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Frobenius流形的一些例子

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摘要
Dubrovin在1991年由数学物理中的2-D拓扑场论的研究中引进了Frobenius流形的概念。这是一类复流形，其上的WDVV方程组决定了其切空间上的Frobenius代数结构。在得到其上的正交坐标系后，我们可更加方便地研究与旋转系数有关的方程组，并在此过程中发现Frobenius流形与某个算子Λ的单值有密切联系，由此得到两者间自然的对应。

关键词：Frobenius流形，WDVV方程组，正则坐标，保单值形变

1 引言

在数学物理对2-D拓扑场论的研究中，我们寻找这样一类多复变函数$F = F(t)$，

$$t = (t^1, ..., t^n)$$

使其三阶导数

$$c_{\alpha\beta\gamma}(t) := \frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\beta \partial t^\gamma}$$

满足下面的性质

1) 规范化:

$$\eta_{\alpha\beta} := c_{1\alpha\beta}(t)$$

构成一个常值非退化矩阵。令

$$\eta^{\alpha\beta} := (\eta_{\alpha\beta})^{-1}.$$ 

我们将使用矩阵$(\eta^{\alpha\beta})$和$(\eta_{\alpha\beta})$来进行升降指标操作。

2) 结合性：对任何$t$，如下函数

$$c^{\gamma}_{\alpha\beta}(t) := \eta^{\gamma^\varepsilon} c_{\varepsilon\alpha\beta}(t)^1$$

在一个以$e_1, ..., e_n$为基的$n-$维空间上定义了一个结合代数$A_t$的结构

∗基数81

¹在本文中，若未作特别说明，均默认当上下脚标同时出现时为对其求和.
\( e_\alpha \cdot e_\beta = c_\alpha^\gamma(t)e_\gamma \)  \hspace{1cm} (1.3)

易知这是一个交换代数。又

\[ c_{1\alpha}(t) = \sum_{\varepsilon=1}^{n} \eta^{\varepsilon}_{\beta \alpha} = \delta^\beta_{\alpha} \]  \hspace{1cm} (1.4)

因此, \( \forall e_\alpha \)，有 \( (e_1,e_\alpha) = c_\alpha^\gamma(t)e_\gamma = ... \)，

从而得到一般的准齐次条件的形式。

3) \( F(t) \) 关于其变量满足准齐次性:

对某些固定数 \( d_1, ..., d_n, d_F \) 成立下式

\[ F(c^{d_1}t^1, ..., c^{d_n}t^n) = c^{d_F}F(t^1, ..., t^n) \]  \hspace{1cm} (1.5)

上式左右两边对 \( t^i \) (\( i = 1, ..., n \)) 求导，得

\[ \partial_i F(c^{d_1}t^1, ..., c^{d_n}t^n)c^{d_i} = c^{d_F}\partial_i F(t^1, ..., t^n) \]  \hspace{1cm} (*)

在对(1.5)式关于 \( c \) 两边求导，得

\[ \sum_{i=1}^{n} \partial_i F(c^{d_1}t^1, ..., c^{d_n}t^n)dc^{d_i}t^i = d_F c^{d_F}F(t^1, ..., t^n) \]  \hspace{1cm} (**)

将 (*) 式代入 (**), 得到

\[ \sum_{i=1}^{n} d_i t^i \partial_i F(t^1, ..., t^n) = d_F F(t^1, ..., t^n) \]

为方便引入 Euler 向量场记号: \( E = E^\alpha(t)\partial_\alpha \)，此时 \( E^\alpha(t) = d_at^\alpha \)，则准齐次条件化为

\[ \mathcal{L}_E F(t) := E^\alpha(t)\partial_\alpha F(t) = d_F \cdot F(t) \]  \hspace{1cm} (1.6)

这里线性向量场 \( E = E(t) = \sum_\alpha d_\alpha t^\alpha \partial_\alpha \) 生成伸缩变换 (1.5)。

注意到李导数 \( \mathcal{L}_E \) 在向量场 \( \partial_i \) (\( i = 1, ..., n \)) 上的作用为

\[ \mathcal{L}_E \partial_i = [E, \partial_i] = [\sum_\alpha d_\alpha t^\alpha \partial_\alpha, \partial_i] = -d_i \partial_i \]  \hspace{1cm} (1.7)

下面的两个对准齐次性和 Euler 向量场的一般化的推广将在之后的讨论中很重要:

1. 在 \( F(t) \) 上加上一个关于 \( t^1, ..., t^n \) 的二次多项式 \( P(t) \) 后，并不改变其三阶导数，故代数 \( A_t \) 保持不变。设 \( P(t) = A_{\alpha\beta}t^\alpha t^\beta + B_\alpha t^\alpha + C \)，则

\[ \mathcal{L}_E P(t) = \sum_{\alpha,\beta} A_{\alpha\beta}(d_\alpha + d_\beta)t^\alpha t^\beta + \sum_\alpha d_\alpha B_\alpha t^\alpha \]

由 \( \mathcal{L}_E (F(t) + P(t)) = d_F (F(t) + P(t)) \) \( \Rightarrow \)

\[ \mathcal{L}_E F(t) - d_F F(t) = \sum_{\alpha,\beta} (d_\beta - d_\alpha) A_{\alpha\beta} t^\alpha t^\beta + \sum_\alpha (d_\beta - d_\alpha) B_\alpha t^\alpha + C d_F \]

设 \( A'_{\alpha\beta} = (d_\beta - d_\alpha) A_{\alpha\beta} \) , \( B'_\alpha = (d_\beta - d_\alpha) B_\alpha \) , \( C' = C d_F \) ，上式化为

\[ \mathcal{L}_E F(t) = d_F F(t) + A'_{\alpha\beta} t^\alpha t^\beta + B'_\alpha t^\alpha + C' \]  \hspace{1cm} (1.8)

从而得到一般的准齐次条件的形式。
特别地，当 $\forall \alpha, \beta$ 均有 $d_F \neq 0$, $d_F \neq d_\alpha$, $d_F \neq d_\alpha + d_\beta$ 时，可由 (1.8) 式反求出多项式 $P(t)$，从而对 $F(t)$ 加上 $P(t)$ 后可消去 (1.8) 式中多余的尾项。

2. 我们考虑更一般的线性非齐次 Euler 向量场

$$E(t) = (q_\beta^\alpha t^\beta + r^\alpha) \partial_\alpha$$

设 $Q = (q_\beta^\alpha)$，当 $Q$ 的特征根都是单的且非零时，此时 $Q$ 可对角化，设存在矩阵 $A$ 使得

$$AQ^{-1} = \text{diag}(d_1, \ldots, d_n)$$

通过 $t = (t^1, \ldots, t^n)$ 作仿射变换

$$(s^1, \ldots, s^n)^T = A(t^1, \ldots, t^n)^T + AQ^{-1}(r^1, \ldots, r^n)^T$$

得到一组新的坐标 $s = (s^1, \ldots, s^n)$，由 $\frac{\partial}{\partial s^\alpha} = \frac{\partial}{\partial t^\alpha} \frac{\partial}{\partial s^\alpha}$，得到

$$\left( \frac{\partial}{\partial t^1}, \ldots, \frac{\partial}{\partial t^n} \right) = \left( \frac{\partial}{\partial s^1}, \ldots, \frac{\partial}{\partial s^n} \right) A$$

从而在新坐标下 Euler 向量场 $E$ 化为

$$E = \left( \frac{\partial}{\partial t^1}, \ldots, \frac{\partial}{\partial t^n} \right) (Q(t^1, \ldots, t^n)^T + (r^1, \ldots, r^n)^T)$$

$$= \left( \frac{\partial}{\partial s^1}, \ldots, \frac{\partial}{\partial s^n} \right) A \left( AQ^{-1}(s^T - AQ^{-1}(r^1, \ldots, r^n)^T) + (r^1, \ldots, r^n)^T \right)$$

$$(1.10)$$

$$= \sum_\alpha d_\alpha s^\alpha \partial_\alpha$$

具有之前的简单形式。

当 $Q$ 的一些特征根是零时，一般来说不能通过对 $t$ 进行仿射变换来消去 (1.9) 式中的非齐次项。在此情形下，对可对角化 $Q$ 可经由对 $t$ 作类似前面的仿射变换，将 Euler 向量场约化为下面较简单的形式:

$$E(t) = \sum_{\alpha|d_\alpha \neq 0} d_\alpha t^\alpha \partial_\alpha + \sum_{\alpha|d_\alpha = 0} r^\alpha \partial_\alpha$$

(1.11)

易见次数 $d_1, \ldots, d_n, d_F$ 在相差一个非零因子的意义下是唯一的，以后我们仅考虑 $d_1 \neq 0$ 的情形，因此我们可对这些次数正规化并总假设

$$d_1 = 1.$$  

(1.12)

在物理文献中，这些正规化的次数通常由 $q_1 = 0, q_2, \ldots, q_n$ 和 $d$ 参数化，使得

$$d_\alpha = 1 - q_\alpha, \quad d_F = 3 - d.$$  

(1.13)
2 WDVV 方程组和 Frobenius 流形

2.1 WDVV 方程组及其解的约化

引言中提到的代数 $A_t$ 的结合性给出了函数 $F(t)$ 需要满足的 WDVV 方程组。由代数乘法的结合性，有

$$\forall \alpha, \beta, \gamma, (e_\alpha \cdot e_\beta) \cdot e_\gamma = e_\alpha \cdot (e_\beta \cdot e_\gamma)$$

将 (1.3) 式代入上式得

$$c^\gamma_{\alpha \beta} c^\beta_\gamma \theta = c^\gamma_{\beta \gamma} \theta c^\beta_\gamma \alpha, \forall \alpha, \beta, \gamma, \theta \in \{1, ..., n\}$$

上式两边乘以 $\eta_\delta$ 并对 $\iota$ 求和，得到

$$c^\gamma_{\alpha \beta} c_\gamma \theta \delta = c^\gamma_{\beta \gamma} \theta c_\gamma \alpha \delta, \forall \alpha, \beta, \delta, \theta \in \{1, ..., n\}$$

即

$$\frac{\partial^3 F(t)}{\partial \alpha^2 \partial \beta^2 \partial \epsilon^2} \eta^\gamma_{\alpha \beta} \frac{\partial^3 F(t)}{\partial \theta^2 \partial \alpha^2 \partial \epsilon^2} \eta^\gamma_{\beta \gamma} \frac{\partial^3 F(t)}{\partial \delta^2 \partial \epsilon^2 \partial \delta} = \frac{\partial^3 F(t)}{\partial \theta^2 \partial \alpha^2 \partial \epsilon^2} \eta^\gamma_{\beta \gamma} \frac{\partial^3 F(t)}{\partial \delta^2 \partial \epsilon^2 \partial \alpha} \eta^\gamma_{\gamma \alpha}, \forall \alpha, \beta, \delta, \theta \in \{1, ..., n\}$$ (2.1)

上述方程组再加上决定其伸缩约化的准齐次性条件 (1.8) 及指明函数 $F(t)$ 对变量 $t_1$ 的依赖性的规范化条件 (1.1)，称为 Witten-Dijkgraaf-E. Verlinde-H. Verlinde (WDVV) 方程组，它的一个解称为自由能函数。

观察发现此方程组关于坐标 $t_1, ..., t_n$ 的线性变换保持不变，我们可用下面的推导更加显式地描述 WDVV 方程组。

引理 2.1 由 Euler 向量场 $E$ 生成的伸缩变换作用在矩阵 $(\eta_{\alpha \beta})$ 上为如下线性共性变换

$$L_E \eta_{\alpha \beta} = (d_F - d_1) \eta_{\alpha \beta}$$ (2.2)

上式中 $d_F$ 和 $d_1$ 如 (1.6, 7) 两式中定义。

证明：对等式 (1.8) 两边分别关于 $t^n, t^\alpha$ 和 $t^\beta$ 求导，对等式左边，依次得到

$$\partial_1 L_E F(t) = \sum_\rho (\partial_1 E^\rho) \partial_\rho F(t) + \sum_\rho E^\rho \partial_1 \rho F(t)$$

由 $\partial_1 E^\rho = \delta^\rho_1$，有

$$\partial_1 L_E F(t) = \partial_1 F(t) + \sum_\rho E^\rho \partial_1 \rho F(t)$$

$$\partial_\alpha \partial_1 L_E F(t) = \partial_\alpha F(t) + \sum_\rho q^\rho_\alpha \rho \partial_1 F(t) + \sum_\rho E^\rho \eta_{\rho \alpha}$$

$$\partial_\beta \partial_\alpha \partial_1 L_E F(t) = \eta_{\alpha \beta} + q^\rho_\alpha \eta_{\rho \beta} + q^\rho_\beta \eta_{\rho \alpha}$$

再对等式右边求导，最后得到

$$q^\rho_\alpha \eta_{\rho \beta} + q^\rho_\beta \eta_{\rho \alpha} = (d_F - d_1) \eta_{\alpha \beta} = (2 - d) \eta_{\alpha \beta}$$

将度量 $< , >$ 看作 $0, 2$—张量，由李导数在张量上的作用法则，有
\[ \mathcal{L}_E \eta_{\alpha\beta} = \mathcal{L}_E \langle \partial_\alpha, \partial_\beta \rangle = \nabla E \eta_{\alpha\beta} + \langle \nabla \partial_\alpha E, \partial_\beta \rangle + \langle \partial_\alpha, \nabla \partial_\beta E \rangle = 0 + \langle (\partial_\alpha E^\rho) \partial_\rho, \partial_\beta \rangle + \langle \partial_\alpha, (\partial_\beta E^\rho) \partial_\rho \rangle = q_\alpha^\rho \eta_{\beta\rho} + q_\beta^\rho \eta_{\alpha\rho}. \]

从而引理得证。 \[ \square \]

**推论 2.1**

1) 若 \( \eta_{11} = 0 \) 且 \( E(t) \) 的所有特征根都是单的，则通过对坐标 \( t^n \) 的线性变换，矩阵 \( (\eta_{\alpha\beta}) \) 能被约化为反对角线形式

\[ \eta_{\alpha\beta} = \delta_{\alpha+\beta,n+1} \quad (2.4) \]

2) 在上述坐标下，对某个关于变量 \( (t^2, ..., t^n) \) 的函数 \( f(t^2, ..., t^n) \), \( F(t) \) 可约化为下

\[ F(t) = \frac{1}{2}(t^1)^2 t^n + \frac{1}{2} t^1 \sum_{\alpha=2}^{n-1} t^\alpha t^{n-\alpha+1} + f(t^2, ..., t^n) \quad (2.5) \]

**证明**：由 \( E(t) \) 的所有特征根都是单的，不妨设 \( Q = (q_{\alpha\beta}^\rho) = \text{diag}(d_1, ..., d_n) \). 由引理 2.1, 有

\[ q_\alpha^\rho \eta_{\beta\rho} + q_\beta^\rho \eta_{\alpha\rho} = (2 - d)\eta_{\alpha\beta} \]

从而对矩阵 \( Q(\eta_{\alpha\beta}) = (q_{\alpha\beta}^\rho) \eta_{\rho\alpha} \), 有

\[ (q_\alpha^\rho \eta_{\beta\rho}) + (q_\beta^\rho \eta_{\alpha\rho})^T = (2 - d)(\eta_{\alpha\beta}) \]

\[ \Rightarrow \text{diag}(d_1, ..., d_n)(\eta_{\alpha\beta}) + (\eta_{\alpha\beta})\text{diag}(d_1, ..., d_n) = (2 - d)(\eta_{\alpha\beta}) \]

\[ \Rightarrow (d_\alpha + d_\beta)\eta_{\alpha\beta} = (2 - d)(\eta_{\alpha\beta}) \]

由 \( d_1, ..., d_n \) 互不相同及矩阵 \( (\eta_{\alpha\beta}) \) 非退化，故 \( (\eta_{\alpha\beta}) \) 每一列恰有一个元素非零。

否则设 \( \exists \alpha, \gamma, \alpha \neq \gamma, \text{使得 } \eta_{\alpha\beta}, \eta_{\gamma\beta} \neq 0 \), 则 \( d_\alpha + d_\beta = 2 - d = d_\gamma + d_\beta \Rightarrow d_\alpha = d_\gamma \), 与 \( d_1, ..., d_n \) 互不相同矛盾！

又由矩阵 \( (\eta_{\alpha\beta}) \) 对称及 \( \eta_{11} \neq 0 \), 易知可在保持坐标分量 \( t^1 \) 不变的情形下通过调换其余坐标的次序，使得度量矩阵 \( (\eta_{\alpha\beta}) \) 具有反对角线的形式，即

\[ (\eta_{\alpha\beta}) = \begin{pmatrix} \eta_{11} & \eta_{12} & \cdots & \eta_{1,n-1} \\ \eta_{2,1} & \ddots & \cdots & \eta_{2,n-1} \\ \vdots & \cdots & \ddots & \eta_{n-1,2} \\ \eta_{n,1} & \cdots & \cdots & \eta_{n,n-1} \end{pmatrix} \]

再对上面得到的每个坐标作适当的伸缩变换，则可将上面矩阵中的反对角线元素均化为 1, 最后得到 \( \eta_{\alpha\beta} = \delta_{\alpha+\beta,n+1} \). 至此推论第 1) 部分得证。

由上面的证明过程知，和

\[ d_\alpha + d_{n-\alpha+1} = 2 - d \quad (2.6) \]
不依赖 \( \alpha \),且

\[
d_F = 2d_1 + d_n
\]  \hspace{1cm} (2.7)

若这些次数被规范化使得 \( d_1 = 1 \),并设 \( d_\alpha = 1 - q_\alpha \),则

\[
d_F = 3 - d
\]  \hspace{1cm} (2.8)

其中 \( q_1, ..., q_n, d \) 满足

\[
q_1 = 0, \quad q_n = d, \quad q_\alpha + q_{n-\alpha+1} = d.
\]  \hspace{1cm} (2.9)

下面证明命题的第 2) 部分, 先设在 1) 中得到的坐标下 \( F(t) \) 有如下形式

\[
F(t) = \frac{1}{2}(t_1^2)t^n + \frac{1}{2} t_1 \sum_{\alpha=2}^{n-1} t_\alpha t^n - \alpha + 1 + f(t_2, ..., t^n)
\]  \hspace{1cm} (2.10)

下证 \( f(t) = f(t_1, ..., t^n) \) 不依赖于变量 \( t_1 \), 实际上, 由 \( \eta_{\alpha\beta} = \delta_{\alpha+\beta,n+1} \),推出

\[
\partial_{t_\alpha t_\beta} f(t) = 0, \quad \forall \alpha, \beta \in \{1, ..., n\}.
\]

设函数 \( g := \partial f \), 则 \( g \) 的所有二阶偏导数均恒为零, 故 \( g \) 为 \( t = (t_1, ..., t^n) \) 的一次函数, 即 \( \exists a_1, ..., a_n, b \in \mathbb{C} \), 使得

\[
g(t) = a_1 t_1 + \cdots + a_n t^n + b
\]

对 \( g(t) \) 关于变量 \( t_1 \) 积分得

\[
f(t) = \frac{1}{2}a_1(t_1^2) + bt_1 + t_1 \sum_{\alpha=2}^{n} a_\alpha t_\alpha + h(t_2, ..., t^n)
\]

注意到在相差 \( t \) 的一个二次多项式的意义下 \( F \) 的三阶导数保持不变, 故 \( f(t) \sim h(t_2, ..., t^n) \) 为只依赖于变量 \( t_2, ..., t^n \) 的函数。从而有

\[
F(t) = \frac{1}{2}(t_1^2)t^n + \frac{1}{2} t_1 \sum_{\alpha=2}^{n-1} t_\alpha t^n - \alpha + 1 + f(t_2, ..., t^n) = \frac{1}{2} t_1 \sum_{\alpha=1}^{n-1} t_\alpha t^n - \alpha + 1 + f(t_2, ..., t^n)
\]  \hspace{1cm} \Box

2.2 低维 WDVV 方程组的解的例子

例 2.1 下面以 \( n = 3 \) 为例求出 WDVV 方程组的约化为 (18) 形式的解。

此时, 设

\[
F(t) = \frac{1}{2}(t_1^2)t^3 + \frac{1}{2} t_1 (t_2^2) + f(t_2,t_3)
\]

为记号简便, 下设 \( t_2 = x, \ t_3 = y, \ f(t_2,t_3) = f(x,y) \) 此时度量矩阵 \( (\eta_{\alpha\beta}) = (\eta^\alpha_\beta) =
\begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}
\)

由此得到

\[
e_2 \cdot e_2 = f_{xxy} e_1 + f_{xxx} e_2 + e_3
\]  \hspace{1cm} (2.11)

\[
e_2 \cdot e_3 = f_{xyy} e_1 + f_{xxy} e_2
\]  \hspace{1cm} (2.12)

\[
e_3 \cdot e_3 = f_{yyy} e_1 + f_{xyy} e_2
\]  \hspace{1cm} (2.13)
又由代数 $At$ 的结合性，$n = 3$ 时只需考虑 $(e_2 \cdot e_2) \cdot e_3 = e_2 \cdot (e_2 \cdot e_3)$ 及 $(e_3 \cdot e_3) \cdot e_2 = e_3 \cdot (e_3 \cdot e_2)$，将 (2.11, 12, 13) 代入这两个条件中，均得到相同的关于 $f$ 的微分方程

$$(f_{xx} f_{xy})^2 = f_{yy} + f_{xxx} f_{xyy}$$ (2.14)

易见上面方程是 $n = 3$ 时 $f$ 关于代数的结合性所需满足的唯一方程。最后还要考虑 $F(t)$ 满足的准齐次性条件：当 $d_1 + d_3 = 2d_2 = d_F - d_1 = 2 - d \Rightarrow d_3 = 1 - d, d_2 = 1 - \frac{d}{2}$，故 Euler 向量场

$$E(t) = t^1 \partial_1 + ((1 - \frac{d}{2}) t^2 + r^2) \partial_2 + ((1 - d) t^3 + r^3) \partial_3$$

当 $d \neq 1, 2$ 时，通过对 $t^2, t^3$ 平移变换可将 $E(t)$ 化为

$$E(t) = t^1 \partial_1 + (1 - \frac{d}{2}) t^2 \partial_2 + (1 - d) t^3 \partial_3$$

将上式代入 (1.8) 式，并记 (1.8) 式中右边关于 $t$ 的二次多项式尾项为 $P(t)$，整理后得到

$$(1 - \frac{d}{2}) t^2 \frac{\partial}{\partial t} f(t^2, t^3) + (1 - d) t^3 \frac{\partial}{\partial t} f(t^2, t^3) = (3 - d) f(t^2, t^3) + P(t)$$

由上式知多项式 $P(t)$ 是只依赖于 $t^2, t^3$ 的函数，将 $t^2, t^3$ 替换为 $x, y$ 并记 $q = -\frac{1 - d}{2}$，则上式化为

$$xf_x - qyf_y = (q + 4)f + P(x, y)$$ (2.15)

先考虑 $P(x, y) = 0$ 时的齐次解，由特征线法，解得

$$f(x, y) = x^{4 + q} \phi(yx^q)$$

另外易验证在相差一个二次图式 $Q(x, y) = ax^2 + bxy + cy^2 + vx + ey + s$ 的情形下，函数 $f(x, y) = x^{4 + q} \phi(yx^q) + Q(x, y)$ 满足方程

$$xf_x - qyf_y = (q + 4)f - (q + 2)ax^2 - (2q + 3)bxy -(3q + 4)cy^2$$

$$- (q + 3)vx - (2q + 4)ey - (4 + q)s$$

$$= (q + 4)f + \tilde{P}(x, y)$$

显然 $q$ 不可能为 $-2$，下面按不同的 $q$ 使得多项式 $\tilde{P}(x, y)$ 中不同的项的系数分别为零的情形进行讨论:

① 当 $q \neq -\frac{3}{2}, -\frac{5}{2}, -3, -4$ 时，即 $d \neq -2, -1, 4, 3$ 时，

$$\forall$$ 二次多项式 $P(x, y)$，总能找到合适的 $a, b, c, v, e, s$ 使得 $f(x, y) = x^{4 + q} \phi(yx^q) + Q(x, y)$ 是 (28) 的解，又相差一个二次多项式意义下视为等同，故此情形下 WDVV 方程组的解

$$F(t) = \frac{1}{2} (t^1)^2 t^3 + \frac{1}{2} t^1 (t^2)^2 + (t^2)^4 \phi((t^2) qt^2).$$ (2.16)

② 当 $q = -\frac{3}{2}$ 时，由 $2q + 3 = 0$，形如 $x^\frac{5}{2} \phi(yx^{-\frac{3}{2}}) + Q(x, y)$ 的函数只能是 (28) 式中 $P(x, y)$ 不含 $xy$ 项的解，为此，需考虑方程

$$xf_x + \frac{3}{2} yf_y = \frac{5}{2} f + bxy$$
的解。设
\[ f(x, y) = x^\frac{5}{2} \phi(yx^{-\frac{3}{2}})u(x, y) \]
则 \( u \) 满足方程
\[ xu_x + \frac{3}{2} yu_y = \frac{byx^{-\frac{3}{2}}}{\phi(yx^{-\frac{3}{2}})} \]
再由特征线法解得
\[ u(x, y) = \frac{byx^{-\frac{3}{2}}}{\phi(yx^{-\frac{3}{2}})} \log x + \theta(yx^{-\frac{3}{2}}) \]
\[ \Rightarrow f(x, y) = bxy \log x + x^{\frac{3}{2}}(\phi \cdot \theta)(yx^{-\frac{3}{2}}) \]
仍记 \( \phi \cdot \theta \) 为 \( \phi \)，并可找到多项式 \( Q(x, y) \) 使得 \( f(x, y) + Q(x, y) \) 为方程 (28) 在 \( q = -\frac{3}{2} \) 时的解，又在相差一个时的解，又在相差一个 \( t \) 的二次多项式的意义下视为等同，最后得到此情形下 WDVV 方程组的解为
\[ F(t) = \frac{1}{2}(t^1)^2t^3 + \frac{1}{2}t^1(t^2)^2 + bt^2t^3 \log t^2 + (t^2)^{\frac{3}{2}} \phi(t^3(t^2)^{-\frac{3}{2}}). \]  \( \text{(2.17)} \)
③ 当 \( q = -\frac{4}{3} \) 时，类似情形②中的讨论，最后得到此情形下 WDVV 方程组的解为
\[ F(t) = \frac{1}{2}(t^1)^2t^3 + \frac{1}{2}t^1(t^2)^2 + c(t^3)^2 \log t^2 + (t^2)^{\frac{3}{2}} \phi(t^3(t^2)^{-\frac{3}{2}}). \]  \( \text{(2.18)} \)
④ 当 \( q = -3 \) 时，类似情形②中的讨论，最后得到此情形下 WDVV 方程组的解为
\[ F(t) = \frac{1}{2}(t^1)^2t^3 + \frac{1}{2}t^1(t^2)^2 + vt^2 \log t^2 + r^2 \phi(t^3(t^2)^{-3}). \]  \( \text{(2.19)} \)
⑤ 当 \( q = -4 \) 时，类似情形②中的讨论，最后得到此情形下 WDVV 方程组的解为
\[ F(t) = \frac{1}{2}(t^1)^2t^3 + \frac{1}{2}t^1(t^2)^2 + s \log t^2 + \phi(t^3(t^2)^{-4}). \]  \( \text{(2.20)} \)
⑥ 当 \( d = 1 \) 时，Euler 向量场可约化为
\[ E(t) = t^1 \partial_1 + \frac{1}{2} t^2 \partial_2 + r^3 \partial_3 \]  \( \text{(2.21)} \)
代入到 (1.8) 式中，仍设 \( f(t^2, t^3) = f(x, y) \)，并简记 \( r^3 = r \)，整理后得到
\[ \frac{1}{2}xf_x + rf_y - 2f = P(t) - \frac{1}{2}r^3(t^1)^2 \]
由于上式左边是变量 \( (t^2, t^3) = (x, y) \) 的函数，故 \( P(t) = \frac{1}{2}r^3(t^1)^2 + P_1(t^2, t^3) \)，其中
\[ P_1(t^2, t^3) = P_1(x, y) \] 是 \( x, y \) 的二次多项式。下面考虑如下方程的解
\[ \frac{1}{2}xf_x + rf_y - 2f = P_1(x, y) \]  \( \text{(2.22)} \)
解得 \( \forall \) 多项式 \( P_1(x, y) \)，\( \exists \) 二次多项式 \( Q(x, y) \) 使得上面方程的解为
\[ f(x, y) = x^4 \phi(y - 2r \log x) + Q(x, y) \]
又在相差一个 $t$ 的二次多项式的意义下视为等同，最后得到此情形下 WDVV 方程组的解为

$$F(t) = \frac{1}{2}(t^1)^2t^3 + \frac{1}{2}t^1(t^2)^2 + x^4\phi(y - 2r\log x) \quad (2.23)$$

⑦ 当 $d = 2$ 时，Euler 向量场可约化为

$$E(t) = t^1\partial_1 + r\partial_2 - t^3\partial_3 \quad (2.24)$$

代入到 (1.8) 式中，仍设 $f(t^2, t^3) = f(x, y)$，整理后得到

$$rf_x - yf_y - f = P_1(x, y)$$

解得 $\forall$ 多项式 $P_1(x, y)$，取二次多项式 $Q(x, y)$ 使得上述方程的解为

$$f(x, y) = y^{-1}\phi(x + r\log y) + Q(x, y)$$

又在相差一个 $t$ 的二次多项式的意义下视为等同，最后得到此情形下 WDVV 方程组的解为

$$F(t) = \frac{1}{2}(t^1)^2t^3 + \frac{1}{2}t^1(t^2)^2 + y^{-1}\phi(x + r\log y) \quad (2.25)$$

下面考虑 $F(t)$ 所满足的代数结合性条件，即 $(f_{xxy})^2 = f_{yyy} + f_{xxx}f_{xyy}$。

仍按照前面得到的各种情形进行讨论。当 $d \neq 1, 2$ 时，

① 当 $q \neq -\frac{3}{2}, -\frac{4}{3}, -3, -4$ 时，即 $d \neq -2, -1, 4, 3$ 时，$f(x, y) = x^{4+q}\phi(yx^q)$，令 $z = yx^q$，对 $f$ 求三阶偏导数得

$$f_{xxx} = [(4 + q)(3 + q)(2 + q)\phi + qz(7q^2 + 27q + 26)\phi' + q^2z^2(6q + 9)\phi'' + q^3z^3\phi''']x^{1+q} = h_1(z)x^{1+q}$$

$$f_{xxy} = [(4q^2 + 14q + 12)\phi' + z(5q^2 + 7q)\phi'' + q^2z^2\phi''']x^{2+2q} = h_2(z)x^{2+2q}$$

$$f_{xyy} = [(4 + 3q)\phi'' + qz\phi''']x^{3+3q} = h_3(z)x^{3+3q}$$

$$f_{yyy} = \phi''''(z)x^{4+4q}$$

将上述各式代入 (2.14) 式中，得到变量 $z$ 的函数 $\phi$ 满足如下非线性常微分方程

$$(h_2(z))^2 = \phi''''(z) + h_1(z)h_3(z) \quad (2.26)$$

② 当 $q = -\frac{3}{2}$ 时，为便于区分记

$$f_1(x, y) = bxy\log x + x^5\phi(yx^{-\frac{3}{2}}) = bxy\log x + f(x, y)$$

这里 $f_1(x, y)$ 为 $F(t)$ 的尾项，$f(x, y)$ 如①中形式，并保持意义不变地延用①中记号 $h_i(z)$，$i = 1, 2, 3$。将 $f_1$ 各三阶偏导数代入 (2.14) 式中，整理后得到 $\phi$ 满足方程

$$(h_2(z) + b)^2 = \phi''''(z) + h_1(z)h_3(z) - bzh_3(z) \quad (2.27)$$
③ 当 $q = -\frac{4}{3}$ 时，类似②中的讨论，最后得到此情形下 $\phi$ 满足方程
\[(h_2(z))^2 = \phi'''(z) + h_1(z)h_3(z) + 2c(h_1(z) + 2zh_2(z) + z^2h_3(z))\] (2.28)

④ 当 $q = -3$ 时，类似②中的讨论，最后得到此情形下 $\phi$ 满足方程
\[(h_2(z))^2 = \phi'''(z) + (h_1(z) - v)h_3(z)\] (2.29)

⑤ 当 $q = -4$ 时，类似②中的讨论，最后得到此情形下 $\phi$ 满足方程
\[(h_2(z))^2 = \phi'''(z) + (h_1(z) + 2s)h_3(z)\] (2.30)

⑥ 当 $d = 1$ 时，$f(x,y) = x^4\phi(y - 2r \log x)$，其各三阶偏导数为
\[
\begin{align*}
  f_{xxx} &= 2x(12\phi - 26r\phi' + 18r^2\phi'' - 4r^3\phi''') \\
  f_{xxy} &= x^2(12\phi' - 14r\phi'' + 4r^2\phi''') \\
  f_{xyy} &= x^3(4\phi''' - 2r\phi''') \\
  f_{yyy} &= x^4\phi'''
\end{align*}
\]

将上面各式代入 (2.14) 式中，整理后得到 $\phi$ 满足方程
\[
0 = \phi''' + 8r^3\phi'' + 8r^2\phi'\phi'' + 128r\phi'\phi''' - 48r\phi\phi'''
+ 96\phi\phi' - 144(\phi')^2 - 52r^2(\phi'')^2
\] (2.31)

⑦ 当 $d = 2$ 时，$f(x,y) = y^{-1}\phi(x + r \log y)$，类似情形⑥求出其各三阶偏导数代入 (2.14) 式中，整理后得到 $\phi$ 满足方程
\[
\phi'''(r^3 + 2\phi' - r\phi'') - (\phi'')^2 - 6r^2\phi'' + 11r\phi' - 6\phi = 0
\] (2.32)

综上，$n = 3$ 时针对 $d$ 取值的各种情形，WDVV 方程组的解可通过解上面各种情形对应的 ODE 而得到。

但当维数 $n$ 更高时，WDVV 方程组将包含很多个方程，即使是验证它们之间的相容性也是很困难的，更难于给出其解的明确表达。

### 2.3 Frobenius代数及其简单性质

为定义 Frobenius 流形，我们先来定义与之相关的 Frobenius 代数。

**定义 2.1** 一个 $\mathbb{C}$ 上的代数 $A$ 称为(交换的) Frobenius 代数，若
1) $A$ 是一个 $\mathbb{C}$ 上的交换结合代数且有单位元 $e$；
2) $A \times A$ 上有一个 $\mathbb{C}$ 非退化内积 $\langle, \rangle: A \times A \to \mathbb{C}$，$(a, b) \leftrightarrow \langle a, b \rangle$，且关于代数的乘法具有如下不变性：
\[
\langle ab, c \rangle = \langle a, bc \rangle, \forall a, b, c \in A.
\]
一个简单的 Frobenius 代数的例子如下:

例 2.2 A 为 n 个一维代数的直和，设 $A_i = C e_i$，$A = \oplus_{i=1}^n A_i$。则 $\{e_1, ..., e_n\}$ 为 A 的一组基，定义基上的乘法为

$$e_ie_j = \delta_{ij}e_i, \quad i, j = 1, ..., n$$  \hspace{1cm} (2.33)

则 $\forall i \neq j$，由内积不变性，有

$$<e_i, e_j> = <e_i e_i, e_j> = <e_i, e_i e_j> = <e_i, 0> = 0$$  \hspace{1cm} (2.34)

从而一组 $<e_i, e_i>$，$i = 1, ..., n$ 为确定此种形式的 Frobenius 代数的一组参数。另外易验证 A 中无幂零元，故 A 的极大可解理想 $\text{Rad}(A) = 0$，从而 A 是半单代数。

实际上，有一类 Frobenius 代数本质上都具有上述例子的形式，即有下面的引理

引理 2.2 C 上无幂零元的 Frobenius 代数均具有例 2.2 的形式。

**证明:** 设 A 为 n 维无幂零元的 Frobenius 代数，e 为其乘法单位元。将 e 扩充为 A 的一组基 $\{f_1 = e, ..., f_n\}$。$\forall f_i, \quad i = 1, ..., n$，定义 A 上的线性变换 $L_i : A \rightarrow A, \quad a \mapsto f_i \cdot a$

