；而这中文的内容还是数学，不学数学的中国人，只能在一旁默默吃饭，干瞪眼儿。一来二去，就剩下数学的小圈子。但小圈子也会有数学领域的八卦，各个教授的师承，讲课研究的好坏，还有什么学界装13指南。在他人眼中，我们几个人就是大学霸。聊数学训练自己的表达，锻炼思维，好处多多，也不赘述了。想起在清华时，楼梯道边上有华罗庚的话，最能表达我的心境：“见面少叙寒暄话，多
把艺术谈几声”。法国教授的研究水平整体很高，教课的态度也认真。按自己的思路讲，还会把自己写的讲义放到网上。可是某几个老师的课，却听不懂。以为是自己的法语太差，听懂大致意思，却错过关键点，为此而深深的苦恼。后来问法国人，他们也不懂，才释然。但有时，所讲的数学我懂，讲到其中的想法、哲学时，就云里雾里；尤其是讲到好玩的段子，法国人都笑了，自己却不知所云。学生选课相当自由，想上课就去听，过了考试就能有学分。总之，巴黎是个很不错的学习数学的地方。
毕业感言

毛逸翔*

四年的本科时光看似很长，其实很短，一转眼便已从我的身边溜走。清华在我身上留下了很多东西，但却已成为了一段我再也回不去的回忆。现在在书桌前坐定，希望能够写下一些文字，回顾四年大学生活的片段。

能在清华度过我青春最美好的四年，我感觉我是很幸运的。本科四年的时光，是最无拘无束的。清华园里有太多的优秀的人，与其他的同学相处，让我成长为了我现在的样子。在学生会担任职务时跟学长还有同学们身上学到的为人处世的态度，在辩论队讨论问题时从朋友身上学到的思维角度和思维方式，在棒球队训练、比赛时从教练和队友身上学到的努力和坚持，在数学的学习上从身边的大神们身上学到的执著和钻研，这一切的一切都让我受益匪浅，也将成为我一生的财富。

我走到今天就读数学博士的原因说来也是一连串巧合。当初大一大二的我本来是抱着打好数学基础的信念来学习数学的，所以上课做作业都很认真，学得也比较扎实。到了大三的时候，一次偶然的机会，我参加了数学中心的一场应用数学的报告，从而让我对数学产生了兴趣。在大三下，我又很幸运地选了法国讲席团的《高等概率》课以及数学中心的罗涛老师开设的《偏微分方程(1)》。在两节课上，老师的认真负责和深入浅出的讲解把随机过程和偏微分方程的知识讲得非常透彻，也让我喜欢上了那部分的知识。自那个时候起，我有了自己喜欢的方向。也正是对这个方向追寻，支持着我走到现在。

不过说到底，生活还是有太多的可能性。或许你本身并不知道自己想要学什么，却误打误撞走进了数学的殿堂，并喜欢上了这里；又或许你本身抱着成为数学家的心态来到这里，却发现这里的数学与你想象的并不相同，失望而迷茫。一切的一切都不可预知，不过这正是生活有趣的地方。在清华，在你们脚下的这方土地，只要你愿意去尝试，愿意去探索，总有适合你的资源和平台在等着你。

Anyway，请跟从自己的内心，过自己想要的生活。享受在清华的生活吧!

*基数02
毕业感言

张胜寒*

不知不觉就毕业了，在园子里还没待够就这么离开，多少还是有些舍不得。
清华，对于来自高考大省河南的我来说，是一个长久以来的梦。但当梦想化为现实以后，我又开始有些迷茫。身边的竞考生都是神一般的存在，自己却是小白一个，什么都不懂，什么都不懂。那时候，真的不太确定之后能走的路、想走的路、该走的路。

一直觉得，那些从一开始就目标明确的人是幸福的。他们可以很早的开始筹划，一步一步踏踏实实的朝前迈进。而我，就要绕很多弯弯道道。不过学数学的一大好处，就是多管点事，学好数学总是没错的。因为不管之后从事什么工作，一个良好的数学基础都会带来莫大的帮助。所以呢，尽管仍有困惑和忧虑，我就按照自己的节奏，一步一步的走下去，并在不断前进的同时，思考着最终该前进的方向。

学数学的另一大好处，就是不需要像工科院系那样学的杂而深。数学课少而难，需要投入不少时间去自习。但是课程之外，合理安排的话，也还是可以有很多自由支配的时间。于是我参加了山野协会，开始了户外生涯，平时在操场跑步刷圈，周末则去郊外拉练或者野营。在系外结识了很多高年级的同学，和他们交谈，能明显感受到数学的重要性，这也坚定了我学好数学的决心。但到了大三，学习了一些研究生课程，明显感受到以前所学课程开始紧密联系起来，才开始真正感受到了数学的魅力，从而下定决心从事数学研究。

这几年，随着数学中心的建立和壮大，清华数学在日益的变强。请来了很多大师来授课作报告，很多以前都不怎么开的课程，也都逐渐开设了起来。Ben Andrews教授的微分几何，是促使我对几何感兴趣的直接原因。而于今于品老师等中心的年轻老师学习，也都极大的开拓了我的视野。除了基础数学以外，中心也有一些做应用数学和统计的老师。所以有志于从事科研工作的同学，非常建议多去中心走走，找中心教授们聊聊的什么的。

最后，愿各位清华数学人都能学有所成，朝着自己的梦想不断前进。

*基数01

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