下证 $\forall i, L_i$ 是半单的。易知

$$\forall i, j, \quad L_i \circ L_j = L_j \circ L_i$$

即 $\{L_i\}_{i=1}^n$ 之间两两交换，故它们有非零的公共特征向量 $x_1$。设

$$L_i x_1 = \lambda_i^1 x_1, \quad \lambda_i^1 x_1 \in C, \quad \forall i = 1, ..., n$$

令 $W_1 = \{ x \in A : <x, x_1> = 0 \}$，则 $W_1$ 为 A 的一个 $n-1$ 维线性子空间。下证 $x_1 \notin W_1$。不妨，由 $x_1$ 为公共特征向量，故

$$\forall y \in A, \exists \alpha_y \in C, \text{ s.t. } yx_1 = \alpha_y x_1$$

$$\Rightarrow \forall y \in A, \quad <x_1, y> = <x_1, x_1 y> = <x_1, \alpha_y x_1> = 0$$

由内积非退化 $\Rightarrow x_1^2 = 0$，这与 A 不含幂零元矛盾！故 $x_1 \notin W_1$。从而 $A = Cx_1 \oplus W_1$ (作为线性空间的直和)。又因为 $\forall x \in W_1$，有

$$<L_i x, x_1> = <f_i x, x_1> = <x, f_ix> = <x, \lambda_i^1 x_1> = 0$$

故 $L_i(W_1) \subset W_1$。故 $\forall i = 1, ..., n$，$L_i|_{W_1}$ 为 $W_1$ 上的线性变换，且两两交换，故在 $W_1$ 中它们有非零的公共特征向量 $x_2$，类似前面，可得出 $W_1$ 的 $n-2$ 维线性子空间 $W_2 = \{ x \in W_1 : <x, x_2> = 0 \}$，同理可证 $x_2 \notin W_2$，从而 $W_1$ 有直和分解 $W_1 = Cx_2 \oplus W_2$，同样也有 $L_i(W_2) \subset W_2, \forall i = 1, ..., n$。

如此进行下去，最终得到 $\{L_i\}_{i=1}^n$ 的 n 个公共特征向量 $x_1, ..., x_n$，易见它们构成 A 的一组基。又

$$\forall x \in A, \quad x = \alpha_1 f_1 + \cdots + \alpha_n f_n \Rightarrow xx_i = (\alpha_1 \lambda_i^1 + \cdots + \alpha_n \lambda_n^1)x_i = \alpha x x_i$$
故对任意 $i$, 存在 $\mu_i \in C$, 使得 $x_i = \mu_i^{-1} x_i \Rightarrow e_i, \ldots, e_n$ 是 $A$ 的一组基，且

\[ e_i^2 = (\mu_i^{-1})^2 \mu_i x_i = \mu_i^{-1} x_i = e_i, \quad \forall i = 1, \ldots, n \]

当 $i \neq j$ 时，由前面讨论知对任意 $r, s \in C$, 使得 $e_i, e_j = e_i, r e_j$, 而 $e_i, e_j$ 线性无关，故 $r = s = 0$。从而这组基满足条件中的 (2.33) 式。

另外，当 $i \neq j$ 时，不妨设 $i < j$, 则 $e_j \in W_i \Rightarrow <e_j, e_i> = 0$, 故这组基也满足条件中的 (2.34) 式。并且 $\forall i = 1, \ldots, n$, 

\[ <e_i, e_i> = e_i, e_i, e_i \]

由于 $e_i, e_i$ 线性无关，故 $r = s = 0$。从而这组基满足条件中的 (2.33) 式。

定义 2.2 Frobenius 代数被称作分次的，若存在一个线性算子 $Q: A \to A$ 和数 $d$, 使得

\[
Q(a \cdot b) = Q(a) \cdot b + a \cdot Q(b) \tag{2.35}
\]

\[
<Q(a), b> + <a, Q(b) >= d <a, b> \tag{2.36}
\]

算子 $Q$ 和数值 $d$ 分别称作此 Frobenius 代数的分次算子和 charge。

注 2.1 由上面定义，可以进一步考虑分次交换结合环 $R$ 上的分次 Frobenius 代数 $(A, <,>)$。此时我们有两个分次算子 $Q_R: R \to R$ 及 $Q_A: A \to A$，它们满足如下性质:

\[
Q_R(\alpha \beta) = Q_R(\alpha) \beta + \alpha Q_R(\beta), \quad \forall \alpha, \beta \in R \tag{2.37}
\]

\[
Q_A(ab) = Q_A(a)b + aQ_A(b), \quad \forall a, b \in A \tag{2.38}
\]

\[
Q_A(\alpha a) = Q_R(\alpha) a + \alpha Q_A(a), \quad \forall \alpha \in R, a \in A \tag{2.39}
\]

\[
Q_R <a, b> + d <a, b> >= <Q_A(a), b> + <a, Q_A(b) > \tag{2.40}
\]

对于一个非零数 $k$, 可定义含单位元 $e$ 的代数的重标度: 我们按如下方式改变代数的乘法法则和单位元

\[ a \cdot b \mapsto ka \cdot b, \quad e \mapsto ke \tag{2.41} \]

2.4 Frobenius流形及简单例子

做好了上面的铺垫，我们接下来定义 Frobenius 流形。其基本想法是将代数 $A_t$ 看作与切空间 $T_t M$ 等同

\[ A_t \ni e_\alpha \leftrightarrow \partial_\alpha = \frac{\partial}{\partial \alpha} \in T_t M, \quad \forall \alpha = 1, \ldots, n \]

从而建立起了 Frobenius 代数纤维丛与流形 $M$ 上的切丛 $TM$ 之间的对应。

定义 2.3 设 $M$ 是一个 $n$-维光滑流形，称 $M$ 为 Frobenius 流形，若对所有 $t \in M$，在某点 $t$ 处的切空间均有指定的 Frobenius 代数的结构，且光滑地依赖于参数 $t$，并满足下列条件:
FM1. 不变内积$<,>_t$是$M$上的一个平坦度量。记$\nabla$为流形$(M, <, >)$上的Levi-Civita联络，单位向量场$e$是此联络的协变常量

$$\nabla e = 0$$  \hspace{1cm} (2.42)

这里所说的度量的平坦性，是指Riemann曲率张量消失，即存在局部坐标系$(t^1,...,t^n)$使得度量矩阵$(<\partial_\alpha, \partial_\beta>)$在此坐标系下为常值。

FM2. 令$c$是如下定义的$M$上的对称3-张量

$$c(u,v,w):= <u \cdot v, w>$$  \hspace{1cm} (2.43)

我们要求4-张量$(\nabla_c)_{\alpha\beta}$关于4个向量场$u,v,w,z \in TM$也是对称的。

FM3. $M$上固定了一个线性向量场$E \in TM$，即

$$\nabla(\nabla E) = 0$$  \hspace{1cm} (2.44)

并且相应的微分同胚单参数子群在度量$<,>$及Frobenius代数$T_tM$上的作用分别为共形变换和重标度。观察到$M$上向量场空间$TM$是$M$上光滑解析函数空间$C^\infty(M)$上的Frobenius代数，我们可以用分次Frobenius代数的观点来刻画这一点。为此，我们取这两个空间上的线性算子

$$Q_{C^\infty(M)} := E \hspace{1cm} (2.45)$$
$$Q_{TM} := id + ad_E \hspace{1cm} (2.46)$$

它们按照之前的定义2.2和注2.1在$TM$上引入了一个分次环$C^\infty(M)$上给定charge$d$的分次Frobenius代数结构。

$E$称为Frobenius流形上的Euler向量场。切空间$T_tM$上的协变常量算子

$$Q = \nabla E(t) = \frac{\partial E^\alpha}{\partial t^\beta} \partial_\alpha \otimes dt^\beta \in TM \otimes TM^* = End(TM)$$  \hspace{1cm} (2.47)

称为Frobenius流形上的分次算子。算子$Q$的特征根$d_1,...,d_n$是$M$上的常值函数，$id - Q$的特征根$q_\alpha = 1 - d_\alpha, \alpha = 1,...,n$称为$M$的伸缩维数。特别地，由(2.42)知单位向量场$e$是$Q$的对应特征根为1的特征向量。

条件FM3还有无穷小量的表达形式

$$\nabla_\gamma(\nabla_\beta E^\alpha) = 0$$  \hspace{1cm} (2.48)$$
$$\mathcal{L}_E C^\gamma_{\alpha\beta} = c^\gamma_{\alpha\beta}$$  \hspace{1cm} (2.49)$$
$$\mathcal{L}_E e = -e$$  \hspace{1cm} (2.50)$$
$$\mathcal{L}_E \eta_{\alpha\beta} = (2 - d) \eta_{\alpha\beta}$$  \hspace{1cm} (2.51)$$

这里$\mathcal{L}_E$是沿着Euler向量场的李导数。另外，由前面的分次代数刻画，(2.49,51)可化为如下形式
\[
\mathcal{L}_E(u \cdot v) = \mathcal{L}_E u \cdot v + u \cdot \mathcal{L}_E v + u \cdot v \quad (2.52)
\]
\[
\mathcal{L}_E <u, v> = <\mathcal{L}_E u, v> + <u, \mathcal{L}_E v> + (2 - d) <u, v> \quad (2.53)
\]

其中 \(u, v\) 是 \(M\) 上任意光滑向量场。

明确了相关概念后，我们考虑 WDVV 方程组的解与 Frobenius 流形之间的关系，有如下引理：

引理 2.3 任何定义在 \(t \in M\) 的一个邻域上的 WDVV 方程组 \((d_1 \neq 0)\) 的解按下面公式给出了此区域中的 Frobenius 流形结构：

\[
\partial_\alpha \cdot \partial_\beta := c^{\gamma}_{\alpha \beta}(t) \partial_\gamma \quad (2.54)
\]
\[
<\partial_\alpha, \partial_\beta> := \eta_{\alpha \beta} \quad (2.55)
\]

其中

\[
\partial_\alpha := \frac{\partial}{\partial t^\alpha} \quad (2.56)
\]

单位向量场

\[
e := \partial_1 \quad (2.57)
\]

以及 Euler 向量场 \((1.8)\)。

反之，局部上任何 Frobenius 流形对某个 WDVV 方程组的解具有 \((2.54 \sim 57)\) 的结构，同时 Euler 向量场有 \((1.9)\) 式结构。

证明 由于 WDVV 方程组的一个解按 \((2.54, 55)\) 两式在切空间上定义了乘法和度量，显然度量为常值，故坐标 \((t^1, \ldots, t^n)\) 是平坦的。从而在此坐标下协变导数就是相应的偏导。由于单位向量场 \(e = \partial_1\)，故 \(\forall \alpha\)，我们有

\[
\nabla_\alpha e = (\partial_\alpha 1) \partial_1 = 0 \Rightarrow \nabla e = 0
\]

从而 FM1 得证。又 \((2.43)\) 式中张量 \(c\) 有分量

\[
c(\partial_\alpha, \partial_\beta, \partial_\gamma) = c^{\gamma}_{\alpha \beta}(t) = \partial_\alpha \partial_\beta \partial_\gamma F(t) \quad (2.58)
\]
\[
\Rightarrow (\nabla_\alpha c)(\partial_\beta, \partial_\gamma) = \partial_\beta c(\partial_\alpha, \partial_\beta, \partial_\gamma) = \partial_\alpha \partial_\beta \partial_\gamma \partial_\delta F(t) \quad (2.59)
\]

由 \(F\) 解析，其偏导数与具体求导顺序无关，故 4-张量 \((\nabla z c)(u, v, w)\) 是完全对称的。FM2 得证。

现在我们证明 FM3，只需要证明 \((2.52, 53)\) 两式成立。此时 Euler 向量场 \(E\) 是线性的，算子 \(Q = \nabla E\) 是对角化常值矩阵

\[
\nabla E = diag(1 - q_1, \ldots, 1 - q_n) \quad (2.60)
\]
对齐性方程 \( F(\lambda^{1-q_1} t^1, \ldots, \lambda^{1-q_n} t^n) = \lambda^{3-d} F(t^1, \ldots, t^n) + \text{quadratic} \) 两边关于 \( t^\alpha, t^\beta, t^\gamma \) 求三阶偏导数，得到

\[
\partial_\alpha \partial_\beta \partial_\gamma F(\lambda^{1-q_1} t^1, \ldots, \lambda^{1-q_n} t^n) = \lambda^{\alpha+\beta+\gamma} \partial_\alpha \partial_\beta \partial_\gamma (c_{\alpha \beta \gamma}(t))
\]

对 (2.61) 式两边关于 \( \lambda \) 求导得

\[
\sum \partial_\varepsilon \partial_\alpha \partial_\beta \partial_\gamma F(\lambda^{1-q_1} t^1, \ldots, \lambda^{1-q_n} t^n)(1 - q_\varepsilon) \lambda^{-q_\varepsilon} t^\varepsilon = (q_\alpha + q_\beta + q_\gamma - d) \lambda^{\alpha+\beta+\gamma} \partial_\alpha \partial_\beta \partial_\gamma (c_{\alpha \beta \gamma}(t))
\]

将 (2.62) 式代入上式并整理得到

\[
L_E c_{\alpha \beta \gamma}(t) = \sum \varepsilon (1 - q_\varepsilon) \lambda^{-q_\varepsilon} t^\varepsilon (q_\alpha + q_\beta + q_\gamma - d) c_{\alpha \beta \gamma}(t)
\]

由此再加后推论 1.1 的证明中得到的等式 \((q_\alpha + q_\beta - d) \eta_{\alpha \beta} = 0\)，最终我们得到分次函数环上的分次 Frobenius 代数的定义式 (2.52, 53)，条件 FM3 得证。

反过来，在一个 Frobenius 流形上，我们可以选取局部平坦坐标 \( (t^1, \ldots, t^n) \)，使得不矢度量 < , > 在此坐标下为标量。将对称性条件 (2.43) 对 \( u = \partial_\alpha, v = \partial_\beta, \) \( w = \partial_\gamma, z = \partial_\delta \) 应用，得到 \( \partial_\delta c_{\alpha \beta \gamma}(t) \) 关于 \( \alpha, \beta, \gamma, \delta \) 是对称的。这与张量 \( c_{\alpha \beta \gamma}(t) \) 本身的对称性一起，推出局部存在一个函数 \( F(t) \) 使得

\[
c_{\alpha \beta \gamma}(t) = \frac{\partial^3 F(t)}{\partial \alpha \partial \beta \partial \gamma}
\]

又由单位向量场 \( e \) 的协变常性，我们可对坐标做一个线性变换使得 \( e = \partial_1 \)，故 (2.57) 式成立。下证函数 \( F(t) \) 满足 (1.8) 式。由 (2.44) 式，推出 Euler 向量场 \( E(t) \) 是一个具有 (1.9) 式形式的线性向量场。由重标度的定义，在坐标 \( t^\alpha \) 下，我们有

\[
[\partial_1, E] = \partial_1 \Rightarrow \partial_1 E^\alpha = \delta^\alpha_1
\]

故算子 \( Q = \nabla E \) 作用在 \( \partial_1 \) 上有

\[
\nabla E(\partial_1) = (\partial_\beta E^\alpha) \partial_\alpha \otimes dt^\beta(\partial_1) = (\partial_1 E^\alpha) \partial_\alpha = \partial_1
\]

即 \( \partial_1 \) 是算子 \( Q = \nabla E \) 的从属于特征根 1 的特征向量，从而 \( d_1 = 1 \)。由 Frobenius 流形定义中的条件 FM3 的 (2.52, 53) 式，将 \( u = \partial_\alpha, v = \partial_\beta \) 带入后，利用等式 \( L_E \partial_\alpha = -(\partial_\alpha E^\nu) \partial_\nu = -q^\alpha_\nu \partial_\nu \) 展开，分别得到

\[
L_E \eta_{\alpha \beta} = q^\alpha_\nu \eta_{\nu \beta} + q^\beta_\nu \eta_{\nu \alpha} = (2 - d) \eta_{\alpha \beta}
\]

\[
L_E c^\gamma_{\alpha \beta} = 2 E^\nu \partial_\nu c^\gamma_{\alpha \beta} - q^\nu_\alpha c^\gamma_{\nu \beta} + q^\nu_\beta c^\gamma_{\nu \alpha} = c^\gamma_{\alpha \beta}
\]

此时 \( \nabla E \) 是对角化的，即 \( q^\alpha_\beta = \delta_{\alpha \beta}(1 - q_\alpha) \)，则上面两式分别化为

\[1\] 2关于李导数在张量上的作用：设 \( T \) 是 \( m \times n \) 张量，即 \( T : (T^m)^n \times (TM)^n \rightarrow C^\infty(M) \)，对 \( \alpha_1, \ldots, \alpha_m \in T^M, X_1, \ldots, X_n \in TM, f_1, \ldots, f_m, g_1, \ldots, g_n \in C^\infty(M) \)，满足 \( T(f_1, \alpha_1, \ldots, f_m, \alpha_m, g_1, X_1, \ldots, g_n, X_n) = f_1 \cdots f_m g_1 \cdots g_n T(\alpha_1, \ldots, \alpha_m, X_1, \ldots, X_n) \)。对 \( M \) 上光滑向量场，则 \( (L_E T)(\alpha_1, \ldots, \alpha_m, X_1, \ldots, X_n) := L_E T(\alpha_1, \ldots, \alpha_m, X_1, \ldots, X_n) - \nabla_{T(\alpha_1, \ldots, \alpha_m, X_1, \ldots, X_n) \alpha_1}(E) - \cdots - \nabla_{T(\alpha_1, \ldots, \alpha_m, X_1, \ldots, X_n) \alpha_m}(E) + T(\alpha_1, \ldots, \alpha_m, \nabla X_1, E, X_2, \ldots, X_n) + \cdots + T(\alpha_1, \ldots, \alpha_m, X_1, \ldots, X_{n-1}, \nabla X_n) E) \)。此处 \( c^\gamma_{\alpha \beta} \) 实际上是 1,2 张量 \( T \)，其在基上的定义为 \( T(dt^\gamma, \partial_\alpha, \partial_\beta) = dt^\gamma(\partial_\alpha \cdot \partial_\beta) \)。
使用 (2.68) 式降低 (2.69) 式中 γ 的指标，得到

\[ E^\varepsilon \partial_\varepsilon c_{\alpha\beta\gamma} = (q_\alpha + q_\beta - q_\gamma) c_{\alpha\beta\gamma} \]

上式对 \( t^\alpha, t^\beta, t^\gamma \) 积分三次，最后得到函数 \( F \) 满足准齐次性条件 (1.8) 式，引理得证。 \( \square \)

**定义 2.4** 两个 Frobenius 流形的(局部)微分同胚 \( \phi : M \to \tilde{M} \) 称为(局部)等价，若 \( \phi^* \)是相应的两个流形上不变度量的线性共形变换，即存在非零常数 \( c \) 使得

\[ \phi^* <, >_{\tilde{M}} = c^2 <, >_M \]

并且 \( \phi_* \) 是切空间代数之间的同构

\[ \phi_* : T_t M \to T_{\phi(t)} \tilde{M}, \forall t \in M \]

以后将相互等价的 Frobenius 流形构成的等价类视为一个对象进行研究。

**几个经典的 Frobenius 流形的例子**

**例 2.3** (平凡 Frobenius 流形) 令 \( A \) 是一个分次 Frobenius 代数，即一组权 \( q_1, ..., q_n \) 被赋予到基向量 \( e_1, ..., e_n \)。使得下面两组常数 \( \{ c_{\alpha\beta\gamma} \}_{\alpha,\beta,\gamma=1}^n \) 和 \( \{ \eta_{\alpha\beta} \}_{\alpha,\beta=1}^n \) 满足

\[ c_{\alpha\beta\gamma} = 0 \text{ for } q_\alpha + q_\beta \neq q_\gamma \]

以及对某个数 \( d \) 有

\[ \eta_{\alpha\beta} = 0 \text{ for } q_\alpha + q_\beta \neq d \]

在代数 \( A \) 中有

\[ e_\alpha e_\beta = c_{\alpha\beta\gamma} e_\gamma, \eta_{\alpha\beta} = < e_\alpha, e_\beta > \]

上述公式在 \( M = A \) 上定义了一个 Frobenius 流形的结构。相应的自由能函数 \( F(t) \) 是一个三次多项式

\[ F(t) = \frac{1}{6} c_{\alpha\beta\gamma} t^\alpha t^\beta t^\gamma \]

设 \( t = t^\alpha e_\alpha \)，则 \( t^3 = t^\alpha t^\beta t^\gamma e_\alpha e_\beta e_\gamma \)，从而还有

\[ F(t) = \frac{1}{6} < t^3, e > \]

其中 \( e = e_1 \) 为单位元。坐标 \( t^\alpha \) 的次数为 \( d_\alpha = 1 - q_\alpha \)，函数 \( F \) 的次数为 \( d_F = 3 - d \)。

**例 2.4** (Frobenius 流形的直积) 设 \( M', M'' \) 分别是 \( n \)-维和 \( m \)-维的两个 Frobenius 流形，则它们的直积 \( M' \times M'' \) 有自然的 Frobenius 流形的结构，若它们的伸缩维度满足约束
若各自的平坦坐标 \((t_1',...,t_n'),(t_1'',...,t_m'')\) 被规范化为 (2.5) 式的形式，即

\[
F'(t_1',...,t_n') = \frac{1}{2} (t_1')^2 t_1'' + \frac{1}{2} t_1' \sum_{\alpha=2}^{n-1} t_{\alpha}' t_{\alpha}-\alpha+1' + f'(t_2',...,t_{n'})
\]

(2.78)

\[
F''(t_1'',...,t_{m''}) = \frac{1}{2} (t_1'')^2 t_1'' + \frac{1}{2} t_1'' \sum_{\beta=2}^{m-1} t_{\beta}' t_{\beta}-\beta+1'' + f''(t_2'',...,t_{m''})
\]

(2.79)

并且 \(d_1' = d_1'' = 1\)，由 (2.77) 式表明 \(d_n' = d_m''\)，则乘积流形的前位函数 \(F\) 有如下形式

\[
F(t_1',t_2',...,t_{n-1}',t_{n''},...,t_{m-1}',t_N,t''') =
\]

\[
= \frac{1}{2} t_1' t_2'' + t_1' t_1'' + \frac{1}{2} t_1' \sum_{\alpha=2}^{n-1} t_{\alpha}' t_{\alpha}-\alpha+1' + \frac{1}{2} t_1'' \sum_{\beta=2}^{m-1} t_{\beta}' t_{\beta}-\beta+1'' + f'(t_2',...,t_{n-1}',\frac{1}{2} (t_N + \hat{t}'')) + f''(t_2'',...,t_{m-1}'',\frac{1}{2} (t_N - \hat{t}''))
\]

(2.80)

其中 \(N = n + m\)，

\[
t_1 = \frac{t_1' + t_1''}{2}, \quad \hat{t}_1 = \frac{t_1' - t_1''}{2}, \quad t_N = t_{n''} + t_{m''}, \quad t'' = t_{n''} - t_{m''}
\]

3 Frobenius流形上的形变联络的和两个度量

3.1 形变联络的平坦性与 Frobenius流形的联系

处理 Frobenius 流形的一个首要工具是 Levi-Civita 联络 \(\nabla\) 的形变

\[
\nabla_u(z)v := \nabla_u v + z u \cdot v
\]

(3.1)

这里 \(u, v\) 是 Frobenius 流形 \(M\) 上的两个向量场， \(z\) 是这个形变联络依赖的参数。

我们将此推广，得到直积 \(M \times C\) 上的一个亚纯联络，其定义由如下公式给出:

\[
\tilde{\nabla}_u \frac{d}{dz} = 0, \quad \tilde{\nabla}_u \frac{d}{dx} = 0, \quad \tilde{\nabla}_u v = \partial_z v + E \cdot v - \frac{1}{z} \mu v
\]

(3.2)

其中

\[
\mu := 1 - \frac{d}{2} E = diag(\mu_1,...,\mu_n), \quad \mu_a := q_a - \frac{d}{2}
\]

(3.3)

这里 \(u, v\) 是 \(M \times C\) 上的切向量场，且沿着 \(C\) 的分量为零。观察发现这是一个对称联络。

引理 3.1 联络 \(\nabla(z)\) 是平坦的 \(\Leftrightarrow\) 代数 \(c_{\alpha \beta}(t)\) 是结合的且局部存在函数 \(F(t)\) 使得 \(c_{\alpha \beta}(t) = \partial_\alpha \partial_\beta \partial_z F(t)\)。

证明: 在平坦坐标下，由 \([\partial_\alpha, \partial_\beta] = 0 \Rightarrow R(\partial_\alpha, \partial_\beta) = [\nabla_\alpha, \nabla_\beta] - \nabla_{[\partial_\alpha, \partial_\beta]} = [\nabla_\alpha, \nabla_\beta]\)，故形变联络的曲率的消失性在分量下的表达为
∀ γ, 0 = R(∂α, ∂β)∂γ = [∇α(z), ∇β(z)]∂γ = [z(∂βcεγαζ − ∂αcεγβζ) + z²(cδβεζcαδζ − cδαεζcβδζ)]∂ε
⇒ ∂βcεγαζ = ∂αcεγβζ, cδβεζcαδζ = cδαεζcβδζ

故以 cαβζ(t) 作为基向量的乘法系数的代数是结合的，又由张量 cαβζ = ηγεζcαβ 的对称性，故 ∂βcαγδζ = ∂αcβγδζ 也是关于 α, β, γ, δ 对称的，从而存在函数 F(t) 使得 cαβζ(t) = ∂α∂β∂ζζ(t). 引理得证。

注 3.1 对余切向量 ξ = ξαdtα + 0dz，形变联络在其分量上作用的表达式为

\[\nabla_α ξ_β = \partial_α ξ_β - z c_α βγ ξ_γ \tag{3.4}\]

\[\nabla_α ξ_β = \partial_α ξ_β - E^γ c_α βγ ξ_α + \frac{1}{z} M_β^α ξ_ε \tag{3.5}\]

其中 M_β^α 为线性算子 μ = 1 - \frac{d}{2} - \nabla E 在相应位置的矩阵元。考虑方程组 \nabla ξ = 0 (其中协变导数只是关于变量 t 的)，若假设其相容性，即 ∂_α∂_αξ_β = ∂_α∂_ξ_β, ∀ α, ε, β, 由

\[\partial_αξ_β = z c_α βγ(t)ξ_γ \Rightarrow \partial_α∂_αξ_β = z ∂_αc_α βγ(t)ξ_γ + z^2 c_α βγ(t)ξ_α(t)ξ_ε \]

交换 α, ε 的顺序，由相容性得到 ∂_αc_α βε = ∂_αc_α βε 及 c_α βε c_ε αβ = c_ε αβ c_α εβ，即性率的消失性。这给出了结合性方程组的一个“Lax 对”。∀ z 此方程组有 n 维解空间。这组解与形变联络 \nabla(z) 的平坦坐标密切相关。也即，存在一组独立函数 ũ_1(t, z), ..., ũ_n(t, z)，使得在这组新坐标下，形变联络对应的协变导数就是相应的偏导数

\[\nabla_α(z) = \frac{\partial}{\partial t_α(t, z)} \tag{3.6}\]

下面我们来看这组新的平坦坐标 ũ 与方程组 \nabla ξ = 0 的解的联系。首先，设 ũ 是关于形变联络 \nabla 的一组平坦坐标，并设一组函数 \{ζ_β = η_β ũ_α\}_α β=1，下证它们构成方程组

\[\nabla ξ = 0 \] 的基础解系。

由于 ũ 是平坦坐标，故在此坐标下协变导数就是相应的偏导数。设 \nabla_α = \partial_α，于是一方面有

\[\nabla_α, η_α = \partial_α, η_α + z η_α, η_α = z c_α γζ η_σ = z c_α γζ η_ε \hat{δ}_ε\]

另一方面，也有

\[\nabla_α, η_α = \nabla_α, η_α = ζ_α β^\beta η_γ η_δ η_β η_δ η_ε = η_ε (\hat{δ}_θ η_γ) \hat{δ}_ε = η_ε (\hat{δ}_θ η_γ) \hat{δ}_ε = η_ε (\hat{δ}_θ η_γ) \hat{δ}_ε\]

对比上述两式，我们得到 ∂_αξ_ε = z c_α γζ η_ε，即 ∀ ε, η_ε 是方程组 \nabla ξ = 0 的一个解，又 (ξ_ε^β) 构成坐标变换的 Jacobi 矩阵，其行列式非零，故 ξ_1, ..., ξ_n 是方程组的一个基础解系。

反之，若 ξ_1, ..., ξ_n 是方程组 \nabla ξ = 0 的一个基础解系，则其中每一个解都是前面得到的基础解系 ξ_1, ..., ξ_n 的线性组合，即存在非退化常值矩阵 P = (P_β^α)，使得
从而有 \((\zeta^\alpha_\beta) = (\xi^\alpha_\beta)P\)

从而有 \(\zeta^\alpha_\beta = \xi^\alpha_\beta P^\alpha_\gamma = \partial_\beta(\xi^\gamma_\alpha)P^\alpha_\gamma\)，记 \(\tilde{s}^\alpha := P^\alpha_\gamma\)，则

\((\tilde{s}^1, ..., \tilde{s}^n) = (\tilde{t}^1, ..., \tilde{t}^n)\)

由矩阵 \(P\) 非退化，知 \(\tilde{s} = (\tilde{s}^1, ..., \tilde{s}^n)\) 也是形变联络 \(\tilde{\nabla}\) 的一组平坦坐标，且 \(\zeta^\alpha_\beta = \partial_\beta \tilde{s}^\alpha\)。

进一步，方程组 \(\tilde{\nabla}\xi = 0\) 加上含有参数 \(z\) 的方程后，整体的相容性等价于 WDVV 方程组。即有如下命题

命 3.2 WDVV 等价于方程组 \(\tilde{\nabla}\xi = 0\) 与下面方程的相容性

\[
 z \partial_z \xi \alpha = z E^\gamma(t)c^\beta_{\gamma\alpha}(t)\xi \beta + Q^\gamma_\alpha \xi \gamma
\]

其中 \(Q^\alpha_\alpha = \partial_\alpha E^\alpha\)。

**证明:** 由方程组 \(\tilde{\nabla}\xi = 0\) 的相容性，已有 \(c^\alpha_{\beta\gamma}\) 的结合性和 \(\partial_\beta c^\gamma_{\alpha\beta}\) 的对称性，由添加进方程 \(z \partial_z \xi \alpha = z E^\gamma(t)c^\beta_{\gamma\alpha}(t)\xi \beta + Q^\gamma_\alpha \xi \gamma\) 后整体有相容性，故 \(\forall i = 1, ..., n\)，有

\[
 \partial_i \partial_z \xi \alpha = \partial_z \partial_i \xi \alpha
\]

具体计算这两项，得到

\[
 \partial_i \partial_z \xi \alpha = Q^\gamma_\alpha c^\beta_{\gamma\alpha} \xi \beta + \partial_z \partial_i \xi \alpha = c^\beta_{\gamma\alpha} \xi \beta + c^\beta_{\gamma\alpha} Q^\gamma_\alpha \xi \gamma + z E^\gamma(t)c^\beta_{\gamma\alpha} \xi \beta
\]

由之前已得的结合性 \(c^\beta_{\gamma\alpha} c^\gamma_{\alpha\beta} = c^\beta_{\gamma\alpha} c^\gamma_{\alpha\beta}\) 及对称性 \(\partial_i c^\gamma_{\alpha\beta} = \partial_\beta c^\gamma_{\alpha\beta}\)，我们得到下面等式

\[
 Q^\gamma_\alpha c^\beta_{\gamma\alpha} + E^\gamma(t) \partial_i c^\gamma_{\alpha\beta} + Q^\gamma_\alpha c^\beta_{\gamma\alpha} = c^\beta_{\gamma\alpha} + c^\gamma_{\alpha} Q^\gamma_\alpha
\]

上式即为 \(L_E(\partial_i \cdot \partial_\alpha) = L_E \partial_i \cdot \partial_\alpha + \partial_i \cdot L_E \partial_\alpha + \partial_\alpha \cdot \partial_i \cdot \partial_\alpha\) 的分量形式，由此得到

\[
 L_E c^\beta_{\gamma\alpha} = c^\beta_{\gamma\alpha}
\]

因此相应的自由度数函数 \(F(t)\) 为 WDVV 方程组的解。又上面的推导过程反过来也成立，故两者等价，命题得证。□

### 3.2 Frobenius流形上的新度量

另一个处理 Frobenius 流形的重要工具是其上定义的一个新度量。为得到此度量，先考虑在余切丛 \(T^* M\) 上定义的一个度量，即作为 1-形式的内积。对两个 1-形式 \(\omega_1\) 和 \(\omega_2\)，我们定义

\[
 (\omega_1, \omega_2)^* = i_E(\omega_1 \cdot \omega_2)
\]

（度量上带脚标 * 是为强调这是一个在余切丛 \(T^* M\) 上的内积）。这里 \(i_E\) 是 1-形式关于向量场 \(E\) 的收缩算子，即
由于在 $TM$ 与 $T^*M$ 之间有基于内积 $<,>$ 的同构 $TM \leftrightarrow T^*M$, $\eta^{\alpha \beta} \partial_\beta \leftrightarrow dt^\alpha$, 我们可定义两个 $1$-形式 $\omega_1, \omega_2$ 的乘积为其在 $TM$ 中对应元素 $X, Y$ 的乘积 $X \cdot Y$ 再映回到 $T^*M$ 中对应的元素，即

$$(\omega_1 \cdot \omega_2)(-) = <X \cdot Y, >$$

于是我们得到新度量的具体定义

$$(\omega_1, \omega_2)^* = i_E(\omega_1 \cdot \omega_2) = <X \cdot Y, E>$$

（3.10）

于是是在平坦坐标 $t^\alpha$ 下，度量 $(,)^*$ 的分量为

$$g^{\alpha \beta}(t) := (dt^\alpha, dt^\beta)^* = <\eta^{\alpha \iota} \partial_\iota \cdot \eta^{\beta \jmath} \partial_\jmath, E^\iota \partial_\iota > = \eta^{\alpha \iota} \eta^{\beta \jmath} E^\iota e^\gamma_{\gamma \iota \jmath} = E^\iota (t) c^\alpha_{\iota \beta}(t)$$

（3.11）

其中 $c^\alpha_{\iota \beta}(t) := \eta^{\alpha \sigma} c^\beta_{\sigma \iota}(t)$. 当算子 $\nabla_E$ 可对角化时，设 $E = E^\iota \partial_\iota = (1 - q_t) t^\iota \partial_\iota$, 则

$$g^{\alpha \beta}(t) = E^\iota c^\alpha_{\iota \beta} = E^\iota \eta^{\alpha \iota} \eta^{\beta \jmath} c^\iota_{\iota \jmath} = \eta^{\alpha \iota} \eta^{\beta \jmath} E^\iota \partial_\iota F_{ij}$$

（3.12）

对 (1.8) 式两边关于 $t^i, t^j$ 求二阶偏导数，得到

$$L_E F_{ij} = (1 - d + q_i + q_j) F_{ij} + A_{ij}$$

（3.13）

代入 (3.12) 式中，得到

$$g^{\alpha \beta}(t) = (1 - d + q_i + q_j) \eta^{\alpha \iota} \eta^{\beta \jmath} F_{ij}(t) + \eta^{\alpha \iota} \eta^{\beta \jmath} A_{ij}$$

$$= (1 + d - q_\alpha - q_\beta) \eta^{\alpha \iota} \eta^{\beta \jmath} F_{ij}(t) + \eta^{\alpha \iota} \eta^{\beta \jmath} A_{ij}$$

（3.14）

引理 3.3 对充分小的 $t^1 \neq 0$, 度量 (3.9) 在 $t^1$-轴附近是非退化的。

证明: 我们有 $c^1_{\alpha \beta}(t) = \eta^{\alpha \sigma} c^\beta_{\sigma 1}(t) = \eta^{\alpha \sigma} \delta^\beta_\sigma = \eta^{\alpha \beta}$, 故对很小的 $t^2, ..., t^n$, 有

$$g^{\alpha \beta}(t) \simeq t^1 c^\alpha_{\iota \beta} + A^{\alpha \beta} = t^1 \eta^{\alpha \beta} + A^{\alpha \beta}$$

（3.15）

故对充分小的 $t^1$, 矩阵 $(g^{\alpha \beta})$ 是非退化的。引理得证。 □

由余切丛上的度量我们可以定义一个切丛上的新的度量。我们已有 $T^*M$ 上的度量

$$g^{ij}(t) = (dt^i, dt^j)^*$$

其中 $(g^{ij})$ 是可逆对称矩阵，其逆矩阵 $(g_{ij}) := (g^{ij})^{-1}$ 指定了流形 $M$ 上的一个度量，也即切丛 $TM$ 上的一个非退化内积 $(,)$, 定义如下

$$(\partial_i, \partial_j) := g_{ij}(t), \quad \partial_i := \frac{\partial}{\partial t^i}$$

（3.16）

设此度量的 Levi-Civita 联络为 $\nabla$, 由于此度量不一定是平坦的，设

$$\nabla_k \partial_i = \Gamma^s_{ki} \partial_s$$

（3.17）

$$\text{20}$$
其中 $\Gamma^k_{ij} = \frac{1}{2} g^{ks}(\partial_i g_{js} + \partial_j g_{is} - \partial_s g_{ij})$ 是 Christoffel 符号，关于指标 $i,j$ 对称。且 $\forall k$, $\nabla_k$ 对度量 $g$ 还要满足条件

$$\nabla_k g_{ij} := \partial_k g(\partial_i, \partial_j) - g(\nabla_k \partial_i, \partial_j) - g(\partial_i, \nabla_k \partial_j) = \partial_i g_{kj} - \Gamma^s_{kj} g_{si} = 0 \quad (3.18)$$

由协变导数在余切丛上作用为 $\nabla_k dt^i = -\Gamma^i_{ks} dt^s$, 上式也等价于 $\nabla_k$ 对余切丛上的度量满足

$$\nabla_k g_{ij} := \partial_k g_{ij} + \Gamma^i_{ks} g_{sj} + \Gamma^j_{ks} g_{si} = 0 \quad (3.19)$$

为方便，我们使用反变形式的分量，即定义

$$\Gamma^i_{jk} := (dt^i, \nabla_k dt^j)^* = (dt^i, -\Gamma^j_{ks} dt^s) = -g^{is} \Gamma^j_{ks} \quad (3.20)$$

则方程组 (3.19) 和 $\Gamma^i_{jk} = \Gamma^i_{kj}$ 在反变形式下的表达分别为

$$\nabla_k g_{ij} = \Gamma^i_{ks} g_{sj} + \Gamma^j_{ks} g_{si} = 0 \quad (3.21)$$

为记述方便，引入算子

$$\nabla_i := g_{is} \nabla_s, \quad \nabla_i \xi_k = g_{is} \partial_s \xi_k + \Gamma^i_{ks} \xi_s \quad (3.22)$$

并称算子 $\nabla_i$ 和相应的系数 $\Gamma^i_{jk}$ 为反变联络。由算子 $\nabla_i$ 的非交换性，下式给出此度量的曲率张量 $R^k_{slt}$ 的定义

$$[\nabla_s, \nabla_l] \xi_t = (\nabla_s \nabla_l - \nabla_l \nabla_s) \xi_t = -R^k_{slt} \xi_t \quad (3.23)$$

其中 $R^k_{slt} = \partial_s \Gamma^k_{lt} - \partial_l \Gamma^k_{st} + \Gamma^r_{sl} \Gamma^k_{rt} - \Gamma^r_{lt} \Gamma^k_{rs}$。若度量的所有曲率都消失，则称该度量是平坦的。对一个平坦度量局部上存在平坦坐标系使得在此坐标系下度量是常值且 Levi-Civita 联络的分量消失。反之，若对一个度量存在一个满足上述要求的平坦坐标系，则此度量是平坦的，且此平坦坐标在相差一个常系数仿射变换的意义下是唯一的。此时平坦坐标可由解下面的方程组得到

$$\nabla_i \partial_j p = g^{is} \partial_s \partial_j p + \Gamma^j_{is} \partial_s p = 0, \quad i,j = 1, \ldots, n \quad (3.24)$$

其 $n$ 个线性无关的解 $p^1, \ldots, p^n$ 即为该度量的平坦坐标。若我们选择已正交化的平坦坐标系 $(dp^a, dp^b)^* = \delta^{ab}$，则对此度量和 Levi-Civita 联络的分量，成立下面的公式：

$$g^{ij} = \frac{\partial i}{\partial \nu^a} \frac{\partial j}{\partial \nu^b} \quad (3.25)$$

$$\Gamma^j_{ik} dt^k = \frac{\partial i}{\partial \nu^a} \frac{\partial^2 j}{\partial \nu^a \partial \nu^b} dp^b \quad (3.26)$$

(3.25) 式是由于

$$g^{ij} = (dt^i, dt^j)^* = (\frac{\partial i}{\partial \nu^a} dp^a, \frac{\partial j}{\partial \nu^a} dp^a)^* = \frac{\partial i}{\partial \nu^a} \frac{\partial j}{\partial \nu^a} \delta^{ab} = \frac{\partial i}{\partial \nu^a} \frac{\partial j}{\partial \nu^a}$$

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式是由 \( \Gamma^{ij}_{k} = (dt^{i}, \nabla_{k}dt^{j})^{*} \Rightarrow \)

\[
\Gamma^{ij}_{k} \frac{\partial t^{k}}{\partial p^{b}} = \left( \frac{\partial^{2}t^{i}}{\partial p^{b}\partial p^{s}} \partial_{t}^{s} \right)^{*} = \frac{\partial^{2}t^{i}}{\partial p^{b}\partial p^{s}} \partial_{t}^{s} \nabla \partial_{t}^{i} = \frac{\partial^{2}t^{i}}{\partial p^{b}\partial p^{s}} \delta_{s}^{i} \partial_{t}^{s}
\]

从而有

\[
\Gamma^{ij}_{k} dt^{k} = \Gamma^{ij}_{k} \frac{\partial t^{k}}{\partial p^{b}} dp^{b} = \frac{\partial t^{i}}{\partial p^{b}} \partial_{t}^{i} \frac{\partial^{2}t^{i}}{\partial p^{b}} dp^{b}
\]

是流形上的一个张量。下面我们考虑 Frobenius 流形 \( M \) 上的度量 \((\cdot,\cdot)^{*}\) 和 \(<,>^{*}\)（后者是原来的平坦度量 \(<,>\) 诱导的 \( T^{*}M \) 上的内积）。在这里我们进一步假设 Euler 向量场 \( E \) 在平坦坐标下是线性的（可能非齐次）。关于这两个度量的关系我们有如下命题：

**命题 3.4** Frobenius 流形上的度量 \((\cdot,\cdot)^{*}\) 和 \(<,>^{*}\) 构成一个平坦束，即满足下面两个性质：

1) 度量 \( g^{ij} = g^{ij}_{1} + \lambda g^{ij}_{2} \) 是平坦的，\( \forall \lambda ; \)

2) 1) 中度量的 Levi-Civita 联络具有形式 \( \Gamma^{ij}_{k} = \Gamma^{ij}_{1k} + \lambda \Gamma^{ij}_{2k} \)。

此时称度量 \((\cdot,\cdot)^{*}\) 为 Frobenius 流形的交叉形式。已知 Frobenius 流形的交叉形式，Euler 向量场 \( E \) 和单位向量场 \( e \)，若还有 \( \forall 1 \leq \alpha, \beta \leq n, d_{\alpha} + d_{\beta} - d_{E} + 2 \neq 0 \)，则我们能唯一地重建出这个 Frobenius 流形结构。实际上，对比 \(< dt^{\alpha}, dt^{\beta} >^{*} = \eta^{\alpha\beta} \) 及

\[
\mathcal{L}_{e}(dt^{\alpha}, dt^{\beta})^{*} = \partial_{t} g^{\alpha\beta} - \nabla_{(c^{\alpha\beta})} dt^{\alpha}(e) - \nabla_{(dt^{\alpha})} dt^{\beta}(e)
\]

\[
= \partial_{t}(E^{\alpha}c^{\alpha\beta}) - g^{\beta\gamma} \nabla_{\partial_{t}} c_{\alpha}^{\beta} - g^{\gamma\alpha} \nabla_{\partial_{t}} c_{\beta}^{\alpha}
\]

\[
= (\partial_{t} E^{\alpha} + E^{\alpha} \partial_{t} c_{\alpha}^{\beta}) = c_{1}^{\alpha\beta} + E^{\alpha} \partial_{t} c_{1}^{\alpha\beta} = \eta^{\alpha\beta}
\]

我们可令

\[
<,>^{*} : = \mathcal{L}_{e}(\cdot,\cdot)^{*} \quad (3.28)
\]

然后我们可选取坐标 \( t^{\alpha} \) 为度量 \(<,>^{*}\) 的平坦坐标，并且使 \( E \) 是齐次的。定义

\[
g^{\alpha\beta} := (dt^{\alpha}, dt^{\beta})^{*}, \quad \deg g^{\alpha\beta} := \frac{\mathcal{L}_{E}g^{\alpha\beta}}{\eta^{\alpha\beta}} \quad (3.29)
\]

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我们可以找到函数 $F$ 使得其 Hessian 矩阵的反变分量 $F^{\alpha\beta}$ （如 (3.14) 式中定义）满足如下方程组

$$g^{\alpha\beta} = \deg g^{\alpha\beta} F^{\alpha\beta}$$  \hspace{1cm} (3.30)

由此我们可建立 Frobenius 流形结构。

4 Frobenius 流形上的正则坐标

4.1 正则坐标的构造

为构造 Frobenius 流形上的正则坐标系，我们需要流形还要满足半单性假设。我们称点 $t \in M$ 为半单的，若 Frobenius 代数 $T_t M$ 是半单的（即它无幂零元）。易见半单性是点的开条件。称 Frobenius 流形 $M$ 满足半单性假设，若 $M$ 上一般的点都是半单的。以后我们将称满足半单性假设的 $M$ 为 massive Frobenius 流形。关于半单点有如下性质：

主要引理 在半单点的一个邻域上存在局部坐标 $u^1, ..., u^n$ 使得

$$\partial_i \cdot \partial_j = \delta_{ij} \partial_i , \quad \partial_i = \frac{\partial}{\partial u^i}$$  \hspace{1cm} (4.1)

证明: 由引理 2.2，在半单点 $t$ 的邻域上存在向量场 $\partial_1, ..., \partial_n$，使得 $\partial_i \cdot \partial_j = \delta_{ij} \partial_i$ （即代数 $T_t M$ 的幂等性）。我们需要证明这些向量场成对交换。令

$$[\partial_k, \partial_l] =: f^k_{ij} \partial_k$$  \hspace{1cm} (4.2)

我们重新表述形变联络 $\tilde{\nabla}(z)$ 在基 $\partial_1, ..., \partial_n$ 下的平坦性条件。联络 $\nabla$ 的曲率算子定义为

$$R(X,Y)Z := [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$$  \hspace{1cm} (4.3)

我们定义在基 $\partial_1, ..., \partial_n$ 下 Frobenius 流形上的欧式联络为

$$\nabla_{\partial_i} \partial_j =: \Gamma^k_{ij} \partial_k$$  \hspace{1cm} (4.4)

由 $R(\cdot, \cdot, \cdot)$ 为 3-张量，联络 $\tilde{\nabla}(z)$ 的曲率消失等价于曲率各分量 $R(\partial_i, \partial_j, \partial_k) = 0 , \forall i,j,k$。固定一组 $i,j,k$，将其展开得到（只需考虑 $i \neq j$ 的情形）

$$0 = R(\partial_i, \partial_j) \partial_k = (\partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik}) \partial_l + (\Gamma^l_{jk} \Gamma^r_{il} - \Gamma^l_{ik} \Gamma^r_{jl} - f^l_{ij} \Gamma^r_{lk}) \partial_r$$

$$+ z(\Gamma^l_{jk} \partial_i - \Gamma^l_{ik} \partial_j + (\delta_{jk} - \delta_{ik}) \Gamma^l_{ij} \partial_l - f^l_{ij} \partial_k)$$

（其中上下指标同时出现代表对其求和只对指标 $l, r$ 有意义）特别由 $z$ 的系数必须是零，推出

$$f^l_{ij} \partial_k = \Gamma^l_{jk} \partial_i - \Gamma^l_{ik} \partial_j + (\delta_{jk} - \delta_{ik}) \Gamma^l_{ij} \partial_l$$  \hspace{1cm} (4.5)

当 $k \neq i, j$ 时， (4.5) $\Rightarrow f^l_{ij} \partial_k = \Gamma^l_{jk} \partial_i - \Gamma^l_{ik} \partial_j$ 只有 $f^l_{ij} = 0$ 时。
当 \( k = i \) 时，(4.5) \( \Rightarrow f_{ij}^i \partial_i = \Gamma^i_{ji} \partial_i - \Gamma^i_{ii} \partial_i - \Gamma^i_{lj} \partial_l \)，右边 \( \partial_i \) 的系数为 \( \Gamma^i_{ji} - \Gamma^i_{ij} = 0 \Rightarrow f_{ij}^i = 0 \)。

当 \( k = j \) 时，(4.5) \( \Rightarrow f_{ij}^j \partial_j = \Gamma^j_{ji} \partial_j + \Gamma^j_{ij} \partial_j \)，右边 \( \partial_j \) 的系数为 \( -\Gamma^j_{ij} + \Gamma^j_{ij} = 0 \Rightarrow f_{ij}^j = 0 \)。

综上，\( \forall i, j, k \)，总有 \( f_{kij} = 0 \)，故 \( [\partial_i, \partial_j] = 0 \)，引理得证。□

在一个 Frobenius 流形 \( M \) 上，称上面引理中所述的局部坐标 \( u^1, \ldots, u^n \) 为正则坐标。

将正则坐标 \( u \) 看成是原来坐标 \( t \) 的函数，则有 \( \frac{\partial}{\partial t^\alpha} = \frac{\partial u^i}{\partial x^\alpha} \frac{\partial}{\partial u^i}, \frac{\partial}{\partial x^\beta} = \frac{\partial u^j}{\partial x^\beta} \frac{\partial}{\partial u^j} \)，推出一方面

\[
\frac{\partial}{\partial t^\alpha} \cdot \frac{\partial}{\partial t^\beta} = \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta} \left( \frac{\partial}{\partial u^i} \cdot \frac{\partial}{\partial u^j} \right) = \sum_i \partial_i u^i \partial_j u^j \frac{\partial}{\partial u^i}
\]

另一方面也有

\[
\frac{\partial}{\partial t^\alpha} \cdot \frac{\partial}{\partial t^\beta} = \xi_{\alpha\beta}(t) \frac{\partial}{\partial t^\gamma} = \xi_{\alpha\beta}(t) \partial_\gamma u^i \frac{\partial}{\partial u^i}
\]

\[
\Rightarrow \xi_{\alpha\beta}(t) \partial_\alpha u = \partial_\alpha u \partial_\beta u
\]

(4.6)

故正则坐标可由上面的 PDEs 的解得到。或者等价地，1-形式 \( du \) 必须是代数同态

\[
du : T_i M \rightarrow \mathbb{C}
\]

(4.7)

这是因为

\[
\forall i, \alpha, \beta, du^i \left( \frac{\partial}{\partial u^\alpha} \cdot \frac{\partial}{\partial u^\beta} \right) = du^i (\delta_{i\alpha} \delta_{i\beta}) = \delta_{i\alpha} \delta_{i\beta} = \delta_{i\alpha} \delta_{i\beta} = du^i \left( \frac{\partial}{\partial u^\alpha} \right) du^i \left( \frac{\partial}{\partial u^\alpha} \right)
\]

正则坐标下，关于 Frobenius 流形上的度量，Euler 向量场和单位向量场有如下两个引理

引理 4.1 在半单点邻域的正则坐标下，Euler 向量场 \( E \) 具有如下形式：

\[
E = \sum_i u^i \frac{\partial}{\partial u^i}
\]

(4.8)

证明：设由 \( E \) 生成的重标度参数为 \( k \)，在幂等元 \( \partial_i \) 上的作用为 \( \partial_i \rightarrow k^{-1} \partial_i \)，故对 \( u^i \) 的变化为 \( u^i \rightarrow ku^i \)，即沿着向量场 \( E \) 的流为伸缩，引理得证。□

引理 4.2 不变内积 \( \langle , \rangle \) 在正则坐标下是对角化的

\[
\langle \partial_i, \partial_j \rangle = \eta_{ij}(u) \delta_{ij}
\]

(4.9)

其中 \( \eta_{11}(u), \ldots, \eta_{nn}(u) \) 为非零函数。正则坐标下的单位向量场 \( e \) 具有形式

\[
e = \sum_i \partial_i
\]

(4.10)

此引理显然确。利用上面两个引理，我们得到关于正则坐标成立如下命题

命题 4.3 在半单点的一个邻域上，特征方程
det(g^{\alpha\beta}(t) - u^{\alpha\beta}) = 0 \quad (4.11)

的所有根 $u^1(t),...,u^n(t)$ 都是单的，且它们是此邻域上的正则坐标。相反对，若点 $t$ 处的特征方程的根都是单的，则 $t$ 为 Frobenius 流形上的半单点，且 $u^1(t),...,u^n(t)$ 是点 $t$ 邻域上的正则坐标。

**证明: **由切空间 $TM$ 和余切空间 $T^*M$ 之间的对应关系，有 $du^i \leftrightarrow \eta_{ii}^{-1}\partial_i$，

$$du^i \cdot du^j \leftrightarrow \eta_{ii}^{-1}\eta_{jj}^{-1}\partial_i \cdot \partial_j = \eta_{ii}^{-2}\delta_{ij}\partial_i \leftrightarrow \eta_{ii}^{-1}\delta_{ij}du^i$$

$$\Rightarrow \quad du^i \cdot du^j = \eta_{ii}^{-1}\delta_{ij}du^i$$

故交叉形式

$$g^{ij}(u) = (du^i,dv^j)^* = E^\varepsilon(u)\eta^{i\sigma}c_{\sigma\varepsilon}(u) = u^\varepsilon\eta_{ii}^{-1}c_{\varepsilon\sigma}(u) = u^\varepsilon\eta_{ii}^{-1}\delta_{ij}$$

$$\Rightarrow \quad \det(g^{ij} - \eta^{ij}) = \det diag(g^{11} - \eta^{11},...,g^{nn} - \eta^{nn}) = \left(\prod_i \eta_{ii}\right)^{-1}\prod_i (u - u^i)$$

由此方程 (4.11) 化为

$$\prod_i (u - u^i) = 0 \quad (4.12)$$

从而命题的第一部分得证。为证第二部分，我们考虑 $T_tM$ 上的线性算子 $U = (U^\alpha_\beta(t))$，其中

$$U^\alpha_\beta(t) := g^{\alpha\varepsilon}(t)\eta_{\varepsilon\beta} \quad (4.13)$$

注意到算子 $U$ 在基向量 $\partial_i$ 上的作用为

$$U\partial_i = U^j_i\partial_j = g^{jk}\eta_{ki}\partial_j = E^\varepsilon c_{\varepsilon k}\eta_{ij}\partial_j = E^\varepsilon\eta^{ij}\epsilon_{ij}\delta_{ij}\partial_j = E^\varepsilon\eta^{ij}\epsilon_{ij}\partial_j = E^\varepsilon(\partial_i \cdot \partial_i) = E \cdot \partial_i$$

故 $U$ 的作用就是乘以 Euler 向量场 $E$。又设 $U = (g^{ij})(\eta_{ij}) = AB$，故其特征方程

$$\det(\lambda I - U) = (-1)^n\det(AB - \lambda I) = (-1)^n\det(A - \lambda B^{-1})\det B$$

由 $\det B \neq 0 \Rightarrow \det(\lambda I - U) = 0 \Leftrightarrow \det(A - \lambda B^{-1}) = 0$，而

$$A - \lambda B^{-1} = (g^{ij} - \lambda(\eta^{ij}) = (g^{ij} - \lambda \eta^{ij})$$

由此知算子 $U$ 的特征方程即 (4.11)。由条件，该方程的根都是单的，所以在命题假设下，在 $t$ 点处乘以 $E$ 的算子是半单的。设其互不相同的特征根为 $\lambda_1,...,\lambda_n$，对应特征向量为 $e_1,..,e_n$，即 $Ee_i = \lambda_ie_i, i = 1,...,n$，若 $T_tM$ 中有幂零元 $x \neq 0$，设 $x = x_1e_1 + \cdots + x_ne_n$，且存在正整数 $k$ 使得 $x^k = 0$。$i \neq j$ 时，由 $\lambda_i \neq \lambda_j$ 及

$$\lambda_ie_i e_j = (Ee_i)e_j = E(e_i e_j) = (e_j e_i) = \lambda je_ie_i$$

推得 $e_i e_j = 0$。又设 $E = E^k e_k$，则 $Ee_i = E^k e_k e_i = E^i(e_i)^2 \Rightarrow (e_i)^2 = \frac{\lambda_i}{\mu_i}e_i =: \mu_ie_i$。且由 $\det AB \neq 0 \Rightarrow \lambda_i \neq 0 \Rightarrow \mu_i \neq 0$。计算 $x^k$ 的展开式，利用上面结果，归纳得到

$$x^k = x_1^k \mu_1^{-1}e_1 + \cdots + x_n^k \mu_n^{-1}e_n$$

上式等于零且仅当 $x_1 = \cdots = x_n = 0$，即 $x = 0$，这与之前假设的 $x \neq 0$ 矛盾！故 $T_tM$ 是半单的，命题得证。


### 4.2 正则坐标下的旋转系数

我们使用正则坐标的目的在于将 massive Frobenius 流形的局部分类问题化归为一个 ODE 的可积系统。为得到这样一个系统，我们需要研究在正则坐标下不变度量的性质。再次重申此度量在正则坐标下具有对角形式，即 \( u^1, \ldots, u^n \) 是（局部）欧式空间关于（复）欧式度量 \(<,>\) 的正交曲线坐标。在正交曲线坐标的几何中，我们研究的一个主要对象是旋转系数

\[
\gamma_{ij}(u) := \frac{\partial_j \sqrt{\eta_{ii}(u)}}{\sqrt{\eta_{jj}(u)}}, \; i \neq j
\]  (4.14)

（这里我们事先选取好 \( \sqrt{\eta_{ii}(u)} \) 的一个分支）。

引理 4.4 不变度量的系数 \( \eta_{ii}(u) \) 具有如下形式:

\[
\eta_{ii}(u) = \partial_i t_1(u), \; i = 1, \ldots, n
\]  (4.15)

**证明:** 由度量不变性，对任意两个向量场 \( a, b \)，不变内积 \( <,> \) 具有如下形式

\[
<a, b> = <e \cdot a, b> = <e, a \cdot b> = \omega(a \cdot b)
\]  (4.16)

其中 1-形式 \( \omega \) 的定义为 \( \omega(\cdot):= <e, \cdot> \)，故

\[
\omega = \eta_i dt^i = dt_1 = \sum k \frac{\partial t_1}{\partial u^k} du^k
\]

\[
\eta_i(u) = \langle \frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j} \rangle = \omega\left( \frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j} \right) = \omega\left( \frac{\partial}{\partial u^i} \right) = \partial_i t_1(u)
\]

引理得证。□

我们总结在正则坐标下不变度量的性质为如下命题:

**命题 4.5** 不变度量的旋转系数 (4.14) 是对称的，即 \( \gamma_{ij}(u) = \gamma_{ji}(u) \)，这个度量关于对角平移

\[
\sum_k \partial_k \eta_{ii}(u) = 0, \; i = 1, \ldots, n
\]  (4.17)

是不变的。函数 \( \eta_{ii}(u) \) 与 \( \gamma_{ij}(u) \) 是正则坐标的对应次数分别为 \(-d\) 和 \(-1\) 的齐次函数。

**证明:** 由 (4.15) 式，有

\[
\gamma_{ij}(u) = \frac{\partial_j \sqrt{\eta_{ii}(u)}}{\sqrt{\eta_{jj}(u)}} = \frac{1}{2} \frac{\partial_j \eta_{ii}(u)}{\eta_{jj}(u)} = \frac{1}{2} \frac{\partial_j \partial_i t_1(u)}{\sqrt{\eta_{ii}(u)}\sqrt{\eta_{jj}(u)}} = \frac{1}{2} \frac{\partial_j \partial_i t_1(u)}{\sqrt{\eta_{ii}(u)}\sqrt{\eta_{jj}(u)}}
\]

由此得到 \( \gamma_{ij}(u) \) 的对称性。

又由单位向量场 \( e = \sum k \partial_k \) 的协变常性（即 \( \forall i, \; \nabla \partial_i e = 0 \)），我们有

\[
0 = \nabla \partial_i (\sum_k \partial_k) = \sum_k \nabla \partial_i \partial_k = \sum_k (\Gamma^j_{ik}) \partial_j = (\sum_k \Gamma^j_{ik}) \partial_j \Rightarrow \sum_k \Gamma^j_{ik} = 0, \; \forall j
\]

特别 \( j = i \) 时，有 \( \sum_k \Gamma^i_{ik} = 0 \)。
由 $\eta^{ii} \neq 0$，故
$$\sum_k \partial_k \eta_{ii} = 0, \ i = 1, ..., n$$
进一步，推出
$$0 = \sum_k \partial_k \partial_i t_1(u) = \partial_i \sum_k \partial_k t_1(u) = \partial_i \sum_k \eta_{kk}(u), \forall i$$
故和 $\sum_k \eta_{kk}(u)$ 是常值。

由 (2.53) 式，将其中的 $u, v$ 均换为 $\partial_i$，则左边
$$\mathcal{L}_E < \partial_i, \partial_i > = u^k \partial_i \eta_{ii}$$
右边
$$2 < \mathcal{L}_E \partial_i, \partial_i >= + (2 - d) < \partial_i, \partial_i > = 2 < - \partial_i, \partial_i > + (2 - d) \eta_{ii} = \eta_{ii}$$
结合这两方面得到 $u^k \partial_k \eta_{ii} = - d \eta_{ii} \Rightarrow \eta_{ii}(cu) = c^{-d} \eta_{ii}(u), \forall c$。对左式两边关于 $u^j$ 求导，得
$$\partial_j \eta_{ii}(cu) = c^{-d-1} \partial_j \eta_{ii}(u)$$
$$\Rightarrow \gamma_{ij}(cu) = \frac{1}{2} \frac{\partial_j \eta_{ii}(cu)}{\sqrt{\eta_{ii}(cu) \eta_{jj}(cu)}} = \frac{1}{2} \frac{c^{-d-1} \partial_j \eta_{ii}(u)}{c^{-d-1} \eta_{ii}(u) \eta_{jj}(u)} = e^{-1} \gamma_{ij}(u)$$
至此命题全部得证。\[\Box\]

**推论 4.6** 旋转系数 (4.14) 满足下面的方程组

$$\partial_k \gamma_{ij} = \gamma_{ik} \gamma_{kj}, \ i, j, k \text{ 互不相同} \quad (4.18)$$

$$\sum_{k=1}^{n} \partial_k \gamma_{ij} = 0 \quad (4.19)$$

$$\sum_{k=1}^{n} u^k \partial_k \gamma_{ij} = - \gamma_{ij} \quad (4.20)$$

**证明:** 由 (4.15) 式的对角化度量的平坦性，以及 $i \neq j$ 时，$\Gamma^k_{ij} = 0$ 若 $k \neq i, j$ 和 $\Gamma^i_{ij} = \frac{1}{2} \eta^{ii} \partial_j \eta_{ii}$，对任意两两互异的 $i, j, k$，由
$$0 = R(\partial_i, \partial_j) \partial_k = [\nabla_{\partial_i}, \nabla_{\partial_j}] \partial_k - \nabla_{[\partial_i, \partial_j]} \partial_k = \nabla_{\partial_i} (\nabla_{\partial_j} \partial_k) - \nabla_{\partial_j} (\nabla_{\partial_i} \partial_k)$$
展开后得到 $\partial_j$ 项的系数
$$\partial_i \Gamma^j_{jk} + \Gamma^j_{ik} \Gamma^i_{kj} - \Gamma^i_{ik} \Gamma^j_{jk} = - \gamma_{ij} \quad (4.21)$$
将各 $\Gamma^i_{ij} = \frac{1}{2} \eta^{ii} \partial_j \eta_{ii}$ 式及
\[
\partial_i \Gamma^j_{jk} = \partial_i (\frac{1}{2} \eta^{jj} \partial_k \eta_{jj}) = -\frac{1}{2} (\eta^{jj} \partial_k \eta_{ij}) (\eta^{jj} \partial_i \eta_{jj}) + \frac{1}{2} \eta^{jj} \partial_i \partial_k \eta_{jj}
\]

代入 (4.21) 式，整理后得到

\[
-\frac{1}{4} \eta_{ii} (\partial_k \eta_{jj}) (\partial_i \eta_{jj}) - \frac{1}{4} \eta_{jj} (\partial_k \eta_{ii}) (\partial_i \eta_{jj}) + \frac{1}{2} \partial_i \partial_k \eta_{jj} = \frac{1}{4} \eta_{kk} \eta_{jj} \eta_{ij}
\]

上式两边乘以 \(\sqrt{n_{jj}/n_{ii}}\) 得到

\[
-\frac{1}{4} \eta_{ii} (\frac{\partial_k \eta_{jj}}{\sqrt{n_{ii} n_{jj}}}) (\frac{\partial_i \eta_{jj}}{\sqrt{n_{jj}}}) - \frac{1}{4} \eta_{jj} (\frac{\partial_k \eta_{ii}}{\sqrt{n_{ii} n_{jj}}}) (\frac{\partial_i \eta_{jj}}{\sqrt{n_{jj}}}) + \frac{1}{2} \frac{\partial_i \partial_k \eta_{jj}}{\sqrt{n_{ii} n_{jj} n_{kk}}} = \frac{1}{4} \frac{\partial_i \eta_{kk}}{\sqrt{n_{ii} n_{jj}}} \frac{\partial_k \eta_{jj}}{\sqrt{n_{jj} n_{kk}}}
\]

化简 (4.22) 式的两边，分别得到

\[
\text{LHS} = -\frac{1}{4} (\frac{\partial_i \eta_{jj}}{\sqrt{n_{ii} n_{jj}}}) (\frac{\partial_k \eta_{jj}}{\sqrt{n_{jj}}}) + \frac{1}{2} \frac{\partial_k \eta_{ii}}{\sqrt{n_{ii} n_{jj}}} = \frac{1}{2} \frac{\partial_k \eta_{ii}}{\sqrt{n_{ii} n_{jj}}}
\]

\[
\text{RHS} = \left( \frac{1}{2} \frac{\partial_i \eta_{kk}}{\sqrt{n_{ii} n_{kk}}} \right) \left( \frac{1}{2} \frac{\partial_k \eta_{jj}}{\sqrt{n_{jj} n_{kk}}} \right) = \gamma_{ik} \gamma_{kj}
\]

由此得到旋转系数 \(\gamma_{ij}(u)\) 满足方程组的 (4.18) 式。直接计算 (4.19) 左边，得

\[
\sum_{k=1}^{n} \partial_k \gamma_{ij} = \sum_{k} \left( -\frac{1}{4} \frac{\partial_i \eta_{jj}}{\sqrt{n_{ii} n_{jj}}} (\eta_{jj} \frac{\partial_k \eta_{ii}}{\sqrt{n_{ii} n_{jj}}}) + \frac{1}{2} \frac{\partial_k \eta_{ii}}{\sqrt{n_{ii} n_{jj}}} \right)
\]

\[
= -\frac{1}{4} \frac{\eta_{jj} \partial_i \eta_{jj}}{\sqrt{n_{ii} n_{jj}}} \sum_k \partial_k \eta_{ii} - \frac{1}{4} \frac{\eta_{ii} \partial_i \eta_{jj}}{\sqrt{n_{ii} n_{jj}}} \sum_k \partial_k \eta_{jj} + \frac{1}{2} \frac{1}{\sqrt{n_{ii} n_{jj}}} \partial_i \sum_k \partial_k \eta_{jj}
\]

由命题 4.5 中的 \(\sum_k \partial_k \eta_{ii} = 0\) 知 (4.19) 式成立。

最后同样由命题 4.5 中的 \(\gamma_{ij}(cu) = c^{-1} \gamma_{ij}(u)\)，两边对 \(c\) 求导，并取 \(c = 1\)，得到 (4.20) 式。推论得证。

我们已经证明了任意 massive Frobenius 流形决定了 Darboux-Egoroff 方程组 (4.18, 19) 的一个伸缩不变 (143) 解。反之，我们也可以证明在某种泛型假设下，方程组 (4.18 ~ 20) 的解都局部决定了一个 massive Frobenius 流形。

定义 \(\Gamma(u) := (\gamma_{ij}(u))\) 为方程组 (4.18 ~ 20) 的解矩阵。

引理 4.7 线性方程组

\[
\partial_k \psi_i = \gamma_{ik} \psi_k, \quad i \neq k
\]

(4.23)

\[
\sum_{k=1}^{n} \partial_k \psi_i = 0, \quad i = 1, \ldots, n
\]

(4.24)

对辅助向量值函数 \(\psi = (\psi_1(u), \ldots, \psi_n(u))^T\) 有 \(n\)-维解空间。

证明：为证引理，只需验证该方程组满足相容性，即 \(\forall k \neq l, \forall \psi_i\)，有
\[\partial_l \partial_k \psi_i = \partial_k \partial_l \psi_i \] (4.25)

由前面推论，当 \(i \neq k, l\) 时，
\[\partial_l \partial_k \psi_i = \partial_l (\gamma_{ik} \psi_k) = (\partial_l \gamma_{ik}) \psi_k + \gamma_{ik} (\partial_l \psi_k) = \gamma_{il} \gamma_{lk} \psi_k + \gamma_{ik} \gamma_{lk} \psi_l\]

上式交换 \(k, l\) 保持不变，故此时 (4.25) 式成立。

当 \(i = k\) 或 \(l\) 时，不妨设 \(i = k\)，往证
\[\partial_l \partial_k \psi_k = \partial_k \partial_l \psi_k \] (4.26)

由 (4.23) 式，有
\[\partial_k \psi_k = -\sum_{j \neq k} \partial_j \psi_j = -\sum_{j \neq k} \gamma_{jk} \psi_j\]

\[\Rightarrow \partial_l \partial_k \psi_k = -\sum_{j \neq k} [(\partial \gamma_{jk}) \psi_j + \gamma_{jk} (\partial_l \psi_j)] = -\sum_{j \neq k, l} \gamma_{lj} \gamma_{lk} \psi_j - \sum_{j \neq k, l} \gamma_{lj} \gamma_{jk} \psi_l - \gamma_{lk} (\partial_l \psi_l)\]

\[= (\sum_{j \neq k, l} \partial_j \gamma_{lk} + \partial_k \gamma_{lk}) \psi_l - \sum_{j \neq k, l} (\partial \gamma_{lk}) \psi_j + \gamma_{lk}^2 \psi_k\]

\[= (\partial \gamma_{lk}) \psi_l + \gamma_{lk}^2 \psi_k = \partial_k (\gamma_{lk} \psi_l) = \partial_l \partial_l \psi_k\]

综上，方程组 (4.23, 24) 满足相容性，故有 \(n\)-维解空间。引理得证。 □

在特定的泛型假设下，可选取方程组 (4.23, 24) 的解空间的一组基，对 \(u\) 是齐次的。

我们引入 \(n \times n\) 矩阵
\[V(u) := [\Gamma(u), U] \] (4.27)

其中 \(U := diag(u^1, ..., u^n)\)， \([, , ]\) 为矩阵的对易子。

引理 4.8 矩阵 \(V(u)\) 满足下面的方程组
\[\partial_k V(u) = [V(u), [E_k, \Gamma]]\] (4.28)

其中 \(E_k\) 是矩阵单位，即 \((E_k)_{ij} = \delta_{ik} \delta_{jk}\)。相反地，所有微分方程 (4.18～20) 可由 (4.28) 推出。

证明: 首先，由 (4.19) 式，当 \(i \neq j\) 时，
\[\partial_l \gamma_{ij} + \partial_j \gamma_{ij} = -\sum_{k \neq i, j} \partial_k \gamma_{ij} = -\sum_{k \neq i, j} \gamma_{ik} \gamma_{kj}\]

又由 (4.20) 式，当 \(i \neq j\) 时，
\[u^l \partial_l \gamma_{ij} + u^j \partial_j \gamma_{ij} = -\gamma_{ij} - \sum_{k \neq i, j} u^k \partial_k \gamma_{ij} = -\gamma_{ij} - \sum_{k \neq i, j} u^k \gamma_{ik} \gamma_{kj}\]

结合上面两个等式，得到
\begin{equation}
(u^i - u^j) \partial_i \gamma_{ij} = \sum_{k \neq i,j} (u^j - u^k) \gamma_{ik} \gamma_{kj} - \gamma_{ij}
\end{equation}

当 \( u^i \neq u^j \) 时，有

\begin{equation}
\partial_i \gamma_{ij} = \frac{1}{u^i - u^j} \left( \sum_{k \neq i,j} (u^j - u^k) \gamma_{ik} \gamma_{kj} - \gamma_{ij} \right)
\end{equation}

由 \( V(u) = [\Gamma(u), U] = (\gamma_{ij}(u^j - u^i)) \) 及 \([E_k, \Gamma(u)] = (\delta_{ik} \gamma_{kj} - \delta_{kj} \gamma_{ik})\)，推出

\begin{equation}
[V(u), [E_k, \Gamma(u)] = \left( \sum_l \gamma_{il}(u^l - u^j) \right) \delta_{ik} \gamma_{kj} - \delta_{kj} \gamma_{ik}
\end{equation}

当 \( i \neq j \)，k \neq i, j \) 时，(4.28) 式左右两边分量分别为

\begin{equation}
\text{LHS} = \partial_k (\gamma_{ij}(u^j - u^i)) = (u^j - u^i) \partial_k \gamma_{ij} = (u^j - u^i) \gamma_{ik} \gamma_{kj}
\end{equation}

\begin{equation}
\text{RHS} = \gamma_{ik} \gamma_{kj}(u^j - u^i) = \text{LHS}
\end{equation}

② \( k = i \) 时，将（4.29）式代入，得（4.28）式左右两边分量分别为

\begin{equation}
\text{LHS} = -\gamma_{ij} + (u^j - u^i) \frac{1}{u^i - u^j} \left( \sum_{l \neq i,j} (u^l - u^j) \gamma_{il} \gamma_{lj} - \gamma_{ij} \right) = -\sum_{l \neq i,j} (u^l - u^j) \gamma_{il} \gamma_{lj}
\end{equation}

\begin{equation}
\text{RHS} = \gamma_{il} \gamma_{lj}(u^l - u^j) - \sum_{l \neq i,j} \gamma_{il} \gamma_{lj}(u^j - u^l) = -\sum_{l \neq i,j} \gamma_{il} \gamma_{lj}(u^l - u^j) = \text{LHS}
\end{equation}

③ \( k = j \) 时，类似②情形可证（4.28）式左右两边相等。

当 \( i = j \) 时易验证（4.28）式左右两边均对称。综上，对 \( \forall k \)，（151）式成立，引理得证。

\[ \square \]

**推论 4.9** 关于矩阵 \( V(u) \) 有如下性质:

1) \( V(u) \) 作用在线性方程组 (4.23, 24) 的解空间上是封闭的；

2) \( V(u) \) 的特征根不依赖于 \( u \)；

3) 程组 (4.23, 24) 的一个解 \( \psi(u) \) 关于 \( u \) 是齐次的

\begin{equation}
\psi(\delta u) = \delta^u \psi(u)
\end{equation}

当且仅当 \( \psi(u) \) 是矩阵 \( V(u) \) 的从属于特征根 \( \mu \) 的特征向量

\begin{equation}
V(u) \psi(u) = \mu \psi(u)
\end{equation}

**证明:** 1) 由

\begin{equation}
\partial_k \psi_k = -\sum_{l \neq k} \partial_l \psi_k = -\sum_{l \neq k} \gamma_{lk} \psi_l
\end{equation}

可将方程组 (4.23, 24) 重新写为矩阵形式

\begin{equation}
\partial_k \psi = -[E_k, \Gamma(u)] \psi
\end{equation}
$$\Rightarrow \partial_k (V(u)\psi(u)) = (\partial_k V(u))\psi(u) + V(u)\partial_k \psi(u)$$
$$= [V(u), [E_k, \Gamma(u)]]\psi(u) - V(u)[E_k, \Gamma(u)]\psi(u)$$
$$= -[E_k, \Gamma(u)](V(u)\psi(u))$$

故 $V(u)\psi(u)$ 也是方程组 (4.23, 24) 的解。

2) 由 $V(u)$ 是方程组 (4.23, 24) 中 $n$ 维解空间上的作用，故对任意 $V(u)$ 的特征值函数 $\lambda(u)$，存在方程组 (4.23, 24) 的解 $\psi(u)$ 使得

$$V(u)\psi(u) = \lambda(u)\psi(u)$$

对一个与 $u^k$ 求导得

$$(\partial_k \lambda(u))\psi(u) - \lambda(u)[E_k, \Gamma(u)]\psi(u) = -[E_k, \Gamma(u)]V(u)\psi(u) = -[E_k, \Gamma(u)]\lambda(u)\psi(u)$$

故 $\partial_k \lambda(u) = 0, \forall k \Rightarrow \lambda(u) = \text{const}.$

3) 由 $\partial_i \psi = [\Gamma(u), E_i]\psi$，有

$$\sum_i u^i \partial_i \psi = \sum_i [\Gamma(u), E_i u^i]\psi = [\Gamma(u), \sum_i E_i u^i]\psi = [\Gamma(u), U]\psi = V\psi$$

故 $\psi(cu) = c^\mu \psi(u) \iff \sum_i u^i \partial_i \psi = \mu \psi \iff V\psi = \mu \psi$，推论得证。□

接下来，我们将在方程组 (4.23, 24) 中嵌入一个谱参数，从而可以使用单值形变技术对方程组 (4.18 ~ 20) 进行积分。我们记 $V(u)$ 的 $n$ 个特征根为 $\mu_1, ..., \mu_n$，由于 $V(u)$ 是反对称的，这组特征根可按下面顺序重新排列

$$\mu_\alpha + \mu_{n-\alpha+1} = 0 \quad (4.33)$$

关于矩阵 $V(U)$ 和 Frobenius 流形的联系，有如下命题

**命题 4.10** 对一个与 WDVV 的伸缩不变解相对应的 massive Frobenius 流形，矩阵 $V(u)$ 是可对角化的，其特征向量 $\psi_\alpha = (\psi_{1\alpha}(u), ..., \psi_{n\alpha}(u))^T$ 有如下表达式:

$$\psi_{\alpha \alpha}(u) = \frac{\partial_\alpha t_\alpha(u)}{\sqrt{\eta_{\alpha \alpha}(u)}}, \quad i, \alpha = 1, ..., n \quad (4.34)$$

相应的特征值为

$$\mu_\alpha = q_\alpha - \frac{d}{2} \quad (4.35)$$

反之，令 $V(u)$ 是任意一个方程组 (4.28) 的可对角化的解，且 $\psi_\alpha = (\psi_{\alpha \alpha}(u))$ 是方程组 (4.23, 24) 的解，并满足

$$V\psi_\alpha = \mu_\alpha \psi_\alpha \quad (4.36)$$

则由下面公式

$$\eta_{\alpha \beta} = \sum_i \psi_{i \alpha} \psi_{i \beta}, \quad \partial_\alpha t_\alpha = \psi_{i1} \psi_{i \alpha}, \quad c_{\alpha \beta \gamma} = \sum_i \frac{\psi_{i \alpha} \psi_{i \beta} \psi_{i \gamma}}{\psi_{i1}} \quad (4.37)$$
局部地决定了一个具有伸缩维度 \( q_{\alpha} = \mu_{\alpha} - \mu_1 \), \( d = -2\mu_1 \) 的 massive Frobenius 流形。通过直接计算可验证该命题。

由命题中的构造，知 Darboux-Egoroff 方程组的一个解 \( \gamma_{ij}(u) \) 决定了 \( n \) 个本质上不同（等价意义下）的 WDVV 的解，这来自于在公式 (4.37) 中选择 \( \psi_1 \) 时的自由性。由此，在相差一个容许度量（即保持流形旋转系数不变）的意义下，可建立 massive Frobenius 流形到方程组 (4.18 ～ 20) （其中要求 \( V(u) \) 可对角化）的解之间的一一对应。在引入谱参数 \( z \) 后，相应的辅助函数 \( \psi \) 为 \( \psi = (\psi_1(u,z), \ldots, \psi_n(u,z)) \), 并有如下引理

引理 4.11 方程组 (4.18, 19) 等价于依赖于谱参数 \( z \) 的辅助函数 \( \psi = (\psi_1(u,z), \ldots, \psi_n(u,z)) \) 满足的线性方程组

\[
\begin{align*}
\partial_k \psi_i &= \gamma_{ik} \psi_k, \quad i \neq k \\
\sum_{k=1}^n \partial_k \psi_i &= z \psi_i, \quad i = 1, \ldots, n
\end{align*}
\] (4.38, 39)

的相容性。此引理可通过直接计算验证。此时由公式

\[
\frac{\partial \tilde{t}(t(u), z)}{\sqrt{\eta_{ii}(u)}} = \psi_i(u, z)
\] (4.40)

gives the solution to the equation \( \partial_\alpha \xi_\beta = z \xi_\gamma \) of (4.38, 39)

进一步，通过引入关于 \( z \) 的有理系数微分算子

\[
\Lambda = \partial_z - U - \frac{1}{z} V(u)
\] (4.41)

（其中矩阵 \( U \) 和 \( V(u) \) 如之定义），我们可将辅助函数 \( \psi(u, z) \) 满足的方程组的相容性条件与旋转系数 \( \gamma_{ij}(u) \) 满足的方程组 (4.18, 19) 等价起来，得到如下命题

命 4.12 方程组 (4.18 ～ 20) （或等价地，方程组 (4.28)）等价于线性方程组 (4.38, 39) 和 \( z \) 的微分方程

\[
\Lambda \psi = 0
\] (4.42)

整体的相容性条件。

由引理 4.11，只需再证明此时关于对 \( z \) 求导的相容性，即

\[
\forall \ k, i, \ \partial_z \partial_k \psi_i = \partial_k \partial_z \psi_i
\]

对其直接展开计算便可以验证该命题。借由对算子 \( \Lambda \) 的研究，可以帮助我们从另一个角度刻画 massive Frobenius 流形。
5 算子 $\Lambda$ 的单值

5.1 算子 $\Lambda$ 的 Stokes 矩阵

考虑方程组 (4.38, 39, 42) 的解 $\psi(u, z) = (\psi_1(u, z), ..., \psi_n(u, z))^T$，对固定的一个 $u$，
它是定义在 $\mathbb{C} \setminus \{z = 0, \infty\}$ 上的多值解析函数。这个多值由算子 $\Lambda$ 的
单值描述。现在我们来研究这个算子单值的更多细节。此时我们固定某个向量 $u = (u^1, ..., u^n)$，
仅要求 $\{u^i\}_{i=1}^n$ 互不相同。在这里我们仅专注于微分方程 (4.42) 的 $z-$ 依赖性，而暂不考虑它对
$u$ 的依赖性。

方程 (4.42) 在 $z-$ 扩充复平面 $\mathbb{C} \cup \infty$ 上有两个奇点：正则奇点 $z = 0$ 和非正则奇点
$z = \infty$。其在 $z = 0$ 处（此时算子 $\Lambda \simeq \partial_z - \frac{V}{z}$）的解矩阵具有形式

$$\psi(z) \simeq z^V \psi_0 \quad (5.1)$$

( $\psi_0$ 为任意向量)。若矩阵 $V$ 可对角化且 $\forall \alpha \neq \beta$，差 $\mu_\alpha - \mu_\beta \notin \mathbb{Z}$，
则可建立 (4.42) 的一个基础解系，使得

$$\psi^0_{j\alpha}(z) = z^{\mu_\alpha} \psi_{j\alpha}(1 + o(z)), \quad z \to 0 \quad (5.2)$$

其中 $\alpha$ 是解的序号，向量值函数 $(\psi_{j\alpha})_{j=1}^n$，$\forall \alpha = 1, ..., n$ 是矩阵 $V$ 的从属于特征值
$\mu_\alpha$ 的特征向量。考虑由向量 (5.2) 作为列向量构成的 (4.42) 的解矩阵 $\Psi^0 = \Psi^0(z)$，此矩阵
在 $z = 0$ 周围的单值具有如下形式

$$\Psi^0(z e^{2\pi i}) = 3 \Psi^0(z) \cdot \text{diag}(e^{2\pi i \mu_1}, ..., e^{2\pi i \mu_n}) \quad (5.3)$$

在 $z = \infty$ 处（此时算子 $\Lambda \simeq \partial_z - U$）的单值由一个 $n \times n$ 的 Stokes 矩阵刻画。Stokes
矩阵的一个定义是基于通过规范变换将 $\Lambda$ 约化为正则形式的理论。设矩阵值函数 $g(z)$ 是
$z = \infty$ 附近的解析函数且满足

$$g(z)g(-z)^T = I, \quad g(z) = 1 + O(z) \quad \text{as} \quad z \to \infty \quad (5.4)$$

则 $\Lambda$ 的规范变换为

$$\Lambda \mapsto g^{-1}(z) \Lambda g(z) \quad (5.5)$$

由 Birkhoff 的理论，该变换的轨道的局部坐标由算子 $\Lambda$ 的 Stokes 矩阵决定。为给出
Stokes 矩阵的一个构造性定义，我们要研究方程 (4.42) 的解在 $z = \infty$ 附近的渐进分析。
仍然固定一个向量参数 $(u^1, ..., u^n)$，满足 $u^i \neq u^j, \forall i \neq j$，此时称为非震荡情形。

我们先引入 Stokes 射线的定义，它们是对 $i \neq j$ 定义的具有如下形式的射线 $R_{ij}$

$$R_{ij} := \{z \mid Re(z(u^i - u^j)) = 0, \quad Re(e^{\epsilon z}(u^i - u^j)) > 0 \quad \text{对充分小的} \quad \epsilon > 0\} \quad (5.6)$$

由此定义知 $R_{ij}$ 是与 $u^i - u^j$ 关于直线 $C(1 - i)$ 对称的射线，且 $R_{ij}$ 与 $R_{ji}$ 反向。令
直线 $\ell$ 是一条复平面上过原点且不包含 Stokes 射线的任意一条定向直线，它将复平面

实际上有 $\psi^0_{j\alpha}(ze^{\theta}) = \psi^0_{j\alpha}(z)e^{\mu_\alpha \theta}$. 3
分成两个半平面 $C_{\text{left}}$ 和 $C_{\text{right}}$，在这两个半平面上分别存在 $(4.42)$ 的解矩阵 $\Psi_{\text{left}}(z)$ 和 $\Psi_{\text{right}}(z)$，使得它们各自在 $C_{\text{left}}$ 和 $C_{\text{right}}$ 上解析，且对 $z \in C_{\text{left/right}}$，当 $z \to \infty$ 时，解矩阵有渐进表达

$$\Psi_{\text{left/right}}(z) = \left(1 + O(z^{-1})\right)e^{zU}$$  \hspace{1cm} (5.7)$$

这两个函数可被解析地延拓到半平面的某个扇形邻域上，在 $C_{\text{left}}$ 和 $C_{\text{right}}$ 的扇形邻域的交集上，解矩阵 $\Psi_{\text{left}}(z)$ 和 $\Psi_{\text{right}}(z)$ 由一个线性变换联系起来。

更具体的来说，设直线 $\ell = \{z = \rho e^{i\phi_0} \mid \rho \in \mathbb{R}, \phi_0 \text{固定}\}$，具有自然定向，则存在非退化的常值矩阵 $S_+, S_-$，使得延拓后的 $\Psi_{\text{left/right}}$ 在直线 $\ell$ 交界处有如下关系:

$$\Psi_{\text{left}}(\rho e^{i\phi_0}) = \Psi_{\text{right}}(\rho e^{i\phi_0})S_+ \hspace{1cm} (5.8)$$

$$\Psi_{\text{left}}(-\rho e^{i\phi_0}) = \Psi_{\text{right}}(-\rho e^{i\phi_0})S_- \hspace{1cm} (5.9)$$

边值问题 $(5.8, 9)$ 和渐进性 (5.7) 一起构成了 Riemann-Hilbert 问题的一种特殊情形。

关于矩阵 $S_+, S_-$ 有如下性质

**命题 5.1**

1) 矩阵 $S_\pm$ 满足

$$S_+ = S, \quad s_-=S^T \hspace{1cm} (5.10)$$

$$S \equiv (s_{ij}), \quad s_{ii} = 1, \quad s_{ij} = 0 \text{ 若 } i \neq j \text{ 且 } R_{ij} \subset C_{\text{right}} \hspace{1cm} (5.11)$$

2) 矩阵

$$M : = S^T S^{-1} \hspace{1cm} (5.12)$$

的特征值为

$$(e^{2\pi i \mu_1}, ..., e^{2\pi i \mu_n}) \hspace{1cm} (5.13)$$

其中 $\mu_1, ..., \mu_n$ 是 Frobenius 族的谱值（即矩阵 $V(u)$ 的特征值）。设矩阵 $C$ 的列是 $M$ 的相应的特征向量，则解 $\Psi^0(z)$ 具有如下形式

$$\Psi^0(z) = \Psi_{\text{right}}(z)C \hspace{1cm} (5.14)$$

**定义 5.1** (5.10) 式中的矩阵 $S$ 称为算子 $\Lambda$ 的 Stokes 矩阵。

由 (5.11) 式知，Stokes 矩阵 $S$ 中除对角线外至多有 $n(n - 1)/2$ 个元素非零（实际上关于矩阵对角线对称位置的两个元素至少其中一个为零），从而可对 $u_1, ..., u_n$ 进行适当的排序使得相应的 Stokes 矩阵化为上三角阵。且当直线 $\ell$ 在复平面上旋转穿过一条 Stokes 射线 $R_{ij}$ 时，Stokes 矩阵也会随之改变。由上面命题还知道若矩阵 $M$ 的特征值都是单的，则解 $\Psi^0(z)$ 完全由 Stokes 矩阵 $S$ 决定。

### 5.2 Stokes 矩阵与 Frobenius 矩形的关系

关于 Riemann-Hilbert 问题 $(5.7 \sim 9)$ 的解在无穷附近的渐进展开有更加精细的估计，即当 $z \to \infty$ 时，解有渐进展开式

$$\Psi_{\text{left/right}}(z) = \left(1 + \frac{\Gamma}{z} + O\left(\frac{1}{z^2}\right)\right)e^{zU} \hspace{1cm} (5.15)$$
由此展开式得到 $V = [\Gamma, U]$。

接下来我们假设对一个给定的 Stokes 矩阵 $S$ 以及一个给定的 $\kappa$，Riemann-Hilbert
问题 (5.7 ∼ 9) 有唯一解。由于该问题的可解性是一个开条件，对于 $S$ 我们得到一个局部
良定义的反对称矩阵值函数 $V(u)$，而这恰好就是方程组 (4.28) 的解，下面的命题对此有
更加具体的表述。

命题 5.2 若方程组 (4.42) 的系数矩阵 $V(u)$ 满足方程组 (4.28)，则 Stokes 矩阵 $S$ 不依赖
于 $u$。反之，若 $V(u)$ 的 $u$−依赖性保持矩阵 $S$ 不变，则 $V(u)$ 满足方程组 (4.28)。

根据此命题，算子 $\Lambda$ 的 Stokes 矩阵 $S$ 可看成方程组 (4.28) 的解的局部参数化。至此，
我们已经接近了使用算子 $\Lambda$ 的 Stokes 矩阵 $S$ 明确的给出 WDVV 方程组的解的参数化这一目标。在此之前，还需要明确此对应是建立在什么意义下的。实际上对于两个等价的
massive Frobenius 流形，它们相应的方程组 (4.28) 的解 $V(u) = (V_{ij}(u))$ 可通过下列的坐标变换联系起来:

$$
(u^1, \ldots, u^n)^T \mapsto P(u^1, \ldots, u^n)^T
$$

$$
V(u) \mapsto \epsilon P^{-1} V(u) P \epsilon
$$

其中 $P$ 是这个变换的矩阵表示，$\epsilon$ 是一个对角元为 $\pm 1$ 的任意对角阵。观察到此变换在微分算子 $\Lambda$ 上的作用为

$$
\Lambda \mapsto \epsilon P^{-1} \Lambda P \epsilon
$$

算子 $\Lambda$ 的 Stokes 矩阵 $S$ 的变化为

$$
S \mapsto P^{-1} \epsilon S \epsilon P
$$

另一方面注意到 Frobenius 流形的 Legendre 型坐标变换仅会如 (5.18) 方式改变算子 $\Lambda$，故相应的 Stokes 矩阵 $S$ 的变换如 (5.19) 形式。

总结之前的讨论，我们建立了如下一个局部一一对应

\{ massibe Frobenius 流形模掉 Legendre 型坐标变换 \} $\leftrightarrow$ \{ 微分算子 $\Lambda$ 的 Stokes 矩阵模掉变换 (5.19) \}

在此意义上，可定义该 Frobenius 流形的 Stokes 矩阵为算子 $\Lambda$ 的 Stokes 矩阵模掉变换 (5.19) 后得到的等价类。

6 结论

通过前面的讨论，我们最后得到在具有给定单值的所有具有如下形式

$$
\Lambda = \frac{d}{dz} - U - \frac{1}{z} V
$$

Legendre 型坐标变换 $S_\kappa$ 的定义为对一个给定的 $\kappa = 1, \ldots, n$，坐标变换为

$$
\hat{t}^\alpha = \partial_\kappa F(t), \quad \frac{\partial^2 \hat{t}^\alpha}{\partial \hat{t}^\beta} = \hat{\eta}_{\alpha \beta}, \quad \hat{\eta}_{\alpha \beta} = \eta_{\alpha \beta}.
$$
的微分算子的空间上有一个自然的 Frobenius 流形结构，这个结构在相差一个 Legendre 型坐标变换的意义下是唯一确定的。

相反地，任何一个满足半单性假设的 Frobenius 流形能由算子 Λ 的 Stokes 矩阵模掉一个变换得到的等价类空间而建立，并由此沟通了流形与算子单值的联系。

参考文献

Galois表示及其在Fermat大定理证明中的初步应用

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摘要

本文综述了Galois表示的基本概念, 以及怎样通过模形式和椭圆曲线构造Galois表示, 以及Andrew Wiles给出的Fermat大定理的证明过程中是怎么通过Galois表示将模形式和椭圆曲线联系起来的。

关键词：Galois表示；Fermat大定理；模形式；椭圆曲线

1 引言

1.1 Fermat大定理的历史

Fermat在大约1637年的时候在他读的《算术》的页边写下了著名的Fermat大定理, 用现在的数学语言描述就是:

命题1 (Fermat大定理) 对于 \( n \in \mathbb{Z}, n \geq 3 \), 方程

\[
a^n + b^n = c^n
\]

没有满足 \( a, b, c \in \mathbb{Z}, abc \neq 0 \) 的解。

Fermat的注记写道：“我找到了一个绝妙的证明，但是这里页边空白太小，写不下。”但是现在看来，Fermat大定理的证明远远没有这么简单，人们都认为Fermat当时并没有得到一个一般性的证明。

容易看到，只需要证明 \( n = 4 \)或\( n \)为奇素数的情况就够了。当\( n = 4 \)时证明只用到初等数论([3, §1.5], [7, §2.2]), Fermat本人在1640年左右用他发明的“无穷递降法”证明了。当\( n = 3 \)时证明要难一些，虽然有初等的证明[7, §6.5], 但是比较繁琐，最好的方法是利用\( \mathbb{Z}[\omega] \)的唯一分解性[8, §5.4], \( \omega = \frac{-1 + \sqrt{3}i}{2} \)。\( n = 3 \)的情况最早由Euler在1758–1770年间证明。因此要使Fermat大定理成立, 只需要如下命题成立就够了:

\[
* \text{基数} \ 83
\]

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命题2 设 \( p \geq 5 \) 是素数，则方程

\[
a^n + b^n + c^n = 0
\]

没有满足 \( a, b, c \in \mathbb{Z}, abc \neq 0 \) 的解。

在19世纪，为了证明Fermat大定理，数学家发展了代数数论的很多工具，并证明了此定理的很多特殊的情况，但是一般情况始终没有解决。直到20世纪后期，Gerhart Frey在1985年考虑了某种椭圆曲线，并指出方程(1)解的存在性将导致某种椭圆曲线的存在性，但是另一方面，这样的椭圆曲线有非常不寻常的性质，人们猜想它根本不存在。之后Jean-Pierre Serre, Ken Ribet, Andrew Wiles, Richard Taylor将这一过程中的每一步都填充完整了。最后Fermat大定理在1995年得到证明。

1.2 本文章的目的

本文章将综述Fermat大定理的证明框架，以及Andrew Wiles的工作，对Fermat大定理的证明过程中是怎么通过Galois表示将模形式和椭圆曲线联系起来的。为了让更多的人能了解Fermat大定理的证明过程，本文章将综述模形式、Galois表示和椭圆曲线的定义、基本性质和重要结论，以及它们之间的联系。对于Fermat大定理的证明细节，由于需要用到更深入的内容，因此就不能在这里详细介绍，只是陈述证明中的重要定理和结论，详细步骤可以参见[1], [2]以及其它相关文献。

在1850年左右，Kummer做了大量的工作，他的工作是将Fermat, Euler的方法推广，试图用此来完全证明Fermat大定理。他利用了分圆域的性质，可以证明当 \( p \) 是正则素数时Fermat大定理成立。但是在1915年，K. L. Jensen证明了存在无穷多个素数不是正则素数，因此这个方法并不能完全证明Fermat大定理。到最后Fermat大定理的解决并没有使用Kummer的思路。本文章将不再综述Kummer的工作。如果读者想了解Kummer的工作，可以参阅[3], [9, §3.2.4, §9.1], [8, §6.5, §9.4]等。

2 模形式

我们只介绍经典的，从上半平面 \( \mathfrak{H} := \{ z \in \mathbb{C} | \text{Im} z > 0 \} \) 定义的模形式。即使在上半平面定义的模形式，也有几种不同的等价定义，我们只陈述其中一种。此章内容主要来自[5]和[6]。

\footnote{即奇素数 \( p \) 使得分圆域 \( \mathbb{Q}(\zeta_p) \) 的类数不能被 \( p \) 整除。}
2.1 定义

全模群 $SL_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{Z}, ad - bc = 1 \right\}$ 通过分式线性变换作用在 $\mathbb{H}$ 上:

$$\gamma(z) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ast z := \frac{az + b}{cz + d}$$

其同样通过分式线性变换作用在 $\mathbb{P}^1(\mathbb{Q}) := \mathbb{Q} \cup \{\infty\}$ 上，而且此作用是可分的。容易知道，$\ker(SL_2(\mathbb{Z}) \to \text{Aut}(\mathbb{H})) = \{\pm I\}$，于是此群作用通过 $PSL_2(\mathbb{Z})$ 分解。$SL_2(\mathbb{Z})$ 的任意一个子群也作用在 $\mathbb{H}$ 上。我们定义 $SL_2(\mathbb{Z})$ 的一些子群:

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N} \right\}$$

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}$$

$$\Gamma_N := \Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}$$

其中 $\Gamma(N)$ 称为级为 $N$ 的主余子群 (principal congruence subgroup)。$SL_2(\mathbb{Z})$ 的子群 $G$ 称为余余子群 (congruence subgroup)，若存在正整数 $N$ 使得 $\Gamma(N) \subset G$。如果 $N$ 是最小的正整数，使得 $\Gamma(N) \subset G$，则 $G$ 称为级为 $N$ 的余余子群。由于 $\Gamma(N)$ 是 $SL_2(\mathbb{Z})$ 的指数有限的子群，因此余余子群也是 $SL_2(\mathbb{Z})$ 的指数有限的子群。反过来不一定对，见 [10, Example 2.5]。（在 [6] 中，$\Gamma_0(N)$ 称为级为 $N$ 的同余子群。）

设 $G$ 是 $SL_2(\mathbb{Z})$ 的指数有限的子群，则其在 $\mathbb{H}$ 上的作用诱导商空间 $Y(G) := \mathbb{H}/G$，其为一个 Hausdorff 空间，有一个 Riemann 面结构，但不紧致。可以在 $\mathbb{H}$ 中选取一个连通集，使得其为 $\mathbb{H}/G$ 的等价类代表元集，称为 $\mathbb{H}$ 在 $G$ 下的基本区域，我们也用符号 $\mathbb{H}/G$ 表示。

全模群 $SL_2(\mathbb{Z})$ 的一个基本区域是 (图1)

$$\mathcal{D} = \left\{ x + iy \mid -\frac{1}{2} \leq x < \frac{1}{2}, x^2 + y^2 > 1, \text{若} x > 0; x^2 + y^2 \geq 1, \text{若} x \leq 0 \right\}$$

而且对于群 $G$，其基本区域可以选取为 $\mathcal{D}$ 经过 $SL_2(\mathbb{Z})/G$ 的一个等价类代表元集 (此为有限集) 变换之后的并集。可以选取等价类代表元集，使得用此方法得到的 $G$ 的基本区域是连通集。我们看变换 $\gamma(z) = \frac{ax + b}{cz + d}$ 的不动点。若 $\frac{ax + b}{cz + d} = z$，则 $cz^2 + (d - a)z - b = 0$，判别式 $\Delta = (d - a)^2 + 4bc = (a + d)^2 - 4$。若 $\Delta > 0$，即 $|a + d| < 2$，则 $\gamma$ 在 $\mathbb{R} \setminus \mathbb{Q}$ 上有两个不动点。这时 $\gamma$ 称为双曲变换。若 $\Delta = 0$，即 $|a + d| = 2$，则 $\gamma$ 只有一个不动
点，为\( P^1(\mathbb{Q}) \)中的点，即有理数或\( i\infty \)。这时\( \gamma \)称为抛物变换。若\( \Delta < 0 \)，即\( |a+d| > 2 \)，则\( \gamma \)有两个不动点，其中一个在上半平面\( \mathbb{H} \)中，称为椭圆不动点。这时\( \gamma \)称为椭圆变换。椭圆不动点\( z \)的周期是指其对应椭圆变换在\( \text{PSL}_2(\mathbb{Z}) \)中的阶。椭圆不动点只有两种：
\[ z \in \text{SL}_2(\mathbb{Z}) \ast i \text{或} z \in \text{SL}_2(\mathbb{Z}) \ast (\frac{-1}{2} + \frac{\sqrt{3}}{2}i) \]。两种情况下的周期分别为2和3。

在\( \mathbb{H} / G \)中添加有限个点后可以使得其成为紧致Hausdorff空间\( X(G) := \overline{\mathbb{H} / G} \)。在其之上有紧致Riemann面的结构(因此其上面还有代数曲线的结构)，称为G的模曲线。这些添加进去的点是\( \mathbb{H} / G \)基本区域边界上同时在\( \mathbb{H} \)边界上的点，是\( i\infty \)或者实轴上的有理点。这些点称为G的尖点(cusps)。实际上，G的所有尖点集合就是G作用在\( P^1(\mathbb{Q}) \)上的轨道集合。例如当\( G = \text{SL}_2(\mathbb{Z}) \)时，其在\( P^1(\mathbb{Q}) \)上的作用可迁，仅有的尖点是\( i\infty \)。我们约定\( X(N) := X(\Gamma(N)) \), \( X_0(N) := X(\Gamma_0(N)) \), \( X_1(N) := X(\Gamma_1(N)) \)。

现在我们可以定义模形式:

**定义3** 设\( G = \text{SL}_2(\mathbb{Z}) \)的指数有限的子群，\( k \in \mathbb{Z} \)。一个亚纯函数\( f: \mathbb{H} \rightarrow \mathbb{C} \)称为群\( G \)的权为\( k \)的亚纯模形式，如果对所有\( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G, z \in \mathbb{H} \)，都有
\[
f(\gamma(z)) = f\left( \frac{az+b}{cz+d} \right) = (cz+d)^k f(z).
\]
(有时我们还要求\( f \)在\( G \)的所有尖点处亚纯，定义与下面的在尖点处全纯的定义类似。)

\( f \)称为群\( G \)的权为\( k \)的(全纯)模形式。如果\( f \)在\( \mathbb{H} \)上全纯，而且在\( G \)的所有尖点处全纯，\( f \)称为群\( G \)的权为\( k \)的尖点形式(cusp form)。如果\( f \)满足之上所有条件，而且在\( G \)的所有尖点处取值为0。

我们来解释一下定义中的“在尖点处全纯”的意义。对于\( G = \text{SL}_2(\mathbb{Z}) \)的指数有限的子群，存在最小的正整数\( N \)使得\( z \rightarrow z + N \)在\( G \)中。这时对于\( f \)是群\( G \)的权为\( k \)的亚纯模形式，有\( f(z) = f(z + N) \)。因此如果\( f \)在\( \mathbb{H} \)上全纯，设\( q = \exp(2\pi i \frac{z}{N}) \)，则\( f \)关于\( q \)在去心的单位圆盘内解析。我们说\( f \)在\( i\infty \)处全纯(即解析)，如果\( f \)关于\( q \)在原点处解析(由复分析，要使此条件成立，只需要\( f \)关于\( q \)在原点处连续)。这时\( f \)在原点处有幂级数展开
\[
f(z) = \sum_{n=0}^{\infty} a_n q^n,
\]
称为\( f \)在\( i\infty \)的Fourier展开或\( q \)-展开(q-expansion)。\( a_n = a_n(f) \)称为\( f \)的第\( n \)个Fourier系数。类似地，用一个抛物变换将其它尖点变为\( i\infty \)，可以定义\( f \)在其它尖点处的全纯性和Fourier展开。如果\( f \)是尖点形式，则有\( a_0 = 0 \)成立。反过来，如果\( f \)在每个尖点处的Fourier展开的第0个系数为0，则\( f \)是尖点形式。
群G的权为k的模形式全体构成一个线性空间，记为M_k(G)。尖点形式全体构成一个线性子空间，记为S_k(G)。容易验证权为k和l的模形式相乘得到权为k + l的模形式，因此M(G) := \bigoplus_{k=0}^{\infty} M_k(G)构成一个分次环。

我们有一些方法来构造G上的权为2k的模形式和尖点形式：Poincaré级数和Eisenstein级数。请参见[6, §8–9]。

2.2 模形式空间的维数

根据Riemann-Roch定理，可以计算出模形式空间和尖点形式空间的维数。我们只陈述权为偶数时的情况：

定理4 ([5, Theorem 3.5.1]) SL_2(Z)的指数有限的子群G的权为2k的模形式空间的维数是

$$\dim M_{2k}(G) = \begin{cases} 
0, & \text{若} k \leq -1, \\
1, & \text{若} k = 0, \\
(2k - 1)(g - 1) + \sigma_\infty k + \sigma_2 \left\lfloor \frac{1}{2} k \right\rfloor + \sigma_3 \left\lfloor \frac{2}{3} k \right\rfloor, & \text{若} k \geq 1.
\end{cases}$$

权为2k的尖点形式空间的维数是

$$\dim S_{2k}(G) = \begin{cases} 
0, & \text{若} k \leq 0, \\
g, & \text{若} k = 1, \\
(2k - 1)(g - 1) + \sigma_\infty (k - 1) + \sigma_2 \left\lfloor \frac{1}{2} k \right\rfloor + \sigma_3 \left\lfloor \frac{2}{3} k \right\rfloor, & \text{若} k \geq 2.
\end{cases}$$

其中g是\(\overline{G}/G\)的亏格，\(\sigma_\infty\)是\(\overline{G}/G\)的尖点个数。\(\sigma_n\)是\(G\)在\(\overline{G}/G\)中的周期为n的椭圆不动点个数，n = 2, 3。

作为例子，我们计算SL_2(Z)和\(\Gamma_0(2)\)的情形。

图1是SL_2(Z)的基本区域和椭圆不动点。其中白点表示SL_2(Z)的周期为2的椭圆不动点：\(i\)。黑点表示周期为3的椭圆不动点，有两个：\(\pm \frac{1}{2} + \frac{\sqrt{3}}{2} i\)，但是粘合之后表示的是同一个点。基本区域的标准三角剖分有3个顶点，3条边，2个面，因此亏格\(g = 0\)，以及\(\sigma_\infty = \sigma_2 = \sigma_3 = 1\)。因此当\(k \geq 1\)时

$$\dim M_{2k}(SL_2(Z)) = 1 - k + \left\lfloor \frac{1}{2} k \right\rfloor + \left\lfloor \frac{2}{3} k \right\rfloor = \begin{cases} 
\left\lfloor \frac{k}{6} \right\rfloor + 1, & \text{若} k \equiv 1 \pmod{6}, \\
\left\lfloor \frac{k}{6} \right\rfloor, & \text{若} k \not\equiv 1 \pmod{6}.
\end{cases}$$

同理容易计算当\(k \geq 1\)时，\(\dim S_{2k}(SL_2(Z)) = \dim M_{2k}(SL_2(Z)) - 1\)。
有$\Gamma_0(2) = \Gamma_1(2)$，以及$\Gamma_0(2) \backslash \Gamma$陪集代表元可以选取为$
abla \begin{pmatrix} I, & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \end{pmatrix}$。因此得到图2是$\Gamma_0(2)$的基本区域和椭圆不动点。其中有点表示周期为2的椭圆不动点：$
abla \begin{pmatrix} \frac{1}{2} + \frac{1}{2}i \end{pmatrix}$。灰点表示$\Gamma_0(2)$的与$i\infty$不等价的尖点：0。而$\Gamma_0(2)$的周期为3的椭圆不动点不存在。基本区域的标准三角剖分有5个顶点，9条边，6个面，因此亏格$\sigma = 2, \sigma_1 = 1, \sigma_2 = 0$。因此得到当$k \geq 1$时$\dim M_{2k}(\Gamma_0(2)) = \left\lfloor \frac{k}{2} \right\rfloor + 1$，以及$\dim S_2(\Gamma_0(2)) = 0$，当$k \geq 2$时$\dim S_{2k}(\Gamma_0(2)) = \left\lfloor \frac{k}{2} \right\rfloor - 1$。
是$S_{2k}(G)$上面的Hermite内积。对于$G' \subset G$都是$SL_2(Z)$的有限指数的子群，$f, g \in S_{2k}(G)$，有$(f, g)_{G'} = [G : G'](f, g)_G$。有些书上Petersson内积的定义还要除掉$H/G$的体积，这样的话就有$(f, g)_{G'} = (f, g)_G$成立。[5, §5.4] 在不致引起歧义的情况下，我们将Petersson内积的下标$G$省略，写为$(f, g)$。

我们先定义全模群$SL_2(Z)$上的Hecke算子[6, §15]。对$T : z \mapsto \frac{az+b}{cz+d} \in GL_2^+(Q)$，定义

$$J_T(z) := \frac{dT(z)}{dz} = \frac{ad-bc}{(cz+d)^2},$$

$$(f|_k T)(z) := J_T(z)^k f(T(z)),$$

其中$f : H \to C$。用此记号，我们可以说$SL_2(Z)$的指数有限的子群$G$上的权为$2k$的模形式$f$满足$f|_k T = f, \forall T \in G$。

设$G$是$SL_2(Z)$的指数有限的子群，$H$是$GL_2^+(Q)$的子集，满足$HG = GH = H$，而且其有限的陪集分解：$H = \bigsqcup_j GM_j, M_j \in H$。由$H$可以造出除子群$Div(Y(G))$以及模形式空间$M_{2k}(G)$上的算子，我们都记为$T$：

$$T : Div(Y(G)) \to Div(Y(G)), \quad (z) \mapsto \sum_j (M_j z),$$

$$T : M_{2k}(G) \to M_{2k}(G), \quad f \mapsto \sum_j f|_k M_j,$$

可以验证，这两个定义与陪集代表元$M_j$的选取无关。

如果我们取$G = SL_2(Z), H = H_n = \{ T \in M_2(Z) | \det(T) = n \}$，则$H$满足前面的条件[6, §15, 引理]，陪集代表元可取为$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$，其中$ab = n, b > 0, d = 0, \cdots, d - 1$。其定义出来的算子$T_n$即为$SL_2(Z)$中的一个Hecke算子。记$\mathbb{T}$为由$T_n, n \geq 1$生成的$\mathbb{Z}$-代数，称为Hecke代数。

命题6 ([6, §15, 定理1]) Hecke代数$\mathbb{T}$是一个交换代数，而且

$$T_m T_n = \sum_{d|(m,n)} dT_{mnd^{-2}}.$$ 

特别是，当$(m, n) = 1$时，$T_m T_n = T_{mn}; \quad T_p^n T_p = T_{p^{n+1}} + pT_{p^{n-1}}$。因此$\mathbb{T}$由$p$是素数时的$T_p$生成。

因此我们可以得到$T_n$的生成函数可以写为局部因子的乘积：

$$\sum_{n=1}^{\infty} T_n n^{-s} = \prod_p \left( 1 + p^{1-2s} - p^{-s} T_p \right)^{-1}.$$
设\( f \in \mathcal{M}_{2k}(\text{SL}_2(\mathbb{Z})) \)。利用\( H_n \)的陪集代表元的选取，我们可以通过计算知道\( T_nf \)的Fourier系数与\( f \)的Fourier系数的关系[6, §16, 定理2]:

\[
 a_m(T_nf) = n^{1-k} \sum_{d|m} d^{2k-1} a_{md^{-2}}(f),
\]

特别，\( a_1(T_nf) = n^{1-k} a_n(f) \)。

在Petersson内积下，\( T_n \)是\( \mathcal{S}_{2k}(\text{SL}_2(\mathbb{Z})) \)上面的自伴随算子[6, §16, 定理4]，即\( \forall f, g \in \mathcal{S}_{2k}(\text{SL}_2(\mathbb{Z})) \), \( \langle T_nf, g \rangle = \langle f, T_ng \rangle \)。因此\( \mathcal{S}_{2k}(\text{SL}_2(\mathbb{Z})) \)存在一组正交基\( \{g_1, \cdots, g_r\} \)，使得每个\( g_j = \sum_{n=1}^{\infty} a_n(g_j) q^n \)都是\( T_n \)的特征向量，且\( a_1(g_j) = 1 \)。\( g_j \)称为归一化的特征形式(normalized eigenform)。事实上[6, §17, 定理1]，\( g_j \)是\( T_n \)的特征值为\( n^{1-k} a_n(g_j) \)的特征向量，而且\( a_j(n) \)满足关系式

\[
 a_m(g_j)a_n(g_j) = \sum_{d|(m,n)} d^{2k-1} a_{md^{-2}}(g_j).
\]

对\( \text{SL}_2(\mathbb{Z}) \)的任意指数有限子群\( G \)上的点形式\( f \)，我们可以定义其对应的\( L \)-函数

\[
 L(f, s) := \sum_{n=1}^{\infty} a_n(f) n^{-s}.
\]

设\( f \)是\( \text{SL}_2(\mathbb{Z}) \)上的权为\( 2k \)的归一化的点形式，由其Fourier系数的递推公式，我们得到其\( L \)-函数的局部因子乘积公式[6, §17, 定理2]:

\[
 L(f, s) = \prod_p \left( 1 + p^{2k-1-2s} - p^{-s} a_p(f) \right)^{-1}.
\]

为了定义\( \text{SL}_2(\mathbb{Z}) \)的同余子群上的Hecke算子，我们先定义一些概念。设\( G \)是一个群，\( \Gamma_1, \Gamma_2 \)是其子群，对任意\( g \in G \)。集合\( \Gamma_1 g \Gamma_2 := \{ \gamma_1 g \gamma_2 | \gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2 \} \)称为双陪集(double coset)。我们注意到\( \Gamma_1 \)在\( \Gamma_1 g \Gamma_2 \)上有左乘作用，\( \Gamma_2 \)在\( \Gamma_1 g \Gamma_2 \)上有右乘作用，于是我们有陪集分解

\[
 \Gamma_1 g \Gamma_2 = \bigcup_j \Gamma_1 \beta_j = \bigcup_k \delta_k \Gamma_2
\]

而且元素\( \beta_j, \delta_k \)可分别取在\( g \Gamma_2 \)和\( \Gamma_1 g \)中。

**定义7** ([6, §29]) 假设\( \Gamma_1, \Gamma_2 \)称为可公度的(commensurable)，记为\( \Gamma_1 \approx \Gamma_2 \)，若\( [\Gamma_i : \Gamma_1 \cap \Gamma_2] < \infty, i = 1, 2 \)。设\( \Gamma \)是\( G \)的子群，\( \Gamma \)的可公度化子(commensurator)定义为\( \Gamma := \{ g \in G | g^{-1} \Gamma g \approx \Gamma \} \)。

可以知道，\( \approx \)是一个等价关系，\( \Gamma \)的可公度化子\( \tilde{\Gamma} \)是\( G \)的子群，而且若\( \Gamma_1 \approx \Gamma_2 \)，则\( \tilde{\Gamma}_1 = \tilde{\Gamma}_2 \)。
命题8 若$g \in G$使得$g^{-1} \Gamma_1 g \approx \Gamma_2$，则双陪集$\Gamma_1 g \Gamma_2$的陪集分解只有有限多个元素。特别地，若$g \in \tilde{\Gamma}$，则双陪集$\Gamma g \Gamma$的陪集分解只有有限多个元素。

证明 可以验证，如下两个映射

$$
\varphi : \Gamma_1 \setminus (\Gamma_1 g \Gamma_2) \rightarrow (g^{-1} \Gamma_1 g \cap \Gamma_2) \setminus \Gamma_2,
$$

$$
\Gamma_1 (\gamma_1 \gamma_2) \mapsto (g^{-1} \Gamma_1 g \cap \Gamma_2) \gamma_2,
$$

$$
\psi : (\Gamma_1 g \Gamma_2) / \Gamma_2 \rightarrow g^{-1} \Gamma_1 g / (g^{-1} \Gamma_1 g \cap \Gamma_2),
$$

$$(\gamma_1 \gamma_2) \Gamma_2 \mapsto (g^{-1} \gamma_1 g)(g^{-1} \Gamma_1 g \cap \Gamma_2)$$

是良好定义的，而且是双射。因此由$g^{-1} \Gamma_1 g \approx \Gamma_2$立即得到陪集分解只有有限多个元

下面设$G = GL_2^+(Q)$，$\Gamma_1, \Gamma_2 \subset SL_2(Z)$是两个同余子群。则$\Gamma_1$和$\Gamma_2$总是可公

度的[5, Exercise 5.1.2]，因为它们的交集还是同余子群。设$\Gamma$是一个同余子群，则$g \in GL_2^+(Q)$，$g^{-1} \Gamma g \cap SL_2(Z)$也是同余子群[5, Lemma 5.1.1]。因此容易知道，$\forall g \in GL_2^+(Q)$，$g^{-1} \Gamma_1 g \approx \Gamma_2$，特别$\Gamma = GL_2^+(Q)$。

定义9 ([5, Definition 5.1.3]) 设$\Gamma_1, \Gamma_2$是两个同余子群，$\alpha \in GL_2^+(Q)$，则由双陪

集$\Gamma_1 \alpha \Gamma_2$定义了除子群$\text{Div}(Y(\Gamma_1))$到$\text{Div}(Y(\Gamma_2))$的同态：

$$
T : \text{Div}(Y(\Gamma_1)) \rightarrow \text{Div}(Y(\Gamma_2)),
$$

$$(z) \mapsto \sum_j (M_j z)$$

以及模形式空间$M_{2k}(\Gamma_1)$到$M_{2k}(\Gamma_2)$的算子：

$$
[\Gamma_1 \alpha \Gamma_2]_{2k} : M_{2k}(\Gamma_1) \rightarrow M_{2k}(\Gamma_2),
$$

$$
f \mapsto \sum_j f|_k M_j$$

其中$\{M_j\}$是$\Gamma_1 \alpha \Gamma_2$的陪集代表元：$\Gamma_1 \alpha \Gamma_2 = \bigsqcup_j \Gamma_j M_j$。此定义是良好的，而且不依赖于陪

集代表元的选取，将尖点形式映为尖点形式。

下面我们取$\Gamma_1 = \Gamma_2 = \Gamma_1(N)$，定义$M_{2k}(N) := M_{2k}(\Gamma_1(N))$上面的Hecke算子。第

一种Hecke算子是diamond算子$\langle d \rangle$；对$d \in (Z/NZ)^\times$，$\langle d \rangle f := [\Gamma_1(N) \Gamma_1(N)]_{2k} f$，其

中$M$是任意$\Gamma_0(N)$中的元素，使得其右下角元素模$N$同余于$d$。由于$\Gamma_1(N)$是$\Gamma_0(N)$的正

规子群（其为将右下角元素模$N$映射的核），因此$\Gamma_1(N) \Gamma_1(N)$的陪集分解代表元只有一个元

素，就是$M$。因此我们有$\langle d \rangle f = f|_k M$，而且可以知道此定义与$M$的选取无关。如果$(N,d) \neq 1$，则定义$\langle d \rangle = 0$。容易知道$\langle d_1 \rangle \langle d_2 \rangle = \langle d_2 \rangle \langle d_1 \rangle = \langle d_1 d_2 \rangle$。
设$\chi: (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$是一个模$N$的特征。我们定义$M_{2k}(N)$的$\chi$-特征空间:

$$M_{2k}(N, \chi) := \left\{ f \in M_{2k}(N) \mid f|_M = \chi(d)f, \forall M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \right\}$$

设$1_{N}$是模$N$的平凡特征，显然$M_{2k}(N, 1_{N}) = M_{2k}(\Gamma_0(N))$。由特征空间的定义，$f \in M_{2k}(N, \chi)$当且仅当$\langle d \rangle f = \chi(d)f, \forall d \in (\mathbb{Z}/N\mathbb{Z})^\times$。可以知道，$M_{2k}(N)$是其所有特征空间的直和，而且若$f \in M_{2k}(N)$是所有$\langle d \rangle$的特征向量，则其必定在某个特征空间中。

第二种Hecke算子$T_n$按如下方式定义：我们先定义$T_p$，其中$p$是素数:

$$T_p f := [\Gamma_1(N) \left( \begin{array}{cc} 1 & 0 \\ 0 & p \end{array} \right) \Gamma_1(N)]_{2k} f.$$

我们可以找到一组陪集代表元，从而写出$T_p f$的显式表达式:

**命题10 ([5, Proposition 5.2.1])** 当$p \mid N$时，$\Gamma_1(N) \left( \begin{array}{cc} 1 & 0 \\ 0 & p \end{array} \right) \Gamma_1(N)$的陪集代表元可以取为$
\left( \begin{array}{cc} 1 & b \\ 0 & p \end{array} \right), b = 0, \cdots, p - 1$. 而当$p \nmid N$时陪集代表元除了前面列出的元素之外，还要加上$\left( \begin{array}{cc} p & 0 \\ 0 & 1 \end{array} \right)$，其中$M$是任意$\Gamma_0(N)$中的元素，使得其右下角元素模$N$同余于$p$。

我们看到，当$N = 1$时$(n)$平凡，而$T_p$和我们前面定义的全模群上的Hecke算子相同。

当$n$不是素数时$T_n$是通过递推公式定义的：当$(m, n) = 1$时，$T_m T_n = T_{mn}$;

$$T_p^p T_p = T_p^{p+1} + p \langle p \rangle T_p^{p-1}. $$

可以验证，当$p \neq q$时，$T_p$与$T_q$可交换，而且$T_p$与$\langle d \rangle$可交换。因此$T_n$是良好定义的，而且是交换的。由此递推公式，我们可以得到一般情况下，

$$T_m T_n = \sum_{d \mid (m, n)} d\langle d \rangle T_{mn d^{-2}},$$

以及$T_n$的生成函数可以写为局部因子的乘积:

$$\sum_{n=1}^{\infty} T_n n^{-s} = \prod_p \left( 1 + p^{1-2s} \langle p \rangle - p^{-s} T_p \right)^{-1}. $$

我们看Hecke算子$T_n$在$f$的Fourier系数上的作用。通过$T_p$的显式表达式，以及$T_n$的递推公式，可以得到:

**命题11 ([5, Proposition 5.3.1])** 设$f = \sum_{n=0}^{\infty} a_n(f) q^n \in M_{2k}(N)$，有

$$a_m(T_n f) = n^{1-k} \sum_{d \mid (m, n)} d^{2k-1} a_{mn d^{-2}}(\langle d \rangle f),$$

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特别，若 \( f \in M_{2k}(N, \chi) \)，则有

\[
a_m(T_n f) = n^{1-k} \sum_{d|(m,n)} \chi(d)d^{2k-1}a_{mnd^{-2}}(f).
\]

在 \( S_{2k}(N) := S_{2k}(\Gamma_1(N)) \) 中，当 \((d, N) = 1\) 及 \( p \nmid N \) 时，\( \langle d \rangle \) 和 \( T_p \) 是正规算子[5, Theorem 5.5.3]，而且所有的Hecke算子交换，因此存在 \( S_{2k}(N) \) 的标准正交基，使得它们是所有满足 \((d, N) = 1\) 及 \( p \nmid N \) 的 \( \langle d \rangle \) 和 \( T_p \) 的特征向量[5, Theorem 5.5.4]。

定义 12 ([5, Definition 5.6.1]) \( S_{2k}(N) \) 的旧形式空间 (subspace of oldforms)

\[
S_{2k}(N)^{\text{old}} := \text{span} \left( \bigcup_{M | N} \bigcup_{M < N} \left\{ f |_{\Gamma_1(N)} \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix} \bigg| f \in S_{2k}(M) \right\} \right)
\]

\[
= \text{span} \left( \bigcup_{p \nmid N} S_{2k}(N/p) \cup \left\{ f |_{\Gamma_1(N)} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \bigg| f \in S_{2k}(N/p) \right\} \right)
\]

是级较低的尖点形式到级 \( N \) 的尖点形式的提升。\( S_{2k}(N) \) 的新形式空间 (subspace of newforms) 是旧形式空间的正交补:

\[
S_{2k}(N)^{\text{new}} := \left( S_{2k}(N)^{\text{old}} \right)^{\perp}.
\]

可以知道，新形式空间和旧形式空间是所有Hecke算子 \( \langle d \rangle \) 和 \( T_p \) 的不变子空间[5, Theorem 5.6.2]。因此前面提到的特征向量都落在新形式空间和旧形式空间的正交集中。一个重要结论是说落在新形式空间中的这些特征向量实际上是所有Hecke算子的特征向量[5, Theorem 5.8.2]。这些特征向量称为新形式 (newform)。这时 \( N \) 称为新形式的导子 (conductor)，其一定落在某个 \( \chi \)-特征空间 \( S_{2k}(N, \chi) \) 中，将 \( \chi \) 称为新形式的特征 (character)。

由于新形式是所有Hecke算子的特征向量，根据Hecke算子在Fourier系数上的作用，我们可以知道对一个特征为 \( \chi \) 的归一化的新形式 \( f = \sum_{n=1}^{\infty} a_n q^n \in S_{2k}(N)^{\text{new}} \)，其Fourier系数有关系式

\[
a_m(f)a_n(f) = \sum_{d|(n,m)} \chi(d)d^{2k-1}a_{mnd^{-2}}(f),
\]

以及 \( L \)-函数有局部因子乘积:

\[
L(f, s) = \prod_{n=1}^{\infty} a_n(f)n^{-s} = \prod_p \left( 1 + \chi(p)p^{2k-1-2s} - p^{-s}a_p(f) \right)^{-1}.
\]
2.4 模Jacobian，模Abel簇

此节内容主要来自[2]的§1.2和§1.7。

设Γ是SL₂(ℤ)的指数有限的子群。权为2的尖点形式空间S₂(Γ)中的元素f可以定义一个微分形式fdz，此形式是Γ不变的，因此我们可以由f定义X(Γ)上的微分形式ω_f，而且可以知道这样定义出来的微分形式是全纯的。

我们考虑对偶空间V = S₂(Γ)∨ := Hom(S₂(Γ), ℂ)，此空间维数与S₂(Γ)的维数一样，是X(Γ)的亏格g。我们可以定义同调群H₁(X(Γ)) =: Λ到V的自然映射，把每个闭链c映成线性泛函ϕ_c : S₂(Γ) → ℂ, f → ∫_c ω_f。Λ在V中的像是一个格，即秩为dim_R(V) = 2g的离散Z模。

取定S中的一个点z₀作为基点，我们有Abel-Jacobi映射Φ_{AJ} : X(Γ) → V/Λ，将z ∈ X(Γ)映为线性泛函f → ∫_z^c ω_f。此定义是良好的，即在相差Λ中的元素的意义下，不依赖于积分路径的选取。因此V/Λ为一个实2g维环面，其上面有复结构。事实上，其为X(Γ)的Jacobi簇，是一个Abel簇，我们称其为Γ的模Jacobian，记为J(Γ)。

我们设f = ∑∞_{n=1} a_n q^n是一个Γ上的(归一化)特征形式，K是Q添加其所有Fourier系数得到的扩域。则我们有代数满同态λ_f : T ⊗ Z Q → K，将每个Hecke算子映成其作用在f后得到的尖点形式的第1项Fourier系数。由T在S₂(Γ)上的作用，可以诱导T在V = S₂(Γ)∨上的作用。更进一步，可以知道T在J(Γ) = V/Λ上也有作用。

Shimura由f构造了Q上的模Abel簇A_f，其维数为[K : Q]。我们简要说明一下构造过程：设I_f := ker(λ_f) ∩ T ⊂ T是T的一个理想。则I_f作用在J(Γ)上的像I_f(J(Γ))是一个Q上的连通的子Abel簇，在T作用下稳定。定义A_f := J(Γ)/I_f(J(Γ))，其为一个Q上的Abel簇，且只与f在G_Q作用下的等价类有关。关于模Abel簇的更多信息，可以参见[2]的相关章节，以及其他相关书籍。由模Abel簇，可以由模形式定义Galois表示，参见后面的章节。

3 Galois表示

此章内容主要来自[1]和[5]。

3.1 定义与性质

设Q是Q在C中的代数闭包。绝对Galois群G_Q := Gal(ℂ/Q)上面有Krull拓扑：单位元处的邻域是G_Q的所有有有限指数有限的子群。在此拓扑下，G_Q是一个射影有限(profinite)群(即有限群的逆极限)，特别地，是一个紧拓扑群。

定义13 设A是一个拓扑环。A上的n维Galois表示ρ指的是一个连续群同态:

$$
\rho : G_Q \to GL_n(A).
$$
定义14  Galois表示$\rho : G_Q \to \text{GL}_n(A)$称为不可约，若$\rho$不能写为两个非平凡子表示的直和。若$\rho$作为$\text{Frac}(A)$上的Galois表示不可约，其中$\text{Frac}(A)$是$A$的分式域的代数闭包，则$\rho$称为绝对不可约。

定义15 ([1, Chapter I, 2.1]) 一个系数环(coefficient ring)指的是一个完备的Noether局部环，其剩余域是一个特征为固定素数$p$的有限域。

数学家常研究三种Galois表示[2, §2.1]：第一种是取$A = \mathbb{C}$，称为Artin表示。由于$G_Q$是紧的全不连通(totally disconnected)群，因此这时$\rho$的像只有有限个点，$\ker \rho$是$G_Q$的指数有限的正规子群，其固定域是$\mathbb{Q}$的有限Galois扩张。第二种是取$A = k$为特征$p$的有限域，称为模$p$表示。同样地，$\ker \rho$是$G_Q$的指数有限的正规子群，其固定域是$\mathbb{Q}$的有限Galois扩张。第三种是取$A$为系数环。任意一个系数环$A$都有唯一的连续环同态$\mathbb{Z}/p\mathbb{Z} \to A$（由唯一环同态$\mathbb{Z} \to A$，以及$A$上的$p$-adic拓扑导出），因此这样的表示称为$p$-adic表示。

下面如果不加说明，我们的表示总是指2维$p$-adic Galois表示。

定义16 ([1, Chapter I, 2.2]) 设$(A, m_A, k_A)$是一个系数环，Galois表示$\rho : G_Q \to \text{GL}_2(A)$的剩余表示(residual representation)

$\overline{\rho} : G_Q \to \text{GL}_2(k_A)$

指的是$\rho$与自然映射$\text{GL}_2(A) \to \text{GL}_2(k_A)$的复合。

我们说$\rho$是有限域$k$上的二元Galois表示$\rho_0 : G_Q \to \text{GL}_2(k)$到$A$的提升，若$k = k_A$，且$\overline{\rho} = \rho_0$。两个$\rho_0$到$A$的提升$\rho, \rho'$称为等价，记为$\rho \sim \rho'$，若存在一个$\text{GL}_2(A)$中模$m_A$同余于单位阵的矩阵，使得$\rho'$通过此矩阵共轭之后等于$\rho$。

$\rho_0$到$A$的一个(Galois)形变([Galois] deformation)指的是$\rho_0$到$A$的提升的一个等价类。设$\rho$是$\rho_0$到$A$的一个提升，则$\rho$所在的形变我们也记成$\rho$。

定义17 ([1, Chapter I, 2.3]) 设$\rho : G_Q \to \text{GL}_2(A)$是$A$上的二元Galois表示，则$\rho$的行列式(determinant)

$\det \rho : G_Q \to A^\times$

指的是$\rho$与$\text{GL}_2(A)$上行列式$\det : \text{GL}_2(A) \to A^\times$的复合。

在研究问题的时候，经常会限制考虑的Galois表示，使其的行列式等于指定的1维表示。作为一个例子，我们先定义(p-adic)分圆特征([p-adic] cyclotomic character)$\chi_p : G_Q \to \mathbb{Z}_p^\times$。我们考虑$\mathbb{Q}(\zeta_p) := \bigcup_{n=1}^{\infty} \mathbb{Q}(\zeta_{p^n})$是$\mathbb{Q}$添加所有$p^n$次单位根所得的扩域。其Galois群为

$$
\text{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q}) = \lim_n \text{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q}) \cong \lim_n (\mathbb{Z}/p^n\mathbb{Z})^\times \cong \mathbb{Z}_p^\times
$$

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因此我们可由Galois群中元素的限制\(G_\mathbb{Q} \rightarrow \text{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q}), \sigma \mapsto \sigma|_{\mathbb{Q}(\zeta_{p^\infty})}\)定义\(\chi_p\)如下：
对\(\sigma \in G_\mathbb{Q}\)，定义
\[\chi_p(\sigma) = (m_n + \mathbb{Z}/p^n\mathbb{Z})_{n=1}^{\infty}\]
其中\(m_n \in \mathbb{Z}\)使得\(\sigma(\zeta_{p^n}) = \cdots\)（[1, Chapter I, 2.7]) 我们说\(\rho\)是奇表示，若\(\det(\rho)\)在\(\mathbb{Q}\)中有极点。

定义19 ([1, Chapter I, 2.8]) 我们说\(\rho\)在\(l\)处非分歧，若\(I_l \subset \ker \rho|_{G_{\mathbb{Q}_l}}\)。
如果$\ker \rho$是$G_\mathbb{Q}$的指数有限的子群(例如当$\rho$是Artin表示和模$p$表示), 则$\rho$在$l$处非分歧, 当且仅当$l$在该群的固定域中非分歧。由于当$K/\mathbb{Q}$是固定的有限代数扩张的时候，只有有限个素数在$K$中分歧，因此我们得到这时$\rho$在几乎所有素数处非分歧。

如果$\rho$在$l$处非分歧，则$\rho|_{G_{\mathbb{Q}l}}: G_{\mathbb{Q}l} \to GL_2(A)$可以经过$G_{\mathbb{Q}l}/I_l = Gal(\overline{\mathbb{F}_l}/\mathbb{F}_l)$分解。因此这时我们可以谈论$\rho$在$\text{Frob}_l$处的取值。

定义20 ([1, Chapter I, 2.9]) 我们说$\rho$在$p$处平坦，若对任意$A$的指数有限的理想$I$，将$\rho|_{G_{\mathbb{Q}l}}$模掉$I$得到的表示$G_{\mathbb{Q}p} \to GL_2(A/I)$都可以延拓为$\mathbb{Z}_p$上的有限平坦群概形。

3.2 模Galois表示, Serre猜想

固定一个$\mathbb{Q}_p$之上的素理想$p$。设$f = \sum_{n=1}^{\infty} a_n q^n$是一个权为2，导子为$N$，特征为$\varepsilon$的(归一化的)新形式。设$K_f$是由$\varepsilon$的值以及Fourier系数$\{a_n\}_{n=1}^{\infty}$生成的数域在$p$处的完备化，以及$O_f$是$K_f$的整数环。由Eichler和Shimura的构造(模Abel簇$A_f$上的$p$-adic Tate模所对应的Galois表示)，$f$对应一个二维奇Galois表示：

$$\rho_f: G_{\mathbb{Q}} \to GL_2(O_f)$$

使得对素数$l \nmid pN$，$\rho_f$在$l$处非分歧，而且$\text{Tr}(\rho_f(Frob_l)) = a_l, \text{det}(\rho_f(Frob_l)) = \varepsilon(l)l$。对任意权为$w$的模形式，也可以类似构造Galois表示，只不过最后一个式子变成$\text{det}(\rho_f(Frob_l)) = \varepsilon(l)^{w+1}l^w$。类比这个性质与Hecke算子在Fourier系数上作用的性质(命题11)，我们可以得到模Galois表示的定义：

定义21 ([1, Chapter I, 4.3]) 系数环$A$上的二维奇Galois表示$\rho: G_{\mathbb{Q}} \to GL_2(A)$称为模Galois表示，若存在整数$N > 0$以及同态$\pi: \mathbb{F}'(N) \to A$，其中$\mathbb{F}'(N) \subset \text{End}(\mathcal{S}_2(\Gamma_1(N)))/\text{End}(\mathcal{S}_2(\Gamma_1(1))$的由$(d)$和$T_l, l \nmid pN$生成的Hecke子代数，使得$\rho$在$Np$之外非分歧，以及对任意素数$l \nmid pN$，都有$\text{Tr}(\rho(Frob_l)) = \pi(T_l), \text{det}(\rho(Frob_l)) = \pi((l^l)l$成立。

著名的Serre猜想说的是对任意$\rho$是有限域上的二维奇绝对不可约奇Galois表示，都存在权为2的特征形式$f$，使得$\overline{\rho}_f \sim \rho$，而且对$f$的导子有较好的估计。特别地，$\rho$是模Galois表示。

Serre猜想一直到2008年才被完全证明。不过在当时，Ribet已经证明了一个重要的特殊情况，Serre将其称之为epsilon猜想：

定理22 (Ribet定理, [1, Chapter I, 4.5]) 设$f$是权为2，导子为$Nl$的新形式，其中$l \nmid N$为素数。若$\overline{\rho}_f$绝对不可约，而且如下两个条件有一个成立：

1. $\overline{\rho}_f$在$l$处非分歧；
2. $l = p$且$\overline{\rho}_f$在$p$处平坦，

则存在一个权为2，导子为$N$的新形式$g$，使得$\overline{\rho}_f \sim \overline{\rho}_g$. 


在Fermat大定理的证明中用此定理就足够了。

4 椭圆曲线

此章内容主要来自[1]和[4]。

4.1 定义和性质

定义23 椭圆曲线是指一个二元组 \((E,O)\)，其中 \(E\) 是一个亏格为1的光滑射影曲线，
O是E上的一个点(无穷远点)。我们说 \((E,O)\) 是域 \(K\) 上的椭圆曲线，若 \(E\) 是 \(K\) 上的曲线，
\(O \in E(K)\) 是 \(E\) 上的 \(K\)-有理点。

任意椭圆曲线都可以看成 \(\mathbb{P}^2\) 中的光滑三次曲线, 由如下的方程给出:

\[ E : y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6 \]  (3)

其中 \(x = \frac{X}{Z}, y = \frac{Y}{Z}\)。此方程称为 \(E\) 的Weierstrass方程。这时 \(O = [0 : 1 : 0]\) 是 \(\mathbb{P}^2\) 中的一个无穷远点。如果 \(E\) 是 \(K\) 上的椭圆曲线，则参数 \(a_1, a_2, a_3, a_4, a_6\) 可以取为 \(K\) 中的元素。

定义24 两个由Weierstrass方程定义的椭圆曲线

\[ E : y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6 \]
\[ E' : y^2 + a'_1 xy + a'_3 y = x^3 + a'_2 x^2 + a'_4 x + a'_6 \]

称为同构。若 \(E'\) 可以由 \(E\) 经过如下的变量替换(然后两边除以 \(u^6\))得到:

\[ \psi : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} u^2 x + r \\ u^3 y + u^2 s x + t \end{pmatrix} = \begin{pmatrix} u^2 & 0 \\ u^2 s & u^3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} r \\ t \end{pmatrix} \]

其中 \(u \in K^\times, r, s, t \in K\)。变量替换 \(\psi\) 称为容许的参数替换。

容易验证椭圆曲线的同构是个等价关系，因此我们把同构的椭圆曲线看作等同。容易知道，如果 \(K\) 的特征 \(\text{char}(K) \neq 2, 3\)，则通过容许的参数替换之后，\(a_1, a_2, a_3\) 可以取为0，
因此 \(E\) 的Weierstrass方程有如下形式:

\[ E : y^2 = x^3 + Ax + B \]  (4)

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我们定义椭圆曲线(3)的一些量:

\[ b_2 = a_1^2 + 4a_2, \quad b_4 = 2a_4 + a_1a_3, \quad b_6 = a_3^2 + 4a_6, \]
\[ b_8 = a_1^2a_6 + 4a_2a_6 - a_1a_3a_4 + a_2a_3^2 - a_4^2, \quad c_4 = b_2^2 - 24b_4, \quad c_6 = -b_2^3 + 36b_2b_4 - 216b_6, \]
\[ \Delta = -b_2^2b_8 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6, \]
\[ j = \frac{c_4^3}{\Delta}, \quad \text{若} \Delta \neq 0. \]

对于方程(4),

\[ b_2 = 0, \quad b_4 = 2A, \quad b_6 = 4B, \]
\[ b_8 = -A^2, \quad c_4 = -48A, \quad c_6 = -864B, \]
\[ \Delta = -16(4A^3 + 27B^2), \]
\[ j = -1728 \frac{64A^3}{\Delta} = 1728 \frac{4A^3}{4A^3 + 27B^2}, \quad \text{若} \Delta \neq 0. \]

\(\Delta\)称为椭圆曲线的判别式(discriminant), \(j\)称为椭圆曲线的\(j\)-不变量(j-invariant).容易验证\(4b_8 = b_2b_6 - b_3^2, 1728\Delta = c_4^3 - c_6^2\)，以及在容许的参数替换下，\(c_4 = u^{-4}c_4, c_6' = u^{-6}c_6, \Delta' = u^{-12}\Delta, j' = j\).于是\(j\)在容许的参数替换下保持不变。这也是\(j\)-不变量名字的由来。

**命题25** 设\(E, E'\)是代数封闭域\(K\)上的两个椭圆曲线，则两者同构当且仅当\(j(E) = j(E')\)。

当\(\Delta \neq 0\)时\(E\)上没有奇点，是光滑曲线。当\(\Delta = 0\)时\(E\)是退化的椭圆曲线，其上恰好有一个奇点。这时如果\(c_4 \neq 0\)，则奇点处有两个不同的切线方向，称为节点(node)；如果\(c_4 = 0\)，则奇点处有两个相同的切线方向，称为尖点(cusp)

**4.2 群律**

椭圆曲线\(E\)上的点有一个加法群结构，称为群律(group law)，其上的加法的定义可以有多种不同的看法。设\(P, Q\)是\(E\)上的点，则\(P + Q\)定义为唯一的点\(R\)使得除子\((P) + (Q)\)和\((R) + (O)\)线性等价，即\((P) + (Q) - (R) - (O)\)是主除子。几何上看，三个点相加为零当且仅当三点共线(若有两个点重合，则此直线是经过此点的切线)。因此要计算两个点相加，只需要把这两个点连起来，所得直线与椭圆曲线相交于第三个点，然后再将此点与无穷远点连起来，所得直线与椭圆曲线相交于第三个点，即为所给两个点的相加。用此几何直观很容易得到两点相加的坐标的表达式：设\(P = (x_1, y_1), Q = (x_2, y_2)\)是以方程(4)给
出的椭圆曲线 $E$ 上的两个点，则有

$$x(P + Q) = \begin{cases} 
\frac{(y_2 - y_1)^2 - x_1 - x_2}{x_2 - x_1}, & \text{若 } P \text{ 和 } Q \text{ 是不同点,} \\
\frac{x_4 - 2Ax_1^2 - 8Bx_1 + A^2}{4(x_1^3 + Ax_1 + B)}, & \text{若 } P \text{ 和 } Q \text{ 是相同点。}
\end{cases}$$

类似地可以得到 $y$ 坐标的表达式。这些式子是原来两点坐标的有理函数。由加法就可以得到 $E$ 上的乘 $m$ 映射：

$$[m] : E \rightarrow E, \quad m \in \mathbb{Z}.$$  

如果 $E$ 是退化的椭圆曲线，则前面描述的群律使得 $E$ 上非奇异点的集合 $E^{ns}$ 上面有群结构。当 $E$ 有一个节点的时候，$E^{ns} \cong \mathbb{G}_m$ 有乘法群结构；当 $E$ 有一个尖点的时候，$E^{ns} \cong \mathbb{G}_a$ 有加法群结构。

### 定义26
椭圆曲线 $E_1, E_2$ 间非平凡的保持无穷远点的 Abel 簇的态射 $\phi : E_1 \rightarrow E_2$ 称为同源 (isogeny)。

一个同源 $\phi : E_1 \rightarrow E_2$ 总是一个群同态，即保持加法：$\phi(P + Q) = \phi(P) + \phi(Q)$。同源的核 $\ker \phi$ 总是 $E_1$ 的有限子群。$\phi$ 的次数 (degree) 定义为其作为曲线间的有限射的次数，也等于 $\ker \phi$ 的元素个数。将 $E_1$ 的点都映成 $O$ 的射的次数定义为 $0$。

设同源 $\phi : E_1 \rightarrow E_2$ 的次数为 $m$，则其对应一个对偶同源 (dual isogeny)

$$\hat{\phi} : E_2 \rightarrow E_1$$

使得 $\hat{\phi} \circ \phi = [m]_{E_1}, \phi \circ \hat{\phi} = [m]_{E_2}$。对偶同源有如下性质：$\hat{\phi} = \phi, \hat{\psi + \psi} = \hat{\phi + \psi}, \hat{\phi \circ \psi} = \psi \circ \hat{\phi}$ 以及 $[\hat{m}] = [m]$。由对偶同源我们知道，椭圆曲线间的同源是一个等价关系。

我们考虑 $E$ 的自同态全体 $\text{End}(E)$，其中包含 $E$ 到自己的所有同源，以及零态射。$\text{End}(E)$ 在加法和映射复合下构成一个环，称为 $E$ 的自同态环 (endomorphism ring)。显然对任意 $m \in \mathbb{Z}$，$E$ 中的乘 $m$ 映射都在自同态环中。

### 命题27 ([1, §2.4])
设 $E$ 为域 $K$ 上的椭圆曲线，则自同态环 $\text{End}(E)$ 同构于如下三种情况之一：(1) $\mathbb{Z}$；(2) 某个虚二次域的序 (order)，即乘法封闭的格；(3) 某个四元代数的极大序。最后一种情况只在 $K$ 的特征非零时出现，这时 $E$ 称为超奇异 (supersingular) 椭圆曲线。

如果 $K$ 的特征不为 2 或 3，则 $E$ 的自同构群 $\text{Aut}(E)$ 为如下三种情况之一：(1) $\mu_2$，若 $j(E) \neq 0, 1728$；(2) $\mu_4$，若 $j(E) = 1728$；(3) $\mu_6$，若 $j(E) = 0$。

当 $E$ 的自同态环 $\text{End}(E)$ 严格比 $\mathbb{Z}$ 大的时候，我们称椭圆曲线 $E$ 有复乘 (complex multiplication)。例如当 $j(E) = 0$ 或 1728 的时候，$E$ 有复乘。
4.3 极小判别式, 导子, L-函数

设 $K$ 是一个完备的局部域，$v: K^\times \to \mathbb{Z}$ 是其赋值。$R$ 是其整数环，$p$ 是唯一的极大理想，$k = R/p$ 是剩余域。椭圆曲线 $E/K$ 的极小 Weierstrass 方程是指方程 $(3)$，其中 $a_1, a_2, a_3, a_4, a_6 \in R$ 使得 $v(\Delta)$ 最小。若 $\text{char}(k) \neq 2, 3$，则极小 Weierstrass 方程的系数可以选取使得 $a_1, a_2, a_3$ 为 $0$。

我们有一些判定某 Weierstrass 方程是否是极小的充分条件 [4, Chapter VII, Remark 1.1]。根据容许的参数替换下 $\Delta$ 变化的关系式，两个 $v(\Delta)$ 只能相差 $12$ 的倍数。因此若某 Weierstrass 方程使得 $v(\Delta) < 12$，则其为极小 Weierstrass 方程。同理，若 $v(c_4) < 4$ 或 $v(c_6) < 6$，则其也为极小 Weierstrass 方程。若剩余域的特征 $\text{char}(k) \neq 2, 3$，则逆命题成立：$E$ 的极小 Weierstrass 方程满足 $v(\Delta) < 12$ 或 $v(c_4) < 4$。

设 $E/K$ 有 (3) 形式的方程，$E$ 模 $p$ 的约化 (reduction)，记为 $\bar{E}$，是将 $E$ 的 Weierstrass 方程的系数模掉 $p$ 得到的 $k$ 上的椭圆曲线。即使原来的曲线 $E$ 是非奇异的，$\bar{E}$ 也有可能是奇异的。如果 $\bar{E}$ 非奇异，则我们称 $E$ 有一个好 (good)，或稳定 (stable) 约化。如果 $\bar{E}$ 有一个节点，则称 $E$ 有一个乘性 (multiplicative) 或半稳定 (semi-stable) 约化。这时若节点处的切线方向在 $k$ 中定义，则称 $E$ 为好约化，否则称为非分裂 (non-split) 约化。如果 $\bar{E}$ 有一个尖点，则称 $E$ 有一个加性 (additive) 或不稳定 (unstable) 约化。有时我们把好约化和乘性约化统称为半稳定约化。我们说数域 $K$ 上的椭圆曲线 $E$ 是半稳定的，若对 $K$ 的任意有限素点 $p$，$E$ 在 $K_p$ 上都是半稳定的，即有好约化或者乘性约化。

设 $K$ 是数域，$E/K$ 是椭圆曲线，对每个 $K$ 的素点 $p$ 我们考虑 $E/K_p$ 的一个极小 Weierstrass 方程，设其判别式为 $\Delta_p$。$E/K$ 的极小判别式 (minimal discriminant) 是一个理想：

$$D_{E/K} := \prod_p p^{v_p(\Delta_p)}$$

若 $K$ 的类数为 $1$ (例如 $K = \mathbb{Q}$)，则可以找到 $E$ 的 Weierstrass 方程使得它同时在 $K$ 的所有素点处极小。这个方程的判别式 $\Delta$ 就等于 $E/K$ 的极小判别式，在相差 $K$ 中一个单位的 $12$ 次幂的意义下。

椭圆曲线的极小判别式反映了 $E$ 的坏约化的多少。另一个反映此性质的量是 $E/K$ 的导子 (conductor)，这是一个理想：

$$N_{E/K} := \prod_p p^{f_p(E/K)}$$
其中的指数$f_p(E/K)$为[1, §2.14]:

$$f_p(E/K) := \begin{cases} 
0, & \text{若$E$在$p$处有一个好约化}, \\
1, & \text{若$E$在$p$处有一个乘性约化}, \\
2, & \text{若$E$在$p$处有一个加性约化，且$p \nmid 6$,} \\
2 + \delta_p, & \text{若$E$在$p$处有一个加性约化，且$p | 6$.}
\end{cases}$$

最后一个情况比较复杂，包含一个整数$\delta_p$，我们就不再这里介绍了。由此我们知道，对于半稳定椭圆曲线，其导子就等于有乘性约化的理想$p$的乘积。

椭圆曲线$E/K$的$L$函数$L(E,s)$定义为局部因子的乘积:

$$L(E,s) := \prod_p L_p(q_p^{-s})^{-1},$$

局部因子

$$L_p(T) := \begin{cases} 
1 - a_p T + q_p T^2, & \text{若$E$在$p$处有一个好约化}, \\
1 - T, & \text{若$E$在$p$处有一个分裂乘性约化}, \\
1 + T, & \text{若$E$在$p$处有一个非分裂乘性约化}, \\
1, & \text{若$E$在$p$处有一个加性约化}.
\end{cases}$$

其中$q_p$是$p$的范数，当$E$在$p$处有一个好约化时，定义$a_p = q_p + 1 - |\hat{E}(k_p)|$，其中$k_p$是$K_p$的剩余域。

### 4.4 椭圆曲线与Galois表示

设$E/K$是椭圆曲线。$E$的$m$-torsion点是指$E$的几何点，乘$m$之后为$O$；

$$E[m] := \ker[m] = \{ P \in E(\overline{K}) | [m]P = O \}$$

如果我们只考虑$E$的$K$-有理$m$-torsion点，则记为$E(K)[m]$。

**命题28** 设$E/K$为椭圆曲线。若$\text{char}(K) = 0$或者$\text{char}(K) = p \nmid m$，则有$E[m] \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$；若$\text{char}(K) = p > 0$，则有$E[p^r] \cong \mathbb{Z}/p^r\mathbb{Z}$或$0$。

对一个固定的素数$l$，通过映射$[l]: E[l^{n+1}] \to E[l^n]$，可以给$l$的幂次的torsion点一个逆向系统。其逆极限称为$E$的$(l$-adic)$Tate$模:

$$T_l(E) := \lim_{\leftarrow n} E[l^n]$$
如果$\text{char}(K) \neq l$，则$T_l(E)$是秩为2的自由$\mathbb{Z}_l$模：$T_l(E) \cong \mathbb{Z}_l \times \mathbb{Z}_l$。用此我们可以定义2维$\mathbb{Q}_l$线性空间：$V_l(E) := T_l(E) \otimes \mathbb{Q} \cong \mathbb{Q}_l \times \mathbb{Q}_l$。

设$G_K := \text{Gal}(\overline{K}/K)$是$K$的绝对Galois群。则由于$E[m]$中的点的坐标都是$K$上的乘$m$映射的坐标表达式对应的代数方程的根，因此$G_K$在$E[m]$上有群作用。而且此群作用于$E$上的加法结构相容，因为加法对应的坐标表达式也是个$K$上的有理函数。因此若$\text{char}(K) = 0$或者$\text{char}(K) = p \nmid m$，则我们得到一个二维Galois表示：
$$\rho_{E,m} : G_K \to \text{Aut}(E[m]) \cong \text{GL}_2(\mathbb{Z}/m\mathbb{Z})$$

同样，对于Tate模$T_l(E)$，当$\text{char}(K) \neq l$时有二维Galois表示：
$$\rho_{E,l} : G_K \to \text{Aut}(T_l(E)) \cong \text{GL}_2(\mathbb{Z}_l)$$

此表示的剩余表示正好是$G_K$作用在$E[l]$上得到的Galois表示$\bar{\rho}_{E,l}$。用$\rho_{E,l}$得到线性空间$V_l(E) := T_l(E) \otimes \mathbb{Q}_l \cong \mathbb{Q}_l \times \mathbb{Q}_l$。

**命题29** ([1, §2.7]) Galois表示$\bar{\rho}_{E,m}$的行列式$\det(\bar{\rho}_{E,m})$等于分圆特征：
$$\chi_m : G_K \to \text{Aut}(\mu_m) \cong (\mathbb{Z}/m\mathbb{Z})^\times.$$ 

现在设$K = \mathbb{Q}$，$l$是素数。则有Galois表示$\rho_{E,l} : G_K \to \text{GL}_2(\mathbb{Z}_l)$，及其剩余表示$\bar{\rho}_{E,l} : G_K \to \text{GL}_2(\mathbb{F}_l)$。这些表示有如下的性质：

**命题30** ([1, Chapter I, 2.11]) 设$E$是$\mathbb{Q}$上椭圆曲线，$\rho_{E,p}$是$p$-adic Tate模$T_p(E)$对应的Galois表示，$N_E$是$E$的导子，则$\rho_{E,p}$的行列式等于分圆特征$\chi_p$，且$\rho_{E,p}$在$pN_E$之外非分歧。特别地，$\rho_{E,p}$是奇表示。

若$E$是极小判别式为$\Delta_E$的半稳定椭圆曲线，则剩余表示$\bar{\rho}_{E,p}$有局部性质：若素数$l \neq p$，则$\bar{\rho}_{E,p}$在$l$处非分歧当且仅当$p \mid \text{ord}_l(\Delta_E)$；而且$\rho_{E,p}$在$p$处平坦当且仅当$p \mid \text{ord}_p(\Delta_E)$。

**命题31** ([2, Proposition 2.8 (b), (c)]) 设$E$是$\mathbb{Q}$上椭圆曲线，则$\rho_{E,p}$绝对不可约，而$\bar{\rho}_{E,p}$在几乎所有素数$p$处绝对不可约。如果$E/\mathbb{Q}$没有复乘，则$\rho_{E,p}$在几乎所有素数$p$处都为满射，从而$\bar{\rho}_{E,p}$也是。

**命题32** ([2, Theorem 2.9]) 设$E$是$\mathbb{Q}$上椭圆曲线，则有：(a) 当$p > 163$时$\rho_{E,p}$不可约，(b) 若$E$半稳定，则当$p > 7$时$\rho_{E,p}$不可约。若$E$半稳定，而且$\bar{\rho}_{E,2}$平凡，则当$p > 3$时$\rho_{E,p}$不可约。

### 4.5 模椭圆曲线，Taniyama-Shimura-Weil猜想

$\mathbb{Q}$上的椭圆曲线$E$称为模椭圆曲线(modular elliptic curve)，若存在一个权为2，导子为$N_E$，特征平凡的新形式$f$，使得两者的$L$-函数相等：
$$L(f, s) = L(E, s).$$
模椭圆曲线有多种等价定义：

定理33 ([1, Chapter I, 5.1]) 设$E/Q$是椭圆曲线，以下条件等价：
(1) $E$是模椭圆曲线；
(2) 对所有素数$p$，$\rho_{E,p}$是模Galois表示；
(3) 对某个素数$p$，$\rho_{E,p}$是模Galois表示；
(4) 存在一个的非平凡的$Q$上代数曲线的态射$\pi : X_0(N_E) \to E$；
(5) $E$同源于某个由权为2，导子为$N_E$的新形式$f$对应的模Abel簇$A_f$。

Taniyama-Shimura-Weil猜想是说每个$Q$上的椭圆曲线都是模椭圆曲线。Wiles在1995年证明了此猜想在半稳定椭圆曲线上的情况，由此证明了Fermat大定理，我们将在后面叙述。此猜想最终在1999年被证明，证明的想法基于Wiles的方法。现这称为模性定理(Modularity Theorem)。

5 Fermat大定理证明框架

5.1 椭圆曲线, Galois表示与Fermat大定理

对于任意互素的三个整数构成的三元组$(A, B, C)$使得$A + B + C = 0$，Gerhart Frey考虑了如下形式的椭圆曲线：

$$E_{A,B,C} : y^2 = x(x - A)(x + B)$$

并阐述了$E_{A,B,C}$的一些算术性质与整数三元组$(A, B, C)$的算术性质的关系。在我们的情形中，只需要考虑$(A, B, C) = (a_p, b_p, c_p)$是方程(1)的可能的非平凡解的情况。由于这三个整数互素，不失一般性，可以设$a \equiv -1 \pmod{4}$以及$2 \mid b$。

命题34 ([1, Chapter I, 1.1]) 设$p \geq 5$是素数，$a, b, c$为非零互素整数满足$a^p + b^p + c^p = 0$，则$E_{a_p, b_p, c_p}$是半稳定椭圆曲线，极大判别式$\Delta_{a_p, b_p, c_p} = 2^{-8}(abc)^2p$，导子$N_{a_p, b_p, c_p} = \prod_{l \mid abc} l$。

证明 $E_{a_p, b_p, c_p}$的形如(3)的Weierstrass方程是

$$y^2 = x^3 + (B - A)x^2 - ABx$$

经过容许的参数替换$(u, r, s, t) = (2, 0, 1, 0)$得到

$$y^2 + xy = x^3 + \frac{B - A - 1}{4}x^2 - \frac{AB}{16}x$$

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由于$A = a^p \equiv -1 \pmod{4}$，以及$p \geq 5$，$32 | B = b^p$，因此方程的各项系数都是整数。
此方程的判别式$\Delta = 2^{-8} A^2 B^2 (A + B)^2 = 2^{-8} (abc)^2 p$，以及$c_4 = A^2 + AB + B^2$，$c_6 = (A - B)(A + \frac{B}{2})(A + 2B)$。由于$A$是奇数，$\frac{B}{2}$是偶数，因此$c_6$是奇数，故此方程在2处极小。对素数$l \mid abc$，则$l \nmid \Delta$，故此方程在$l$处极小。若$c \mid l$且方程$l \not\mid l$使得$c \nmid l$，方程写为

$$
\begin{align*}
E &= E_{a, b, c, p} \text{在} l \text{处有好约化。}
\end{align*}
$$

若素数$l \mid abc$，则方程(5)已经是模$l$极小，且$l \mid \Delta = 2^4 (abc)^2 p$，故$E_{a, b, c, p}$在$l$处有好约化。若$l = 2$，则(6)在$\mathbb{F}_2$中约化为$\tilde{E} : y^2 + x y = x^4$或$y^2 + x y = x^3 + x^2$，两者都以$(0, 0)$为奇点，前者写为$y(x + x) - x^3 = 0$，为分裂乘性约化，后者为$(y^2 + x y - x^2) - x^3 = 0$，为非分裂乘性约化。因此$E_{a, b, c, p}$在2处有乘性约化。若2 $\not\mid l \mid ab$，则方程(5)已经是模$l$极小。若$l \mid ab$，则$\tilde{E}$的奇点为$(0, 0)$，方程写为$y^2 - (B - A)x^2 - x^3 = 0$，$l \mid 4(B - A)$，因此奇点为节点。若$l \mid c$，则在$\mathbb{F}_p$中$A + B = 0$，$\tilde{E}$的奇点为$(A, 0)$，方程写为$y^2 - A(x - A)^2 - (x - A)^3 = 0$，$l \mid 4A$，因此奇点为节点。故$E_{a, b, c, p}$在$l$处有乘性约化。得到$E_{a, b, c, p}$是半稳定椭圆曲线，导子$N_{a, b, c, p} = \prod l$。

记$E := E_{a, b, c, p}$，则其诱导一个Galois表示：

$$
\overline{\rho}_{a, b, c, p} := \overline{\rho}_{E, p} : G_Q \to GL_2(\mathbb{F}_p)
$$

Gerhart Frey和Jean-Pierre Serre注意到此Galois表示有不寻常的局部性质[11]：

命题35 ([1, Chapter I, 3.1]) $\overline{\rho}_{a, b, c, p}$是奇的绝对不可约Galois表示，在$2p$之外非分歧，在$p$处平坦。

证明 由命题32(c)得到其为不可约表示，再由Serre的一个定理，由其不可约及$E$半稳定得到其绝对不可约。剩下的断言由命题30得到。

人们怀疑根本不存在$G_Q \to GL_2(\mathbb{F}_p)$的Galois表示满足这些性质，但是这在当时并没有被证明。不过根据Ribet的定理(定理22)，不存在模Galois表示满足这些性质：

证明 如果$\rho_{a, b, c, p} \sim \rho_{a, b, c, p}$是模Galois表示，即$\tilde{E}_{a, b, c, p}$是权为2的新形式，导子为$N_{a, b, c, p} = \prod l$. 因为$\overline{\rho}_{a, b, c, p}$是绝对不可约，在$2p$之外非分歧，在$p$处平坦，根据定理22，存在权为2，导子为2的新形式$g$使得$\overline{\rho}_g = \overline{\rho}_{a, b, c, p}$. 但是我们前面已经算得尖点形式空间$S_2(\Gamma_0(2))$的维数是0，所以根本不存在这样的新形式$g$。

因此要证明Fermat大定理，只需要证明$\rho_{a, b, c, p}$是模Galois表示就够了。

定理36 (Wiles, [1, Chapter I, 5.3]) $Q$上任意半稳定椭圆曲线都是模椭圆曲线。
特别，$E_{ap, bp, cp}$是模椭圆曲线，$\rho_{ap, bp, cp}$是模Galois表示。

### 5.2 Wiles的工作

这一节我们简要介绍Wiles为了证明Fermat大定理所做的一些工作。

我们设素数 $p \geq 3$，$k$ 是特征为 $p$ 的有限域。$\rho_0 : G_Q \rightarrow \text{GL}_2(k)$ 是具有行列式 $\chi_p$ 的Galois表示。

**定义37** ([1, Chapter I, 7.1]) 我们说一个Galois表示 $\rho : G_Q \rightarrow \text{GL}_2(A)$ 在 $p$ 处是ordinary，若 $\rho$ 在惯性群 $I_p$ 的限制在一组适当的基下的表示为 $\rho|_{I_p} = \begin{pmatrix} \chi_p & * \\ 0 & 1 \end{pmatrix}$。我们说 $\rho$ 在素数 $l$ 处半稳定(semistable)，若如下两个条件之一成立:

1. $l = p$ 且 $\rho$ 在 $p$ 处平坦或者 ordinary(或者两者都成立);
2. $l \neq p$ 且在一组适当的基下，$\rho|_{I_p} = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$。

我们说 $\rho$ 是半稳定的，如果它在每个素数处都是半稳定的。

半稳定Galois表示的一个例子：设 $E/Q$ 是半稳定椭圆曲线，则 $\rho_{E,p} : G_Q \rightarrow \text{GL}_2(\mathbb{Z}_p)$ 是半稳定的。接下来我们假设 $\rho_0$ 是半稳定的。

**定义38** ([1, Chapter I, 7.2]) 一个形变类型(deformation type) $D$ 指的是剩余表示 $\rho_0$ 的形变所满足的一组性质。更确切地说，$D$ 是一个从以 $k$ 为剩余域的系数环的范畴到集合范畴的一个函子，对于系数环 $A$，$D(A)$ 是满足性质 $D$ 的 $\rho_0$ 到 $A$ 的形变的全体。

为了证明定理36，我们只考虑一类特殊的形变类型。设 $S = \{ l \neq p | \rho_0 \text{在} l \text{分歧} \}$，以及与 $S$ 不交的素数的有限集合 $\Sigma_D$。我们定义形变类型 $D$ 中的形变 $\rho$ 满足：(1) 具有行列式 $\chi_p$; (2) 在 $S \cup \{ p \} \cup \Sigma_D$ 之外非分歧; (3) 在 $\Sigma_D$ 之外半稳定; (4) 若 $p \in \Sigma_D$ 且 $\rho_0$ 在 $p$ 处平坦，则 $\rho$ 也在 $p$ 处平坦。粗略地说，最后三个条件是说 $\rho$ 在 $\Sigma_D$ 之外与 $\rho_0$ 有相同的局部性质。我们可以知道，若 $\rho_0$ 在 $p$ 处 ordinary，则 $\rho$ 也是。

接下来我们假设 $\rho_0$ 是绝对不可约的。使用Mazur的Galois形变理论，Wiles给每个形变类型 $D$ 对应了一个系数环 $R_D$，称为万有形变环(universal deformation ring)，以及 $\rho_0$ 的类型 $D$ 的万有形变(universal deformation)$\rho_D : G_Q \rightarrow \text{GL}_2(R_D)$。万有形变环和万有形变满足如下的万有性质：对任意的 $\rho_0$ 的具有类型 $D$ 的形变 $\rho : G_Q \rightarrow \text{GL}_2(A)$，都存在唯一的同态 $\pi_A : R_D \rightarrow A$ 使得 $\rho$ 经过 $\rho_D$ 分解：

\[
\begin{array}{ccc}
G_Q & \xrightarrow{\rho_D} & \text{GL}_2(R_D) \\
\downarrow \rho & & \downarrow \exists \pi_A \\
\text{GL}_2(A) & & \\
\end{array}
\]
关于Galois形变理论的介绍，以及万有形变环的构造，可以参阅[1, Chapter VIII, IX, XIII]。

如果假设\(\rho_0\)还是模Galois表示，而且\(\rho_0|_{G_Q(\sqrt{-3})}\)也是绝对不可约的，Wiles定义了另一个系数环\(T_D\)，称为万有形变环(universal modular deformation ring)，以及万有模形变(universal modular deformation)\(\rho_{D,\text{mod}}: G_Q \rightarrow \text{GL}_2(T_D)\)，满足如下的万有性质：对任意的\(\rho_0\)的具有类型\(D\)的形变\(\rho: G_Q \rightarrow \text{GL}_2(A)\)，使得\(\rho\)是模Galois表示，都存在唯一的同态\(\pi_A: T_D \rightarrow A\)使得\(\rho\)经过\(\rho_{D,\text{mod}}\)分解：

\[
\begin{array}{ccc}
G_Q & \xrightarrow{\rho} & \text{GL}_2(T_D) \\
\downarrow & & \downarrow \exists \pi_A \\
\text{GL}_2(A) & \downarrow & \\
\end{array}
\]

Wiles花很大力气证明了\(T_D\)和\(\rho_{D,\text{mod}}\)的存在性，并构造出来。可以参见[1, Chapter VII, X, XII]。

根据万有形变环\(R_D\)的万有性质，存在唯一的环同态\(\varphi_D: R_D \rightarrow T_D\)使得\(\rho_{D,\text{mod}} = \varphi_D \circ \rho_D\)成立：

\[
\begin{array}{ccc}
G_Q & \xrightarrow{\rho} & \text{GL}_2(R_D) \\
\downarrow & & \downarrow \exists \varphi_D \\
\text{GL}_2(T_D) & \downarrow & \\
\end{array}
\]

Wiles证明了一个“主定理”，其一个特殊情况描述了\(\varphi_D\)的性质：

**定理39** ([1, Chapter I, 7.4]) 设\(\rho_0: G_Q \rightarrow \text{GL}_2(k)\)是具有行列式\(\chi_p\)的半稳定、绝对不可约的模Galois表示，而且\(\rho_0|_{G_Q(\sqrt{-3})}\)也是绝对不可约的。则映射\(\varphi_D: R_D \rightarrow T_D\)是完全相交环(complete intersection ring)的同构。

完全相交环的定义比较简单，我们就在此不叙述，可以参见[1, Chapter XI]。我们由此定理可以知道的是，\(\varphi_D: R_D \rightarrow T_D\)是完全相交环的同构，特别地，是环的同构。由\(\varphi_D\)是同构，我们立即得到如下推论：

**命题40** 设\(\rho_0\)满足定理39的条件，则\(\rho_0\)的任意的具有类型\(D\)的形变都是模Galois表示。

定理39的证明是用所谓的“Wiles数值判据”(Wiles' numerical criterion)：

**定理41** ([1, Chapter I, 7.5]) 设\(R, T\)是两个系数环，\(O\)是完备的离散赋值环，以及下
使得图中的各个映射都是满射。设 $I_R := \ker \pi_R, I_T := \ker \pi_T, \eta_T := \pi_T(\text{Ann}_T(I_T))$，则以下结论等价：

1. $\varphi$ 是完全相交环的同构；
2. $I_R/I_R^2$ 是有限集，而且 $|I_R/I_R^2| \leq |O/\eta_T|$；
3. $I_R/I_R^2$ 是有限集，而且 $|I_R/I_R^2| = |O/\eta_T|$。

用此定理，我们设 $f$ 是权为 2 的新形式，$\rho_f : \mathbb{Q} \to \text{GL}_2(O_f)$ 是 $\rho_0$ 的一个具有类型 $D$ 的形变。由 $T_D$ 的万有性，存在唯一的环同态 $\pi_{T_D} : T_D \to O_f$，使得 $\rho_f = \pi_{T_D} \circ \rho_{D,\text{mod}}$。设 $\pi_{R_D} := \pi_{T_D} \circ \varphi_D$，则有以下交换图表：

$$
\begin{array}{c}
R_D \\
\varphi_D \downarrow \downarrow \downarrow \downarrow \downarrow \\
T_D \\
\pi_{R_D} \\
\end{array}
$$

为了证明 $\varphi_D$ 是同构，Wiles 要证明定理 41 的等价条件 (2)。这样就变成了一个纯代数的问题。此证明的框架和具体细节请参见 [1] 的相关章节。

### 5.3 半稳定椭圆曲线的模性定理

这一节我们将综述定理 36 的大体证明过程。

我们前面已经说过，$\mathbb{Q}$ 上半稳定椭圆曲线 $E$ 对应的 Galois 表示 $\rho_{E,p}$ 是半稳定的。由命题 30，我们知道 $\rho_{E,p}$ 以 $\chi_p$ 作为行列式。

Serre 的一个定理说，当 $E$ 半稳定时，对任意素数 $p \geq 3$，剩余表示 $\overline{\rho}_{E,p}$ 为满射或者可约。因此当 $p \geq 3$ 时，由 $\overline{\rho}_{E,p}$ 不可约可以推出其绝对不可约，而且当 $p = 3$ 时，此等价于 $\overline{\rho}_{E,3}|_{G_{\sqrt{-3}}}$ 绝对不可约。因此由命题 40 和模椭圆曲线的等价定义，我们得到下面命题：

**命题 42** ([1, Chapter I, 7.8]) 设 $E/\mathbb{Q}$ 是半稳定椭圆曲线，若对于某个素数 $p \geq 3$，$\overline{\rho}_{E,p}$ 为不可约模 Galois 表示，则 $E$ 是模椭圆曲线。

Wiles 的证法是证明对于 $p = 3$ 或 $p = 5$，以上命题的条件成立。此证明基于以下几个定理，我们不加证明地陈述它们：

**定理 43** ([1, Chapter I, 7.9]) 对任意 $\mathbb{Q}$ 上椭圆曲线 $E$，若 $\overline{\rho}_{E,3}$ 不可约，则 $\overline{\rho}_{E,3}$ 是模 Galois 表示。
此定理也称为Wiles minus epsilon, 是Langlands和Tunnell的一个深刻的定理的推论，与GL2的Langlands纲领有关。关于GL2的Langlands纲领的介绍，以及此定理的更多内容，可以参见[1, Chapter VI]。

**定理44 ([1, Chapter I, 7.10])** 设$E/Q$是半稳定椭圆曲线，使得$\rho_{E,5}$不可约。则存在另一个半稳定椭圆曲线$E'/Q$使得$\rho_{E',3}$不可约，而且$\rho_{E',5} \sim \rho_{E,5}$。

在[1, Chapter XVI]中，我们可以构造出一族椭圆曲线$E'/Q$使得$\rho_{E',3}$不可约，而且$\rho_{E',5} \sim \rho_{E,5}$，而且这些椭圆曲线在5之外都是半稳定的。在这之中我们取$E'$在5-adic意义下充分接近$E$，我们就找到了所求的椭圆曲线。

**定理45 ([1, Chapter I, 7.11])** 设$E/Q$是半稳定椭圆曲线，则$\rho_{E,3}$和$\rho_{E,5}$至少有一个不可约。

如果此定理的结论不成立，则$E[15]$会有一个15阶的Galois不变子群，与[1, Chapter XVI]的Lemma 9 (iv)矛盾。

**定理36的证明** 设$E/Q$是半稳定椭圆曲线。若$\rho_{E,3}$不可约，则由定理43，其为模Galois表示，再由命题42，$E$是模椭圆曲线。

若$\rho_{E,3}$可约，则由定理45，$\rho_{E,5}$不可约，由定理44，存在另一个半稳定椭圆曲线$E'/Q$使得$\rho_{E',3}$不可约，而且$\rho_{E',5} \sim \rho_{E,5}$。因此由前面得到$E'$是模椭圆曲线，故$\rho_{E',5}$是模Galois表示，故$\rho_{E,5}$也是模Galois表示。由命题42，$E$是模椭圆曲线。□

我们在这里提一下，其实只需要Wiles minus epsilon(定理43)以及命题42，利用方程(1)的非平凡解所对应的半稳定椭圆曲线(5)的参数的特殊性，就能得到(5)是模椭圆曲线，从而证明Fermat大定理。此证明的过程请参见[12]。

### 6 结论

到现在为止，我们已经综述了Galois表示、模形式和椭圆曲线的基本概念和性质，以及怎样由模形式和椭圆曲线造出Galois表示，以及两者造出的Galois表示的联系。

为了证明Fermat大定理，我们假设方程(1)存在一个非平凡解，由此我们造出了一个半稳定椭圆曲线，由椭圆曲线造出Galois表示，其有不寻常的局部性质，使得其不可能是模Galois表示。但是由Wiles的半稳定情形的模性定理，此Galois表示一定是模Galois表示，发生了矛盾。因此我们证明了(1)的非平凡解不存在，从而Fermat大定理成立。

为了证明Wiles的半稳定情形的模性定理，我们使用Galois形变的理论，构造了万有形变环以及万有模形变环，并使用Wiles数值判据，通过一个纯代数的问题，证明了在某些情况下万有形变环和万有模形变环的同构性，进而得到某些情况下剩余表示的Galois形
变的模性。我们利用GL_2的Langlands纲领以及其它数学工具，证明了半稳定椭圆曲线一定存在某个剩余表示满足我们需要的性质，因此其Galois形变一定是模Galois表示，进而得到半稳定椭圆曲线的模性。

其它的形如A + B + C = 0的不定方程也可以用类似的方法研究，通过研究由其构造的椭圆曲线的Galois表示的局部性质，可以得到某些方程的解的不存在性。可以参见[11]里面所举的例子，以及[1, Chapter XX]。

在1999年椭圆曲线的模性定理被证明，以及在2008年Serre猜想被证明，使得我们可以断言任意Q上椭圆曲线都是模椭圆曲线，以及其对应的Galois表示的剩余表示的模Galois提升对应新形式的权和导子的更好的估计。因此我们可以对更多的形如A + B + C = 0的不定方程的解做出结论。

Fermat大定理的陈述是如此简单和初等，使得任意一个中学生都能看懂，但是其证明却异常艰难，困扰了无数的数学家。从Fermat大定理的提出，直到其最终被证明，花了超过350年的时间，其间促进了许多数学学科和理论的发展，例如代数数论，包括理想理论、ζ-函数及相关解析理论、类域论，Langlands纲领，以及模形式、椭圆曲线和Galois表示理论，以及椭圆曲线模性定理和Serre猜想的提出和证明。难怪有数学家说Fermat大定理是“下金蛋的母鸡”。

参考文献


局部域与整体域的欧拉特征公式

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摘要

局部域与整体域的欧拉特征公式是由John Tate分别在1962年与1965年给出的。论文目的是关于局部域与整体域的欧拉特征公式的介绍与详细地证明。在本文中，首先我们陈述Galois上同调的基本定义以及相关性质，然后我们将分别介绍与证明局部域的欧拉特征公式与整体域的欧拉特征公式。其证明思路源于Haruzo Hida书中[1,Ch4.4.4和4.4.5]以及J.S.Milne书中[2,Ch1.5]。在学习过程中，我发现Haruzo Hida在证明局部欧拉特征公式中，有些打印错误；这些打印错误导致后面证明中需要做大量的修改；但其大致思路与方向是没有任何问题的。然而在证明整体欧拉特征公式的时候，我发现他在处理等式φ(M) = φ(M*(1))时，他的证明有误，并且这个错误是非平凡的。但我发现他在处理上述等式的方法确实有很大的问题。随后沿着J.S.Milne的证明思路，我用完全不同于他的方法独立证明了φ(M) = φ(M*(1))。最后在3.3小节中给出了反例来说明J.S.Milne证明的错误。

关键词：错误，“*”函子，φ(M) = φ(M*(1))，反例

1 预备知识


1.1 上链与上同调

令G是一个有限群，M是一个G-模。从G-模范畴到阿贝尔群范畴的函子M → MG 是左正合的。这个函子的导出函子用H^n(G,−)来表示。其中H^n(G,M) 被称作群G的上同调群，我们可以用链复形通过如下方法去对它进行计算：

*基数72
首先定义 $C^n(G, M) := \text{Map}(G^n, M)$，其中 $C^n(G, M)$ 中的元素 $f$ 是 $M$ 的上链且有 $n$ 个变元在 $G$ 中的函数。微分映射

$$d_n : C^n(G, M) \to C^{n+1}(G, M)$$

定义如下

$$d_n(f_n)(g_1, \cdots, g_n) = g_1 \cdot f(g_2, \cdots, g_n) + \sum_{i=1}^{n} (-1)^i f(g_1, \cdots, g_1 g_i+1, \cdots, g_{n+1})$$

$$+ (-1)^{n+1} f(g_1, \cdots, g_n),$$

我们可以很容易地直接验证 $d_{n+1} \circ d_n = 0$。此时上同调群定义为

$$H^n(G, M) = H^n(C^\bullet(G, M)) = \ker d_n / \text{Im} d_{n-1}.$$  

当 $G$ 是一个 profinite 群时，$G$ 在离散的阿贝尔群范畴 $C_G$ 上的作用是连续的，其中 $C_G$ 是 $G$-模范畴的子范畴。所以对任意一个（离散）$G$-模 $M$，我们可以定义

$$H^n_c(G, M) := \lim_{\rightarrow H \subseteq G} H^n(G/H, M^H),$$

其中 $H$ 跑遍 $G$ 的所有正规开子群。如果我们想要计算上同调群 $H^n_c(G, M)$，我们还是可以采用上面相同的方法去计算，唯一需要修改的是上链必须是连续的（此时的连续自然由 profinite 的定义给出），这时在不发生混淆的情况下，我们用 $H^n(G, M)$ 去表示 $H^n_c(G, M)$。

1.2 corG/U映射与resG/U映射

令 $U$ 是 $G$ 的一个闭子群，我们有限制映射

$$\text{res}_{G/U} : H^n(G, M) \to H^n(U, M).$$

如果 $U$ 是 $G$ 的一个有限指标开子群，则我们有余限制映射

$$\text{cor}_{G/U} : H^n(U, M) \to H^n(G, M).$$

有了以上的定义，我们将给出下列性质：

命题 1.1 有下列性质成立：

1. $\text{cor}_{G/U} \circ \text{res}_{G/U}(x) = [G : U] x$，其中 $x \in H^n(G, M)$；

2. 如果 $G$ 是一个有限群，$M$ 是一个有限 $G$-模，且 $(|G|, |M|) = 1$。则对所有的正整数 $q$，$H^q(G, M) = 0$. 

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1.3 Inflation和Restriction序列

命題1.2 令U是G的一个正规闭子群，假设对所有的p = 1, 2, · · ·, q−1，H^p(U, M) = 0。则我们有如下正合序列：

0 → H^q(G/U, M_U) → H^q(G, M) → H^0(G/U, H^q(U, M)) → H^{q+1}(G/U, M_U).

1.4 Shapiro引理

令U是G的一个闭子群，其诱导模由Ind^G_U M = Hom_{Z[U]}(Z[G], M)给出。

命題1.3 则我们有如下同构:

H^n(G, Ind^G_U M) = H^n(U, M)

对所有n ≥ 0。

1.5 Tate上同调

令G是一个有限群，M是一个G-模，则Tate上同调群定义如下:

H^n_T(G, M) = \begin{cases} 
H^n(G, M), & n ≥ 1 \\
H^0(G, M), & n = 0 \\
\ker N_m/I_G M, & n = -1 \\
H_{-1}(G, M), & n < -1
\end{cases}

其中N_m : m → \sum_{g \in G} gm, I_G \subset Z[G]由g−1所生成，g ∈ G。

命題1.4 假设G是一个有限循环群，且由g所生成，则

H^n_T(G, M) ∼ H^{n+2}_T(G, M).

推论1.5 同样地，假设G是一个有限循环群且由g所生成，则对所有的n ≥ 1，我们有

H^{2n}(G, M) ∼ H^{0}_T(G, M) = M^G/N_m(M)

以及

H^{2n−1}(G, M) ∼ \ker N_m/I_G M.

命題1.6 如果G是一个循环群，M是一个有限模，则

|H^0_T(G, M)| = |H^1_T(G, M)|.
2 局部欧拉特征公式

定理2.1（局部情形）p是给定的一个有理素数，令K是Q_p的一个有限扩张，令G = Gal(\overline{Q}_p/K)。如果M是一个有限G-模，我们有如下的局部欧拉特征公式:

\[ \frac{|H^0(G, M)| \cdot |H^2(G, M)|}{|H^1(G, M)|} = \frac{|H^0(G, M)| \cdot |H^0(G, M^*(1))|}{|H^1(G, M)|} = ||M||_K, \]

其中M^*(1) = Hom(M, K^\times)，|n|_K = [O_K : nO_K]^{-1}中n为正整数。

其中第一个等式来源于Tate duality。下面我们不加证明的给出此性质，如下:

命题2.2（Tate duality）令M是一个有限生成的离散\Z[G]-模，则对所有的0 ≤ r ≤ 2，有

\[ H^r(G, M^*(1)) \cong H^{2-r}(G, M) \]

其中M^*(1) = Hom(M, K^\times)，N^* = Hom(N, Q/Z)是阿贝尔群N的Pontryagin对偶模。

特别地，如果M是一个有限模，则所有的上同调群H^r(G, M)都是有限群，并且如果r ≥ 3，则H^r(G, M) = 0。

2.1 局部情形的证明

为方便起见，我们用H^n(M)去代替H^n(G, M)。因为M = \bigoplus_l M[l^\infty]，其中l为有理素数。利用上同调群的加性特征，我们只需考虑M = M[l^\infty]这种情形即可。现在我们假设M = M[l^\infty]则H^q(M)是一个有限长度的\Z_l-模。对任意的\Z_l-模N我们有|N| = l^{\text{length}_{\Z_l}(N)}。这是因为任意一个非零单模它一定同构于\Z_l/l。这里length(N)是\Z_l-模N的Jordan-Holder序列的长度。此时，欧拉特征定义如下:

\[ \chi(M) = \chi(G, M) = \sum_{q=0}^{2} (-1)^q \text{length}_{\Z_l} H^q(M), \]

\[ \chi'(M) = \chi'(G, M) = \log_l(||M||_K) = \begin{cases} -[K : Q_p] \text{length}_{\Z_p} M & l = p, \\ 0 & l \neq p. \end{cases} \]

注意观察定理2.1的左右两边，实际上我们只需证明

\[ \chi(M) = \chi'(M) = \begin{cases} -[K : Q_p] \text{length}_{\Z_p} M & l = p, \\ 0 & l \neq p. \end{cases} \]

为了更好地理解这个定理，首先我们来看一种非常特殊的情况: M = Fl = \Z/lZ (G在Fl上是平凡作用)。由Tate Duality知，dim_{\Z_l} H^0(G, Fl) = dim_{\Z_p} \mathbb{F}_l^G = 1,
\[
dim_{\mathbb{F}_l} H^2(G, \mathbb{F}_l) = \dim_{\mathbb{F}_l} H^0(G, \mu_l)^* = \dim_{\mathbb{F}_l} \mu_l^* = \dim_{\mathbb{F}_l} \mu_l(K), \quad \text{其中} \quad \mu_l(K) = \{ z \in K \mid z^l = 1 \}. \quad \text{另一方面, 由Kummer理论, 我们有} \quad \dim_{\mathbb{F}_l} H^1(G, \mathbb{F}_l) = \dim_{\mathbb{F}_l} H^1(G, \mu_l)^* = \dim_{\mathbb{F}_l} \mu_l^*(K) = \dim_{\mathbb{F}_l} \mu_l(K). \quad \text{换句话说, 其中最后一个等式原因如下:}
\]

\[
1 \longrightarrow \mu_l \longrightarrow \mathbb{K}^\times \xrightarrow{\times x^l} \mathbb{K}^\times \longrightarrow 1.
\]

我们有

\[
1 \longrightarrow H^0(G, \mu_l) \longrightarrow H^0(G, \mathbb{K}^\times) \longrightarrow H^0(G, \mathbb{K}^\times) \longrightarrow H^1(G, \mu_l) \longrightarrow H^1(G, \mathbb{K}^\times) \longrightarrow 1,
\]

其中 \( H^1(G, \mathbb{K}^\times) = 1 \)由Hilbert 90定理给出。所以 \( H^1(G, \mu_l) \cong \mathbb{K}^\times / (\mathbb{K}^\times)^l \)。因为 \( \mathbb{K}^\times \cong O_K^\times \times \mathbb{Z} \), \( O^\times \cong O_K \times \mu = O_K \times \mu \cong (K) \times \prod_{q \neq l} \mu_q \cong (K) \), 其中 \( O_K \)是 \( K \)的整数环, 我们有:

\[
\mathbb{K}^\times / (\mathbb{K}^\times)^l \cong \begin{cases} 
\mathbb{Z}/l\mathbb{Z} \oplus \mu_l(K) & l \neq p, \\
\mathbb{Z}/p\mathbb{Z} \oplus O_K/pO_K \oplus \mu_p(K) & l = p.
\end{cases}
\]

下面分两种情形讨论。

当 \( l = p \)时，

\[
\chi(\mathbb{F}_p) = \dim_{\mathbb{F}_p} H^0(G, \mathbb{F}_p) - \dim_{\mathbb{F}_p} H^1(G, \mathbb{F}_p) + \dim_{\mathbb{F}_p} H^2(G, \mathbb{F}_p)
= 1 - \dim_{\mathbb{F}_p}(\mathbb{F}_p \oplus O_K/pO_K \oplus \mu_p(K)) + \dim_{\mathbb{F}_p} \mu_p(K)
= -\dim_{\mathbb{F}_p} O_K/pO_K = -[K: \mathbb{Q}_p] \dim_{\mathbb{F}_p} \mathbb{F}_p = \chi'(\mathbb{F}_p).
\]

当 \( l \neq p \)时，

\[
\chi(\mathbb{F}_l) = 1 - \dim_{\mathbb{F}_l}(\mathbb{F}_l \oplus \mu_l(K)) - \dim_{\mathbb{F}_l} \mu_l(K) = 0 = \chi'(\mathbb{F}_l).
\]

由Tate duality知, 这个定理同样对 \( M = \mu_l = \mathbb{F}_l^*(1) \)也成立, 特殊情况验证完毕。下面我们来考虑一般的情形。

首先, 我们证明如果 \( W_0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0 \)是一个有限\( \mathbb{Z}_l[G] \)-模的正合序列, 则有 \( \chi'(M) = \chi'(N) + \chi'(L) \), \( \chi(M) = \chi(N) + \chi(L) \), \( \chi'(M) = \chi'(M^{ss}) \)和 \( \chi(M) = \chi(M^{ss}) \)成立。其中 \( M^{ss} = \bigoplus_{q=1}^n M_q/M_{q-1} \)是一个 \( \mathbb{Z}_l[G] \)-模, \( 0 = M_0 \subset M_1 \subset \cdots M_n = M \)是一个Jordan-Holder序列, \( M_q \)均为 \( \mathbb{Z}_l[G] \)-模。
事实上，\( \chi'(M) = -[K : \mathbb{Q}_p] \text{length}_{\mathbb{Z}_p} M = -[K : \mathbb{Q}_p] (\text{length}_{\mathbb{Z}_p} L + \text{length}_{\mathbb{Z}_p} N) = \chi'(N) + \chi'(L) \)。由性质1.2我们知道0 → \( H^0(L) \) → \( H^0(M) \) → \( H^0(N) \) → \( H^1(L) \) → \( H^1(M) \) → \( H^1(N) \) → \( H^2(L) \) → \( H^2(M) \) → \( H^2(N) \) → 0是一个正合列。则我们有:

\[
\chi(M) = \sum_{q=0}^{2} (-1)^q \text{length} H^q(M)
= \sum_{q=0}^{2} (-1)^q (\text{length} H^q(L) + \text{length} H^q(N)) \\
= \sum_{q=0}^{2} (-1)^q \text{length} H^q(L) + \sum_{q=0}^{2} (-1)^q \text{length} H^q(N) \\
= \chi(L) + \chi(N).
\]

以及

\[
\chi'(M^{ss}) = \chi' \left( \bigoplus_{q=1}^{n} M_q / M_{q-1} \right) \\
= \sum_{q=1}^{n} \chi'(M_q / M_{q-1}) \\
= \sum_{q=1}^{n} (\chi'(M_q) - \chi'(M_{q-1})) \\
= \chi'(M_n) = \chi'(M).
\]

对于\( \chi(M) = \chi(M^{ss}) \)同理如上。

由于\( M_q / M_{q-1} \)是一个\( \mathbb{F}_l[G] \)-模，所以\( M^{ss} \)也是一个\( \mathbb{F}_l[G] \)-模。因此不失一般性，我们可以直接假设\( M \)本身是一个\( \mathbb{F}_l[G] \)-模。此时\( \dim_{\mathbb{F}_l} M = \text{length}_{\mathbb{Z}_l} M \)。

现在我们回顾下Grothendieck群的基本概念。令\( G \)是一个profinite群，\( E \)是一个给定的域。我们考虑由如下数据所构成的范畴\( \text{Rep}_E(G) \):

1. 范畴里的所有元素都是有限维\( E \)-线性空间并且在离散拓扑下有连续的\( G \)作用；
2. 所有的态射都是\( E[G] \)-线性映射。

\( \text{Rep}_E(G) \)的Grothendieck群\( R_E(G) \)是一个阿贝尔群，它由如下生成元与关系所确定：\( R_E(G) \)是由形如符号\([M]\)的元素所生成，其中\( M \in \text{Rep}_E(G) \)为一个类。这唯一的关系\([M] = [N] + [L] \)由短正合列\( 0 \to L \to M \to N \to 0 \)所确定。

现在我们回到原证明中。我们考虑由所有的有限\( \mathbb{F}_l[G] \)-模所构成的模范畴\( \text{Rep}_{\mathbb{F}_l}(G) \)，它的Grothendieck群是\( R_{\mathbb{F}_l}(G) \)。事实上，我们可以把\( \chi \)和\( \chi' \)看作是
从Grothendieck群$R_{\xi}(G)$到$\mathbb{Z}$的两个函数。我们只需对$R_{\xi}(G)$的生成元验证即可。由于$\mathbb{Z}$是自由扭模，所以我们只需验证$R_{\xi}(G) \otimes \mathbb{Q}$的生成元即可。而$R_{\xi}(G) \otimes \mathbb{Q}$的生成元我们可由如下性质得到:

**命题2.3**（参考[2, lemma 2.10])令$G$是一个有限群，对$G$的任意一个子群$H$，令$\text{Ind}_H^G$是$R_{\xi}(H) \otimes \mathbb{Q} \rightarrow R_{\xi}(G) \otimes \mathbb{Q}$的一个同态，把$H$-模的类映射到对应的$G$-模的类。则$R_{\xi}(G) \otimes \mathbb{Q}$是由$\text{Ind}_H^G$的像所生成，其中$H$跑遍order与$\rho$互素的$G$的所有循环群。

我们选取域$K$的一个有限扩张$F$，使得$\text{Gal}(\mathbb{Q}_p/F)$在$M$上作用平凡。为方便起见, 令$\overline{G} = \text{Gal}(F/K)$。因此，对定理2.1我们只需要验证$R_{\xi}(\overline{G})$的生成元即可。由性质2.3，我们知道$R_{\xi}(\overline{G}) \otimes \mathbb{Q}$是由$\text{Ind}_{\overline{H}}^{\overline{G}}$所生成，其中循环子群的order与$\rho$互素，$\rho : \overline{H} \rightarrow \mathbb{K}^\times$是一个特征，其中$\mathbb{K}$是$\mathbb{F}_l$的一个有限扩张。因此我们可以假设$M = \text{Ind}_{\overline{H}}^{\overline{G}} = \text{Ind}_{\overline{H}}^{\overline{G}} \rho$，其中$H = \text{Gal}(\mathbb{Q}_p/F \overline{H})$。由Shapiro’s引理知$H^q(G, \text{Ind}_{\overline{H}}^{\overline{G}}) \cong H^q(G, \text{Ind}_{\overline{H}}^{\overline{G}} \rho) \cong H^q(\overline{H}, \rho)$，因此$\chi(G, \text{Ind}_{\overline{H}}^{\overline{G}}(\rho) = \chi(\overline{H}, \rho)$。所以我们只需验证$\rho$即可。（换句话，只需验证一维线性空间$V(\rho)$，其中$\overline{H}$是通过$\rho$在其上作用）

![Diagram](attachment:image.png)

因此我们假设$F^\overline{H} = K$，则$\overline{H} = \overline{G}$，$M = V(\rho)$在$K$是一维的以及$\overline{G} = \overline{H}$是一个循环群，且$([\overline{G}], l) = 1$。由性质1.1(2)对所有的$q \geq 1$，我们有$H^q(\overline{G}, M) = 0$ 对所有 $q \geq 1$。因此，我们有如下inflation和restriction序列:

$H^q(G/G', M^{G'}) \longrightarrow H^q(G, M) \longrightarrow H^0(G/G', H^q(G', M)) \longrightarrow H^{q+1}(G/G', M^{G'})$

$0 = H^q(\overline{G}, M) \longrightarrow H^q(\overline{G}, M) \longrightarrow H^0(\overline{G}, H^q(G', M)) \longrightarrow H^{q+1}(\overline{G}, M) = 0$,

其中$G' = \text{Gal}(\mathbb{Q}_p/F)$. 从而对$q = 0, 1, 2$，我们可以得到$H^q(G, M) \cong H^0(\overline{G}, H^q(G'/M))$。

因此

$$H^q(G', M) = H^q(G', \mathbb{K}) = H^q(G', \mathbb{F}_l) \otimes_{\mathbb{F}_l} \mathbb{K} = \begin{cases} \mathbb{K} & q = 0; \\ ((F^x/(F^x)')^\ast \otimes_{\mathbb{F}_l} \mathbb{K} & q = 1; \\ \mu^1_t(F) \otimes_{\mathbb{F}_l} \mathbb{K} & q = 2. \end{cases}$$

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则

$$\chi(G, M) = \dim_{\mathbb{F}_l} \mathcal{K}_G^G - \dim_{\mathbb{F}_l} (((F^\times)/(F^\times)^l)^* \otimes_{\mathbb{F}_l} \mathcal{K})^G + \dim_{\mathbb{F}_l} (\mu^*(F) \otimes_{\mathbb{F}_l} \mathcal{K})^G.$$ 

由于我们已经验证了 $M = \mathbb{F}_l$ 和 $M = \mu$ 这两种情形，因此我们可以假设 $\rho$ 既不是平凡作用也不是分圆特征作用，从而可以得到 $\mathcal{K}_G^G = (\mu^*(F) \otimes_{\mathbb{F}_l} \mathcal{K})^G = 0$。这是因为 Galois 群在 $\mathcal{K}$ 上的作用是由平凡特征所决定以及 $\rho$ 在 $\mu_p$ 上是由分圆特征作用所决定。因此，$\chi(G, M) = -\dim_{\mathbb{F}_l} (((F^\times)/(F^\times)^l)^* \otimes_{\mathbb{F}_l} \mathcal{K})^G$。

如同前面考虑特殊情况一样，我们还是分两种情形讨论。

当 $l = p$ 时，我们只需证明如下等式:

$$\chi(G, M) = -\dim_{\mathbb{F}_l} (((F^\times)/(F^\times)^l)^* \otimes_{\mathbb{F}_l} \mathcal{K})^G = -[K : \mathbb{Q}_p] \dim_{\mathbb{F}_p} M = \chi'(G, M).$$

因为 $\mu$ 是 $F^\times$ 的极大扭子群，则我们有:

$$
\begin{array}{cccc}
1 & \rightarrow & \mu & \rightarrow & F^\times & \rightarrow & F^\times/\mu & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & \mu & \rightarrow & F^\times & \rightarrow & F^\times/\mu & \rightarrow & 1,
\end{array}
$$

其中 $p : x \mapsto x^p$。由蛇形引理我们知道:

$$1 \rightarrow \mu/\mu^p \rightarrow (F^\times)/(F^\times)^p \rightarrow (F^\times/\mu)/(F^\times/\mu)^p \rightarrow 1.$$

下面我们如下定义 “*” 函子：“*” $= \text{Hom}(-, \mathbb{Q}/\mathbb{Z}) = \text{Hom}(-, \mathbb{F}_p)$ (第二个等号仅在这种情形下成立)，我们知道这是个反变左正合函子，因此

$$1 \rightarrow ((F^\times/\mu)/(F^\times/\mu)^p)^* \rightarrow ((F^\times)/(F^\times)^p)^* \rightarrow (\mu/\mu^p)^* \rightarrow \text{Ext}^1((F^\times/\mu)/(F^\times/\mu)^p, \mathbb{F}_p) = 1.$$

我们知道 $\mathcal{K}$ 是平坦的，则

$$1 \rightarrow ((F^\times/\mu)/(F^\times/\mu)^p)^* \otimes_{\mathbb{F}_p} \mathcal{K} \rightarrow ((F^\times)/(F^\times)^p)^* \otimes_{\mathbb{F}_p} \mathcal{K} \rightarrow (\mu/\mu^p)^* \otimes_{\mathbb{F}_p} \mathcal{K} \rightarrow 1.$$

因此

$$1 \rightarrow H^0(\overline{G}, ((F^\times/\mu)/(F^\times/\mu)^p)^* \otimes_{\mathbb{F}_p} \mathcal{K}) \rightarrow H^0(\overline{G}, ((F^\times)/(F^\times)^p)^* \otimes_{\mathbb{F}_p} \mathcal{K}) \rightarrow H^0(\overline{G}, (\mu/\mu^p)^* \otimes_{\mathbb{F}_p} \mathcal{K}).$$

而 $\mu/\mu^p \cong \mu_p(F)$, 则

$$H^0(\overline{G}, (\mu/\mu^p)^* \otimes_{\mathbb{F}_p} \mathcal{K}) = (\mu_p^*(F) \otimes_{\mathbb{F}_p} \mathcal{K})^G = 0.$$

因此

$$H^0(\overline{G}, ((F^\times/\mu)/(F^\times/\mu)^p)^* \otimes_{\mathbb{F}_p} \mathcal{K}) \cong H^0(\overline{G}, ((F^\times)/(F^\times)^p)^* \otimes_{\mathbb{F}_p} \mathcal{K}).$$
换句话说，我们有
\[ \dim_{F_p}(((F^\times)/(F^\times)^p)\otimes_{F_p} K)^G = \dim_{F_p}(((F^\times/\mu)/(F^\times/\mu)^p)\otimes_{F_p} K)^G = \dim_{F_p}(((F^\times/\mu)\otimes_{Z} F_p)^* \otimes_{F_p} K)^G. \]

我们把\( F \)加法赋值用\( v : F^\times \to Z \)来表示，则我们有正合列:

\[ 1 \to O_F^\times/\mu \to F^\times/\mu \to Z \to 0. \]

由于此正合列的项均为自由扭模，张量乘上\( \dim \)有同构，我们可以考虑

\[ \text{Rank}_{Z_p}(\text{Hom}(O_F^\times/\mu \otimes_{Z_p} Z_p, Z_p) \otimes_{Z_p} OL)^G \]

\[ = \dim_{Q_p}(\text{Hom}(O_F^\times/\mu \otimes_{Q} Q_p) \otimes_{Q_p} L)^G. \]

令域\( G \)做导出可约的，则的

所以，因此我们有:

\[ 0 \to \text{Ext}^1(F_p, F_p) = \text{dim} \to \]

\[ 0 \to K^G \to ((F^\times/\mu \otimes_{Z} F_p)^* \otimes_{F_p} K)^G \to ((O_F^\times/\mu \otimes_{Z} F_p)^* \otimes_{F_p} K)^G \to H^1(G, K) = 0. \]

所以，

\[ \dim_{F_p}(((F^\times/\mu \otimes_{Z} F_p)^* \otimes_{F_p} K)^G) = \dim_{F_p}((O_F^\times/\mu \otimes_{Z} F_p)^* \otimes_{F_p} K)^G. \]

现在我们利用下述性质将表示\( \rho \)提升到特征为0的域上去考虑，此时新的表示用\( \tilde{\rho} \)来记。

命理2.4（参考[1, 推论2.7]）令域\( K \)是\( \overline{Q}_p \)的有限扩张，其整数环记为\( O \)。对于\( O \)的极大理想\( m \)，令\( E = O/mO \)。假设\( p \)不整除\( |G| \)以及所有\( G \)的所有不可约\( K \)-表示均为绝对不可约的，则\( G \)的所有不可约\( E \)-表示均为绝对不可约的。同时约化映射\( \rho \to (\rho \mod mO) \)诱导出\( G \)的不可约\( K \)-表示等价类与\( G \)的不可约\( E \)-表示等价类之间的双射，且保维数。

所以我们将选取\( Q_p \)的唯一非分歧扩张\( L \)，其维数是\( \dim_{F_p} K \)。由\( p \)-adic分析的性质，我们有\( O_L/(p) \cong K \)，\( O_L^\times \cong (1 + pO_L) \times K^\times \)，其中\( O_L \)是\( L \)的\( p \)-adic整数环。通过上述性质里的同构，我们可以考虑\( \rho \)在\( O_L^\times \)里取值。我们把这个新特征（或者一维表示）\( \tilde{\rho} : \overline{G} \to O_L^\times \)叫做\( \rho \)的Teichmuller提升。因为\( O_F^\times/\mu \)是自由扭模以及\( (\overline{G}, \rho) = 1 \)，通过性质2.4，对\( \rho \)的唯一的Teichmuller提升，我们有:

\[ \dim_{F_p}((O_F^\times/\mu \otimes_{Z} F_p)^* \otimes_{F_p} K)^G = \dim_{F_p}((O_F^\times/\mu \otimes_{Z} F_p)^* \otimes_{F_p} K)^G \]

\[ = \dim_{F_p}(\text{Hom}(O_F^\times/\mu \otimes_{Z} F_p, F_p) \otimes_{F_p} K)^G \]

\[ = \text{Rank}_{Z_p}(\text{Hom}(O_F^\times/\mu \otimes_{Z_p} Z_p, Z_p) \otimes_{Z_p} OL)^G \]

\[ = \dim_{Q_p}(\text{Hom}(O_F^\times/\mu \otimes_{Q} Q_p) \otimes_{Q_p} L)^G. \]

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由$p$-adic理论，作为$G$-模，我们有$O_F^\times/\mu \otimes Z Q \cong F$。因此

$$\dim_{Q_p}(\text{Hom}((O_F^\times/\mu) \otimes Z Q, Q_p) \otimes_{Q_p} L)^G = \dim_{Q_p}(\text{Hom}(F, Q_p) \otimes_{Q_p} L)^G.$$ 

通过正规基定理，$F \cong K[G] \cong Q_p[G]$. 则

$$\dim_{Q_p}(\text{Hom}(F, Q_p) \otimes_{Q_p} L)^G = \dim_{Q_p}(\text{Hom}(Q_p[G], Q_p) \otimes_{Q_p} L)^G = [K : Q_p]\dim_{Q_p}(\text{Hom}(Q_p[G], Q_p) \otimes_{Q_p} L)^G.$$ 

作为$G$-模，我们有$\text{Hom}(Q_p[G], Q_p) \cong Q_p[G]$，通过$\psi: f \mapsto \sum_{\sigma \in G} a_{\sigma} \sigma$，给出。其中$a_{\sigma} = f(\sigma)$，则

$$[K : Q_p]\dim_{Q_p}(\text{Hom}(Q_p[G], Q_p) \otimes_{Q_p} L)^G = [K : Q_p]\dim_{Q_p}(Q_p \otimes_{Q_p} L)^G$$

$$= [K : Q_p]\dim_{Q_p}(Q_p[G] \otimes_{Q_p} L)^G = [K : Q_p]\dim_{Q_p}(Q_p[G] \otimes_{Q_p} L^0) \ldots \hspace{1cm} (*)$$

$$= [K : Q_p]\dim_{Q_p}(Q_p[G] \otimes_{Q_p} L) = [K : Q_p]\dim_{Q_p}L^0 = [K : Q_p]\dim_{Q_p}L$$

$$= [K : Q_p]\dim_{Q_p}M,$$

其中$Q_p[G] \cong Q_p$以及(*)来自如下同构：

$$Q_p[G] \otimes_{Q_p} L \cong Q_p[G] \otimes_{Q_p} L^0$$

由

$$\sigma \otimes m \mapsto \sigma \otimes \sigma^{-1} m,$$

给出，其中$L^0$是平凡$G$-模且作为$Q_p$-线性空间有$L \cong L^0$。

当$l \neq p$时，我们只需检验$\chi(G, M) = -\dim((F^\times/(F^\times)^*)\otimes_{F_l} K)^G = 0$。同理如上进行讨论后，我们知道$((F^\times/(F^\times)^*)\otimes_{F_l} K)^G \cong ((F^\times/\mu \otimes Z F_l)^* \otimes_{F_l} K)^G$. 但$F^\times/\mu$是$Z_p$-模，而$l$在$Z_p$中不可逆。因此$F^\times/\mu \otimes Z F_l = 0$即$\chi(G, M) = 0$。

注释2.5 事实上，$\dim_{F_p}M^G = \dim_{F_p}(M^*)^G$，因为$G$是一个有限循环群，令$\sigma$是$G$的
生成元，则我们有
\[ \dim_{F_p}(M^*)^G = \dim_{F_p}(\text{Hom}_{F_p}(M, F_p))^G \]
\[ = \dim_{\bar{G}} \text{Hom}_{F_p}(M, F_p) \]
\[ = \dim_{\bar{G}} \text{Hom}_{F_p}(M/(\sigma - 1)M, F_p) \]
\[ = \dim_{F_p} M/(\sigma - 1)M = \dim_{\bar{G}} M^G \]

其中 \( M \) 是 \( F_p[\bar{G}] \)-模。因此，通过这个结论有:
\[ \dim_{F_p}((F^\times/(F^\times)^\sigma \otimes_{F_p} \mathcal{K}))^G = \dim_{F_p}(((F^\times/(F^\times)^\sigma \otimes_{F_p} \mathcal{K})^*)^G \]
\[ = \dim_{F_p} ((F^\times/(F^\times)^\sigma) \otimes_{F_p} \mathcal{K}^*)^G. \]

所以对于证明定理2.1，我们只需验证
\[ \dim_{F_p}((F^\times/(F^\times)^\sigma) \otimes_{F_p} \mathcal{K}^*)^G = [K : \mathbb{Q}_p] \dim_{F_p} M. \]

Haruzo Hida在其证明中，一开始就将“*”函子漏掉（他说是typo错误）去计算。虽说“*”函子最后是不起作用的，但这是完全不可取的，因为没有任何理由能够说明“*”函子可以漏掉，且证明过程中“*”函子确实带来了相当多的障碍。鉴于此，在印老师的指导下以及李彬博士的帮助下，我对其证明做了大量的修正性工作。我们从上面的证明过程中可以看出，这些修正性的工作并不是非常容易。

### 3 整体欧拉特征公式

**定理3.1** (整体情形) 给定一个有限模 \( M \)，令 \( K \) 是 \( \mathbb{Q} \) 的一个有限扩张，\( S \) 是 \( \mathbb{Q} \) 的有限素点集满足包含阿基米德素点以及 \( |M| \) 的所有素因子。\( \Sigma \) 是在 \( S \) 之上 \( K \) 的素点集合，在 \( \Sigma \) 之外都是非分歧的，我们令 \( \mathcal{G}_S = \text{Gal}(K^*/K) \)。若 \( M \) 是一个有限 \( \mathcal{G}_S \)-模，则有:
\[
\frac{|H^2(\mathcal{G}_S, M)| \cdot |H^0(\mathcal{G}_S, M)|}{|H^1(\mathcal{G}_S, M)|} = \prod_{\nu \in \Sigma_{\infty}} \frac{|H^0(G_\nu, M)|}{|M||K_\nu|},
\]
其中 \( |n|_{K_\nu} = n^{[K_\nu : \mathbb{R}]} \)，\( \nu \) 是阿基米德素点，\( G_\nu = \text{Gal}(K_\nu/K_\nu) \)。

在这个定理里，我们知道 \( M \) 是 \( \mathcal{G}_S \)-模，由于 \( G_\nu \) 是 \( \mathcal{G}_S \) 的分解群，我们也可以把 \( M \) 看作 \( G_\nu \)-模。在证明这个定理前，我们将介绍一个非常重要的定理，这个定理想由John Tate和Georges Poitou证明，它是解决整个欧拉特征公式的关键。但我们在这里只对她进行叙述而不加证明，其证明细节在参考文献[1], [2], [4], [8]中均能找到。
命题3.2 令$K$是$\mathbb{Q}$的一个有限扩张，$S$是$\mathbb{Q}$的有限素点集且包含阿基米德素点，$\Sigma$是$S$之上$K$的素点集。固定$S$中的一个素点$p$，若$M$是一个有限$S$-模，且其order为$p$的幂。

则有以下性质:

1. $$H^r(\mathcal{G}_S, M) \cong \prod_{v \in \Sigma(\mathbb{R})} H^r(G_v, M) \quad \text{对所有} \quad r \geq 3,$$
其中$\Sigma(\mathbb{R})$是$K$的阿基米德实素点集；

2. 我们有如下长正合列:

$$0 \to H^0(\mathcal{G}_S, M) \to \prod_{v \in \Sigma_0} H^0(G_v, M) \times \prod_{v \in \Sigma_\infty} H^0(G_v, M) \to H^2(\mathcal{G}_S, M^*(1))^*$$

$$\to H^1(\mathcal{G}_S, M) \to \prod_{v \in \Sigma} H^1(G_v, M) \to H^1(\mathcal{G}_S, M^*(1))^*$$

$$\to H^2(\mathcal{G}_S, M) \to \prod_{v \in \Sigma} H^2(G_v, M) \to H^0(\mathcal{G}_S, M^*(1))^* \to 0.$$

3.1 整体情形的证明

因为$M = \bigoplus_{l} M[l^\infty]$，同样我们可以假设$M = M[l^\infty]$。现在我们对$l > 2$进行证明。令

$$\varphi(M) = \chi(\mathcal{G}_S, M) - \sum_{v \in \Sigma_\infty} (\text{length}_{\mathbb{Z}_l} H^0(G_v, M) - [K_v : \mathbb{R}] \text{length}_{\mathbb{Z}_l} M)$$

则我们只需证明$\varphi(M) = 0$。

由性质3.2(1)，对所有的$q \geq 3$，我们有$H^q(\mathcal{G}_S, M) = \prod_{v \in \Sigma(\mathbb{R})} H^q(G_v, M)$。而$|G_v| = 2$以及$l > 3$，我们有$\gcd(|G_v|, M) = 1$。通过性质1.1(2)，对所有的$q \geq 1$，我们有$H^q(G_v, M) = 0$，因此对所有的$q \geq 3$，有$H^q(\mathcal{G}_S, M) = 0$。

当$\mathbb{Z}_l[\mathcal{G}_S]$-模序列$0 \to L \to M \to N \to 0$是短正合列时，我们有下列有限$\mathbb{Z}_l$-模长正合列:

$$0 \to H^0(\mathcal{G}_S, L) \to H^0(\mathcal{G}_S, M) \to H^0(\mathcal{G}_S, N) \to \cdots \to H^2(\mathcal{G}_S, N) \to 0$$

所以如同局部欧拉特征，我们同样有加性等式$\chi(M) = \chi(L) + \chi(N)$和$\varphi(M) = \varphi(L) + \varphi(N)$。其中$\chi$和$\varphi$同样可看作是从Grothendieck群$R_{\mathcal{G}_S}(G)$到$\mathbb{Z}$的两个函数。
然后由性质3.2(2)，我们有 $\chi(M) + \chi(M^*(1)) = \sum_{v \in \Sigma} \chi_v(G_v, M)$. 其中

$$
\chi_v(G_v, M) = \begin{cases} 
  \sum_{q=0}^{2} (-1)^q \text{length}_{\mathbb{Z}_q} H^q(G_v, M), & v \text{ 是有限素点}; \\
  \sum_{q=0}^{2} (-1)^q \text{length}_{\mathbb{Z}_q} H^0_q(G_v, M), & v \text{ 是阿基米德素点}.
\end{cases}
$$

则

$$
\sum_{v \in \Sigma} \chi_v(G_v, M) = \sum_{v \in \Sigma_\infty} \chi_v(G_v, M) + \sum_{v \in \Sigma_0} \chi_v(G_v, M),
$$

其中 $\Sigma_0$ 是有限素点集。我们知道

$$
\sum_{v \in \Sigma_\infty} \chi_v(G_v, M) = \sum_{v \in \Sigma_\infty} \sum_{q=0}^{2} (-1)^q \text{length}_{\mathbb{Z}_q} H^0_q(G_v, M).
$$

因为 $G_v$ 是一个循环群，由性质1.6知

$$
\text{length}_{\mathbb{Z}_q} H^2_q(G_v, M) = \text{length}_{\mathbb{Z}_q} H^1_q(G_v, M)
$$

$$
\text{length}_{\mathbb{Z}_q} H^0_q(G_v, M) = \text{length}_{\mathbb{Z}_q} H^1_q(G_v, M) = \text{length}_{\mathbb{Z}_q} H^1(G_v, M).
$$

所以

$$
\sum_{v \in \Sigma_\infty} \chi_v(G_v, M) = \sum_{v \in \Sigma_\infty} \text{length}_{\mathbb{Z}_q} H^1(G_v, M).
$$

另一方面，因为对 $v \not\in \Sigma$，我们有 $||M||_K_v = 1$. 通过乘积公式 $\prod_v ||M||_K_v = 1$ 以及前面证明的局部欧拉特征公式，我们有

$$
\sum_{v \in \Sigma_0} \chi_v(G_v, M) = \log \prod_{v \in \Sigma_0} (||M||_K_v) = -\log \prod_{v \in \Sigma_\infty} (||M||_K_v) = -\sum_{v \in \Sigma_\infty} [K_v : \mathbb{R}] \text{length}_{\mathbb{Z}_q} M.
$$

所以，

$$
\varphi(M) + \varphi(M^*(1)) = \chi(M) + \chi(M^*(1))
$$

$$
\sum_{v \in \Sigma_\infty} (\text{length}_{\mathbb{Z}_q} H^0(G_v, M) + \text{length}_{\mathbb{Z}_q} H^0(G_v, M^*(1)) - 2[K_v : \mathbb{R}] \text{length}_{\mathbb{Z}_q} M)
$$

$$
= \sum_{v \in \Sigma_\infty} (\text{length}_{\mathbb{Z}_q} H^1(G_v, M) - \text{length}_{\mathbb{Z}_q} H^0(G_v, M) - \text{length}_{\mathbb{Z}_q} H^0(G_v, M^*(1)) + [K_v : \mathbb{R}] \text{length}_{\mathbb{Z}_q} M)
$$

$$
= 0.
$$

最后一个等式源于下面性质:
命题3.3 (参考[2, 定理2.3(c)]) 令 $K_v = \mathbb{R}$ 或 $\mathbb{C}$, $G_v = \text{Gal}(\overline{K}/K_v)$, $|n|_{K_v} = n[\overline{K}/\mathbb{R}]$。对任意有限 $G_v$-模 $M$, 我们有:

$$\frac{|H^0(G_v, M)| \cdot |H^0(G_v, M^*(1))|}{|H^1(G_v, M)|} = ||M||_{K_v}.$$ 

因此我们只需证明 $\varphi(M) = \varphi(M^*(1))$ 即可。

3.2 对 $\varphi(M) = \varphi(M^*(1))$ 修改证明

由于J.S.Milne的证明有误，下面的证明是在印林生教授的指导与李彬博士的帮助下独立用己的方法对其进行详细的证明。如同局部欧拉特征公式，我们选取 $K$ 的一个有限扩张 $F$, 使得 $G' = \text{Gal}(K'/F)$ 在 $M$ 和 $\mu_l$ 上作用是平凡的。为方便起见，我们令 $G = \text{Gal}(F/K)$。同样用讨论局部的方法，我们可以假设 $G$ 是一个循环群，它的degree与 $l$ 互素；且 $M$ 是一个 $F[l]$-模。则对所有的 $q \geq 3$, 我们有 $H^q(G', M)=0$。因为 $\gcd(|G|, l)=1$, 则对所有的 $q \geq 0$, $H^q(G, M)=0$。同理, 再次运用inflation和restriction序列, 我们有 $H^q(\mathcal{S}, M) \cong H^0(\mathcal{G}, H^q(\mathcal{S}, M))$。

由于我们把 $\varphi$ 看作是从Grothendieck群 $R_{F_1}(G)$ 到 $\mathbb{Z}$ 的函数（或同态）。令

$$\chi': R_{F_1}(\mathcal{G}) \otimes \mathbb{Q} \to R_{F_1}(\mathcal{G}) \otimes \mathbb{Q},$$

$$[M] \mapsto \sum_{i=0}^{2} (-1)^i[H^i(\mathcal{S}', M)]$$

及

$$\theta : R_{F_1}(\mathcal{G}) \otimes \mathbb{Q} \to \mathbb{Q},$$

$$[M] \mapsto \dim_{F_1} M^{\mathcal{G}},$$

因此我们有

$$\chi = \theta \circ \chi'.$$

我们知道 $[H^0(\mathcal{S}', \mu_l)] = [\mu_l]$, 以及

(i) $[H^1(\mathcal{S}', \mu_l)] = [O_{F,S}/l] + [Cl_S(F)[l]]$,

(ii) $[H^2(\mathcal{S}', \mu_l)] = [Cl_S(F)/l] - [F_l] + \bigoplus_{p \in S \setminus S_{\infty}(F)} F_l + \bigoplus_{p \in S_{\infty}(F)} H^0_G(G_p, F_l)$,

其中 $O_{F,S}$ 是 $S$-单位群， $Cl_S(F)[l]$ 是 $S$-理想类群 $Cl_S(F)$ 的 $l$-扭部分。其证明细节可以参考[8,(8.7.4)]。
注意$O_{F,S}/l = \bigoplus_{p \in S(F)} \mathbb{F}_l + [\mu_l] - [\mathbb{F}_l]$及$[\text{Cl}_S(F)/l] = [\text{Cl}_S(F)/l]$，则我们有

$$\chi^\prime([\mu_l]) = \bigoplus_{p \in S_\infty(F)} H^0_T(G_p, \mathbb{F}_l) - \bigoplus_{p \in S_\infty(F)} \mathbb{F}_l.$$ 

对于有限$\mathbb{F}_l[G]$-模$M$，我们有：

(iii) $\chi^\prime(M^\ast(1)) = [M^\ast] \cdot \chi^\prime([\mu_l])$. 参考[2, 引理5.4],

(iv) $[M] \cdot [\mathbb{F}_l[G]] = \dim_{\mathbb{F}_l} M \cdot [\mathbb{F}_l[G]]$

因此，

$$\chi^\prime([M^\ast(1)]) = [M^\ast] \otimes \big( \bigoplus_{p \in S_\infty(F)} (H^0_T(G_p, \mathbb{F}_l) - \mathbb{F}_l) \big)$$

$$= [M^\ast] \otimes \big( \bigoplus_{v \in \Sigma_\infty} \big( \bigoplus_{p|v} (H^0_T(G_p, \mathbb{F}_l) - \mathbb{F}_l) \big) \big).$$

同理，我们有

$$\chi^\prime([M]) = [M(-1)] \otimes \big( \bigoplus_{p \in S_\infty(F)} (H^0_T(G_p, \mathbb{F}_l) - \mathbb{F}_l) \big)$$

$$= [M(-1)] \otimes \big( \bigoplus_{v \in \Sigma_\infty} \big( \bigoplus_{p|v} (H^0_T(G_p, \mathbb{F}_l) - \mathbb{F}_l) \big) \big).$$

其中$M(-1) = (M^\ast(1))^\ast$。因此，我们要证明$\varphi(M) = \varphi(M^\ast(1))$其实就是要证明：

$$-\theta([M^\ast] \otimes \big( \bigoplus_{v \in \Sigma_\infty(R_p \mid v)} (H^0_T(G_p, \mathbb{F}_l) - \mathbb{F}_l) \big) - \sum_{v \in \Sigma_\infty} (\dim_{\mathbb{F}_l} H^0(G_v, M^\ast(1))) - [K_v : \mathbb{R}] \dim_{\mathbb{F}_l} M^\ast(1))$$

$$= -\theta([M(-1)] \otimes \big( \bigoplus_{v \in \Sigma_\infty(R_p \mid v)} (H^0_T(G_p, \mathbb{F}_l) - \mathbb{F}_l) \big) - \sum_{v \in \Sigma_\infty} (\dim_{\mathbb{F}_l} H^0(G_v, M) - [K_v : \mathbb{R}] \dim_{\mathbb{F}_l} M),$$

经整理后，只需证明：

$$\sum_{v \in \Sigma_\infty(R_p \mid v)} (\theta([M^\ast] \otimes (\bigoplus_{p|v} (H^0_T(G_p, \mathbb{F}_l) - \mathbb{F}_l)) + \dim_{\mathbb{F}_l} M^\ast(1)_{G_v})$$

$$= \sum_{v \in \Sigma_\infty(R_p \mid v)} (\theta([M(-1)] \otimes (\bigoplus_{p|v} (H^0_T(G_p, \mathbb{F}_l) - \mathbb{F}_l)) + \dim_{\mathbb{F}_l} M_{G_v}).$$

下面我们将上述素点$v$进行分类讨论。

当$v = \mathbb{C}$，$p = \mathbb{C}$时，有$G$-模同构：

$$\bigoplus_{p|v} \mathbb{F}_l \cong \mathbb{F}_l[G]$$
以及

\[ H^0_T(G_p, F_l) = 0. \]

这时我们可以很容易验证上述左右两边相等，与证明局部情形时完全相同；

当\( v = \mathbb{R}, p = \mathbb{R} \)时，有\( G \)-模同构:

\[ \bigoplus_{p \mid v} F_l \cong F_l[G]. \]

以及

\[ \bigoplus_{p \mid v} H^0_T(G_p, F_l) \cong H^0_T(G_p, F_l) \otimes_{F_l} F_l[G]. \]

此刻处理方法同上。当\( v = \mathbb{R}, p = \mathbb{C} \)时，我们有

\[ H^0_T(G_p, F_l) = 0, \]

这时只需独立的证明如下等式:

\[
\sum_{v \in \Sigma(\mathbb{R})} \left( \theta([M^* \otimes (\bigoplus_{p \mid v, p \text{ is complex}} F_l)]) + \dim_{F_l} M^*(1)^{G_v} \right) = \sum_{v \in \Sigma(\mathbb{R})} \left( \theta([M(-1) \otimes (\bigoplus_{p \mid v, p \text{ is complex}} F_l)]) + \dim_{F_l} M^{G_v} \right).
\]

下面我们将对剩下的每个\( v \)进行独立地证明。

事实上，

\[
\theta([M^* \otimes (\bigoplus_{p \mid v, p \text{ is complex}} F_l)]) = \dim_{F_l} M^*(1)^{G_v}.
\]

显然\( G_v \)是\( G \)的一个子群，定义\( G' = G/G_v \)，则我们有

\[
\theta([M^* \otimes (\bigoplus_{p \mid v, p \text{ is complex}} F_l)]) = \dim_{F_l} ([M^* \otimes (\bigoplus_{p \mid v, p \text{ is complex}} F_l)]^{G_v})^{G'}
\]

\[
= \dim_{F_l} ([M^*]^{G_v} \otimes F_l)^{G'}
\]

\[
= \dim_{F_l} ([M^*]^{G_v} \otimes F_l[G'])^{G'}
\]

\[
= \dim_{F_l} M^{G_v}.
\]

其中最后两个等式分别由(iv)和注释2.5给出。

我们用同样的方法再次讨论有

\[
\theta([M(-1) \otimes (\bigoplus_{p \mid v, p \text{ is complex}} F_l)]) = \dim_{F_l} M^*(1)^{G_v}.
\]

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所以，我们完全证明了 \( l > 2 \) 的情形。
当 \( l=2 \) 时，我们有 \( M^*(1) = M^* \) 因此我们只需要对 \( \psi \) 的加性证明稍微做点修改。这个修改证明是非常容易的，具体可参考文献[1,Ch4.4.5]。
至此，我们完全证明了整体欧拉特征公式。

### 3.3 对J.S.Milne错误证明的说明及其反例

在参考文献[1,Ch4.4.5]与[2,Ch1.5]中，他们直接去证明 \( \chi(M) = \chi(M^*(1)) \) 与 \( \dim_{F_l} M^{G_v} = \dim_{F_l}(M^*(1))^{G_v} \) 这两个等式。综合上面修正性证明与整体欧拉特征公式结论来看，是根本不可能得到这两个等式的。换句话说这两个等式是完全不等的。事实上，我们无法说清楚等式 \( \chi(M) = \chi(M^*(1)) \) 是来源于 \( \chi' \)，同样我们也没办法说明 \( \dim_{F_l} M^{G_v} = \dim_{F_l}(M^*(1))^{G_v} \)。对后一个等式，我们有如下反例：

令 \( K = \mathbb{Q}(\sqrt{3}) \)， \( L = K(\sqrt{-1}) = \mathbb{Q}(\sqrt{3}, \sqrt{-1}) \supset \mu_3 \)：其中 \( L \) 是 \( K \) 的一个循环扩张，且它的 degree 为 2 与 3 互素。易知域 \( K \) 里面上的素数在域 \( L \) 里是非分歧的（利用分歧性质的判别式法则）。所以令 \( M = F_3 \)，则 \( M^*(1) = \mu_3 \)。但我们可以发现 \( \dim_{F_l} M^{G_v} = 1 \neq 0 = \dim_{F_l}(M^*(1))^{G_v} \)。

**参考文献**


Fully Nonlinear Equation

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Abstract

We give an introduction to fully nonlinear elliptic equations. We focus on the proofs of Krylov-Safonov Estimate, Krylov-Evans Theorem and Caffarelli’s $C^{2,\alpha}$ and $W^{2,p}$ regularity results. We may assume the readers have knowledge of linear elliptic equations’ foundation. Also, we leave the Alexandroff-Bakelman-Pucci Maximum Principle’s proof to readers. We mainly based on Caffarelli and Cabre’s book. And we refer readers to Han and Lin’s book and Evans and Gariepy’s book for basic details.

Keywords: Fully nonlinear equation; viscosity solution; A-B-P maximal principle; Harnack Inequality; $C^{2,\alpha}$; perturbation

1 Introduction

Compared with linear and quasilinear elliptic equations, the fully nonlinear elliptic equations have not been fully developed. We are still unable to apply this kind of equations to Geometric Problems as successfully as linear and quasilinear. However, the fully nonlinear elliptic equations’ development maybe is in the centre of elliptic PDEs. Real Monge-Ampere Equations is the core of Optimal Transport. The $C^{2,\alpha}$ Estimate of Complex Monge-Ampere Equation is the core in Calabi-Yau’s Existence. But we still have not got the $C^{2,\alpha}$ Estimate in non concave situation. We have counterexamples to that. We still hope to find some proper assumption to enlarge the assumption of concave. One of the reasons for this is we have good regularity theory for linear and quasilinear equations, we of course want to get that for fully nonlinear. Second is if we want to do fully nonlinear equations on manifold, we first need a good theory on domain. In this article we have no attempts to give any progress on this direction. We just follow Caffarelli and Cabre’s book to show the mature results until 80s’about concave situation.

2 Viscosity Solutions of Elliptic Equations

We consider equations of the form

\[ F(D^2u(x), x) = f(x) \]  

(1)
where $x \in \Omega$ and $u$ and $f$ are functions defined in a bounded domain $\Omega$ of $\mathbb{R}^n$. $F(M,x)$ is a real valued function defined on $\mathbb{S} \times \Omega$, where $\mathbb{S}$ is the space of real $n \times n$ symmetric matrices. We assume that $F$ is uniformly elliptic operator, i.e.,

**Definition 2.1.** $F$ is a uniformly elliptic if there are two positive constants $\lambda \leq \Lambda$ (called ellipticity constants) such that for any $M \in \mathbb{S}$ and $x \in \Omega$

$$\lambda\|N\| \leq F(M + N, x) - F(M, x) \leq \Lambda\|N\| \quad \forall N \geq 0$$

where $\|M\| = \sup_{|x|=1}|Mx|$. We recall that $M \in \mathbb{S}$ can be decomposed as $M = M^+ - M^-$ where $M^+, M^- \geq 0$ and $M^+ M^- = 0$. So we have the following

$$F(M + N, x) \leq F(M, x) + \Lambda\|N^+\| - \lambda\|N^-\| \quad \forall M, N \in \mathbb{S} \quad \forall x \in \Omega$$

**Definition 2.2.** A continuous function $u$ in $\Omega$ is a viscosity subsolution(resp. viscosity supersolution) of (1) in $\Omega$, when the following condition holds if $x_0 \in \Omega$, $\phi \in C^2(\Omega)$ and $u - \phi$ has a local maximum at $x_0$ then

$$F(D^2 \phi(x_0), x_0) \geq f(x_0)$$

(resp. if $u - \phi$ has a local minimum at $x_0$ then $F(D^2 u, x) \geq (resp. \leq, =) f(x)$ in the viscosity sense in $\Omega$ whenever $u$ is a subsolution(resp. supersolution, solution) of (1) in $\Omega$

**Definition 2.3.** Let $0 < \lambda \leq \lambda$. For $M \in \mathbb{S}$, we define Pucci’s extremal operators:

$$\mathcal{M}^-(M, \lambda, \Lambda) = \mathcal{M}^-(M) = \lambda \sum_{e_i > 0} e_i + \Lambda \sum_{e_i < 0} e_i$$

$$\mathcal{M}^+(M, \lambda, \Lambda) = \mathcal{M}^+(M) = \Lambda \sum_{e_i > 0} e_i + \lambda \sum_{e_i < 0} e_i$$

where $e_i = e_i(M)$ are the eigenvalues of $M$

**Definition 2.4.** Let $f$ be a continuous function in $\Omega$ and $\lambda \leq \Lambda$ two positive constants. We denote by $\mathcal{S}(\lambda, \Lambda, f)$ the space of continuous functions $u$ in $\Omega$ such that $\mathcal{M}^+(D^2 u, \lambda, \Lambda) \geq f(x)$ in the viscosity sense in $\Omega$

we also define

$$\mathcal{S}(\lambda, \Lambda, f) = \mathcal{S}(\lambda, \Lambda, f) \cap \mathcal{S}(\lambda, \Lambda, -|f|)$$

$$\mathcal{S}^*(\lambda, \Lambda, f) = \mathcal{S}(\lambda, \Lambda, -|f|) \cap \mathcal{S}(\lambda, \Lambda, |f|)$$

### 3 Alexandroff Estimate and Maximum Principle

We start defining the concept of convex envelope of a continuous function. A function $L$ defined in $\mathbb{R}^n$ is said to be affine if

$$L(x) = l_0 + l(x),$$

where $l_0 \in \mathbb{R}$ and $L$ is a linear function.
Theorem 3.1. Let $v$ be a continuous function in an open convex set $A$. The envelope of $v$ in $A$ is defined by

$$
\Gamma(v)(x) = \sup_L \{L(x) : L \leq v \text{ in } A, \text{ } L \text{ is affine}\}
$$

for $x \in A$

We have that $\Gamma(v)$ is a convex function in $A$. The set $\{v = \Gamma(v)\} = \{x \in A : v(x) = \Gamma(v)(x)\}$ is called the (lower) contact set of $v$. The points in the contact set are called contact points.

Theorem 3.2. Let $\Omega$ be a bounded domain of $\mathbb{R}^n$ and $f$ be a continuous and bounded function in $\Omega$. Assume that $u \in S(\lambda, \Lambda, f)$ in $\Omega$, $u$ is continuous in $\overline{\Omega}$ and $u \geq 0$ on $\partial \Omega$.

Then

$$
\sup_{\Omega} u \leq C \text{diam}(\Omega) \|f^+\|_{L^\infty(\Omega \cap \{u = \Gamma_u\})}
$$

Here $C$ is a universal constant, $\text{diam}(\Omega)$ is the diameter of $\Omega$ and $\Gamma_u$ is the convex envelope in $B_{2d}$ of $-u^-$, where $B_d$ is a ball of radius $d = \text{diam}(\Omega)$ such that $\Omega \subset B_d$ and we have extended $u \equiv 0$ outside $\Omega$.

We do not give the details of the proof, since it depends on the deep fact that Lipschitz functions is almost everywhere differentiable and convex function is almost everywhere twice functional differentiable. We just need the results. If anyone have interests in these two theorem, we strongly recommend them to [EG]. The following result is the maximum principle for viscosity solutions.

Corollary 3.3. Assume that $u \in C(\overline{\Omega})$. Then

1. $u \in S(\lambda, \Lambda, 0)$ and $u \leq 0$ on $\partial \Omega$ imply $u \leq 0$ in $\Omega$.

2. $u \in S(\lambda, \Lambda, 0)$ and $u \geq 0$ on $\partial \Omega$ imply $u \geq 0$ in $\Omega$.

4 Krylov-Safonov’s Harnack Inequality

Early in 60’s, De Giorgi, Nash and Moser gave the Harnack Inequality for divergence elliptic equations. Until 80’s, Krylov and Safonov gave the Harnack Inequality for non-divergence elliptic equations.

If the readers are familiar with De Giorgi-Nash-Moser Iteration, they can find the divergence elliptic equations can take test functions and use Energy Method to get the final result. The key point is that the derivatives can be controlled by solution itself and by Sobolev and Poincaré Inequality we can get solution being controlled by derivatives. Then we can bootstraps and iterations.

When we focus on non divergence situation, we lost the important Energy Method. But Krylov and Safonov geniously use the Alexandroff (who is Perelman’s advisor) Maximum Principal to get the estimate.

We have confidence that one day we will do further just as Krylov and Safonov did when people had divergence form but no non-divergence form.

We try to give a completed proof of this theorem.
4.1 Two Tools

Sometimes the solution may not be directly use Alexandroff Maximum Principle. We need to adjust it to the right form. Then what we get is the set of solution which get big values can not be too large in the measure sense. I call it "Pull down test function".

**Definition 4.1.**

\[ Q_l(x_0) = \prod_{i=1}^{n} (x_i^0 - \frac{l}{2}, x_i^0 + \frac{l}{2}) \]

and \( Q_l = Q_l(0) \)

**Lemma 4.2.** Given \( 0 < \lambda \leq \Lambda \) constants, there exists a smooth function \( \varphi \) in \( \mathbb{R}^n \) and universal positive constants \( C \) and \( M > 1 \) such that (recall \( \overline{Q}_1 \subset \overline{Q}_3 \subset B_{2\sqrt{n}} \))

\[ \varphi \geq 0 \quad \text{in} \quad \mathbb{R}^n \setminus B_{2\sqrt{n}} \quad (2) \]

\[ \varphi \leq -2 \quad \text{in} \quad Q_3 \quad (3) \]

\[ M^{-}(D^2\varphi, \lambda, \Lambda) \leq C\xi \quad \text{in} \quad \mathbb{R}^n \quad (4) \]

where \( 0 \leq \xi \leq 1 \) is a continuous function in \( \mathbb{R}^n \) with \( \text{supp} \xi \subset \Omega \) Moreover, \( \varphi \geq -M \) in \( \mathbb{R}^n \)

**Proof.** Consider the universal constant \( \alpha = \max\{1, (n-1)\Lambda/\lambda - 1\} \). We have that \( B_{1/4} \subset B_{1/2} \subset Q_1 \subset Q_3 \subset B_{3\sqrt{n}/2} \subset B_{2\sqrt{n}} \). Define

\[ \varphi(x) = M_1 - M_2|x|^{-\alpha} \quad \text{in} \quad \mathbb{R}^n \setminus B_{1/4}; \]

we choose \( M_1 \) and \( M_2 \) positive such that

\[ \varphi|_{\partial B_{2\sqrt{n}}} \equiv 0 \quad \text{and} \quad \varphi|_{\partial B_{3\sqrt{n}/2}} \equiv -2 \]

(2) therefore holds; it is possible to extend \( \varphi \) smoothly to all \( \mathbb{R}^n \) such that (3) holds. This extension depends only on \( n, \lambda, \Lambda \) We check(4). We first get

\[ M(D^2\varphi)(x) \leq C(n, \lambda, \Lambda) \quad \text{for} \ |x| \leq 1/4 \]

Because of the rotational symmetry of \( \varphi \), for \( |x| \geq 1/4 \), we only need to consider at the point \( (r, 0, \cdots, 0) \),

\[ \partial_{ij}\phi = 0 \quad \text{if} \ i \neq j, \]

\[ \partial_{11} = -M_2\alpha(1+\alpha)r^{-\alpha-2}, \]

\[ \partial_{ii}\varphi = M_2\alpha r^{-\alpha-2} \quad \text{for} \ i > 1 \]

so we have

\[ M^+(D^2\varphi)(x) = M_2[\lambda(n-1)\alpha|x|^{-\alpha-2} - \lambda\alpha(1+\alpha)|x|^{-\alpha-2}] \]

\[ = M_2[\alpha|x|^{-\alpha-2}(\lambda(n-1)) - \lambda(1+\alpha))] \leq 0 \]

We take \( 0 \leq \xi \leq 1 \) smooth such that \( \xi \equiv 1 \) in \( B_{1/4} \), \( \xi \equiv 0 \) outside \( B_{1/2} \) so that (4) holds \( \Box \)
Now we turn to a corollary of the Calderon-Zygmund cube decomposition. The key point of Calderon-Zygmund lemma is "Critical Concentration", which means the mass first time concentrates, so it does not concentrate too much.

**Definition 4.3.** Let \( Q_1 \) be the unit cube. We split it into \( 2^n \) cubes of half side. We do the same splitting with each one of these \( 2^n \) cubes and we iterate this process. The cubes obtained in this way are called dyadic cubes.

If \( Q \) is a dyadic cube different from \( Q_1 \), we say that \( \tilde{Q} \) is the predecessor of \( Q \), if \( Q \) is one of the \( 2^n \) cubes obtained from dividing \( \tilde{Q} \).

**Lemma 4.4.** Let \( A \subset B \subset Q_1 \) be measurable sets and \( 0 < \delta < 1 \) such that
(a) \( |A| \leq \delta \),
(b) If \( Q \) is a dyadic cube such that \( |A \cap Q| > \delta|Q| \), then \( \tilde{Q} \subset B \)
Then \( |A| \leq \delta|B| \)

**Proof.** We have
\[
\frac{|Q_1 \cap A|}{|Q_1|} = |A| \leq \delta
\]
We divide \( Q_1 \) into \( 2^n \) dyadic cubes. If \( Q \) is one of these \( 2^n \) subcubes of \( Q_1 \) and satisfies \( |Q \cap A|/|Q| \leq \delta \), we then split \( Q \) into \( 2^n \) dyadic cubes. We iterate this process. In this way we pick a family \( Q^1, Q^2, \ldots \) of dyadic cubes(different from \( Q_1 \)) such that
\[
\frac{|Q^i \cap A|}{|Q^i|} > \delta, \quad \forall i.
\]
Use the fact that any point in a measurable set has density 1. We conclude \( A \subset \bigcup Q^i \), except for a set of measure zero.
Also we have
\[
\frac{|\tilde{Q}^i \cap A|}{|\tilde{Q}^i|} \leq \delta \quad \forall i.
\]
Since \( \frac{|Q \cap A|}{|Q|} > \delta \) and (b) holds. We have that \( \tilde{Q}^i \subset B \). Hence,
\[
A \subset \bigcup_{i \geq 1} \tilde{Q}^i \subset B
\]
Then
\[
|A| \leq \delta|B|
\]
\[\square\]

### 4.2 Harnack Inequality and Normalization

We introduce the Harnack Inequality for viscosity solutions and state the normalization of the argument. In the following proofs we often scale the solution for simplification. In order to conserve the elliptic constants we need some standard process.
Theorem 4.5. Let \( u \in S^*(\lambda, \Lambda, f) \) in \( Q_1 \) satisfy \( u \geq 0 \) in \( Q_1 \), where \( f \) is continuous and bounded in \( Q_1 \). Then
\[
\sup_{Q_1/2} u \leq C \left( \inf_{Q_1/2} u + \|f\|_{L^n(Q_1)} \right),
\]
where \( C \) is a universal constant.

Theorem 4.6. If \( u \in \mathcal{S}(\lambda, \Lambda, f) \), then consider
\[
x = x_0 + \frac{1}{K} y, \quad y \in Q_1, \quad x \in Q = Q_{1/K}(x_0)
\]
and the function
\[
\tilde{u}(y) = \frac{u(x)}{E}
\]
then \( \tilde{u}(y) \in \mathcal{S}(\lambda, \Lambda, f(x)/(K^2 E)) \).

Now we use this process to normalize the Harnack Inequality to a constant one.

Lemma 4.7. Let \( u \in \mathcal{S}^*(\lambda, \Lambda, f) \) in \( Q_{4\sqrt{n}} \), \( u \in C(Q_{4\sqrt{n}}) \) satisfy \( u \geq 0 \) in \( Q_{4\sqrt{n}} \), (5)
\[
\inf_{Q_3} \leq 1 \quad \text{and} \quad \|f\|_{L^n(Q_{4\sqrt{n}})} \leq \epsilon_0 \quad (7)
\]
then
\[
|\{u \leq M\} \cap Q_1| > \mu. \quad (8)
\]

4.3 Main steps of proof

We first get a density decay result for subsolution. Then for solution we have two side controls of decay and increase. From the density result to uniform bound we need a bootstrap blowing up.

Lemma 4.8. There exist universal constants \( \epsilon_0 > 0, 0 < \mu < 1 \) and \( M > 1 \), such that if \( u \in \mathcal{S}(|f|) \) in \( Q_{4\sqrt{n}} \), \( u \in C(Q_{4\sqrt{n}}) \) and \( f \) satisfy
\[
u > 0 \quad \text{in} \quad Q_{4\sqrt{n}}, \quad (5)\]
\[
\inf_{Q_3} \leq 1 \quad \text{and} \quad (6)
\]
\[
\|f\|_{L^n(Q_{4\sqrt{n}})} \leq \epsilon_0 \quad (7)
\]
then
\[
|\{u \leq M\} \cap Q_1| > \mu. \quad (8)
\]
\textbf{Proof.} Take $\varphi$ as in (4.2) and define $\omega = u + \varphi$. Recall (2)(3) and (4) we have
$$
\omega \in \mathcal{S}(|f| + C\xi) \quad \text{in} \quad B_{2\sqrt{n}}
$$
We see easily we can apply the Alexandroff Maximum Principle to $\omega$ in $B_{2\sqrt{n}}$ and get
$$
1 \leq C \left( \int_{\{\omega = \Gamma_\omega\} \cap B_{2\sqrt{n}}} (|f| + C\xi)^n \right)^{1/n}
$$
$$
\leq C\|f\|_{L^n(Q_{4\sqrt{n}})} + C|\{\omega = \Gamma_\omega\} \cap Q_1|^{1/n}
$$
(9) (10)
Taking $\epsilon_0$ small enough, (10) and (7) imply
$$
\frac{1}{2} \leq C|\{\omega = \Gamma_\omega\} \cap Q_1|^{1/n}
$$
$$
\leq C|\{u \leq M\} \cap Q_1|^{1/n}
$$
\hfill \square

\textbf{Lemma 4.9.} Let $u$ be as in (4.8). Then
$$
|\{u > M^k\} \cap Q_1| \leq (1 - \mu)^k
$$
for $k = 1, 2, 3, \ldots$, where $M$ and $\mu$ are as in (4.8)
As a consequence, we have that
$$
|\{u \geq t\} \cap Q_1| \leq dt^{-\epsilon} \quad \forall t > 0,
$$
where $d$ and $\epsilon$ are positive universal constants

\textbf{Proof.} For $k=1$ (4.9) just (4.8). Suppose now that (4.9) holds for $k-1$, and let
$$
A = \{u > M^k\} \cap Q_1, \quad B = \{u > M^{k-1}\} \cap Q_1.
$$
(4.9) will be proved if we show that
$$
|A| \leq (1 - \mu)|B|.
$$
We want to apply (4.4). If we have show that a dyadic cube $Q$ satisfy
$$
|A \cap Q| > (1 - \mu)|Q|
$$
then $Q \subset B$. We can use the (4.4) to get
$$
|A| \leq (1 - \mu)|B|.
$$
If there is a dyadic cube $Q$ satisfy
$$
|A \cap Q| > (1 - \mu)|Q|
$$
but \( \overline{Q} \not\subset B \).

We denote \( Q = Q_{1/2}(x_0) \) and \( \tilde{x} \in \tilde{Q} \) such that \( u(\tilde{x}) \leq M^{k-1} \). Now we use (4.6) take \( K = \frac{1}{2} \) and \( E = M^{k-1} \) then by (4.8) we have

\[
\mu < |\{ \tilde{u}(y) \leq M \} \cap Q_1| = 2^m |\{ u(x) \leq M^K \} \cap Q|.
\]

Hence \( |Q \setminus A| > \mu |Q| \)

The following is easy. \( \square \)

If we turn to supersolution, we see that the set where supersolution takes small value will have a lower bound of measure. Hence if we combine we may get the bootstrap blowing up argument.

**Lemma 4.10.** Let \( u \in S(-|f|) \) in \( Q_{4\sqrt{n}} \). Assume that \( f \) satisfies (7) and \( u \) satisfies (4.9)

Then there exist universal constants \( M_0 > 1 \), and \( \sigma > 0 \) such that, for \( \epsilon \) as in (4.9) and \( \nu = M_0/(M_0 - \frac{1}{2}) > 1 \), the following holds: if \( j > 1 \) is an integer and \( x_0 \) satisfies

\[
|x_0|_\infty \leq \frac{1}{4}
\]

and

\[
u^j M_0,
\]

then

\[
Q_j := Q_{l_j}(x_0) \subset Q_1 \quad \text{and} \quad \sup_{Q_j} u \geq \nu^j M_0,
\]

where \( l_j = \sigma M_0^{-\epsilon/n} \nu^{-\epsilon j/n} \).

**Proof.** Take \( \sigma > 0 \) and \( M_0 > 1 \) such that

\[
\frac{1}{2} \sigma^n > d2^j (4\sqrt{n})^n
\]

and

\[
\sigma M_0^{-\epsilon/n} + dM_0^{-\epsilon} \leq \frac{1}{2},
\]

with \( d \) and \( \epsilon \) as in (4.9). Since \( l_j = \sigma M_0^{-\epsilon/n} \nu^{-\epsilon j/n} \) so \( l_j \leq 1/2 \) combined with (11) we have

\[
Q_{l_j/(4\sqrt{n})(x_0)} \subset Q_{l_j}(x_0) = Q^j \subset Q_1.
\]

(13)

We suppose that \( \sup_{Q_j} u < \nu^j M_0 \) then we use the (4.9) and (13) we have the density decay that

\[
|\{ u \geq \nu^j M_0 \} \cap Q_{l_j/(4\sqrt{n})(x_0)}| \leq |\{ u \geq \nu^j M_0 \} \cap Q_1| \leq d\nu^{-j} \left( \frac{M_0}{2} \right)^{-\epsilon}.
\]

(14)

Now we turn the solution "up to down" to get the density increase. We use a similar form of (4.6) that

\[
x = x_0 + \frac{l_j}{4\sqrt{n}} y. \quad y \in Q_{4\sqrt{n}}. \quad x \in Q_j = Q_{l_j}(x_0)
\]
and the function
\[ v(y) = \nu M_0 - \frac{u^{\nu_j^{-1}}(x_0)}{(\nu - 1)M_0}. \]
then we figure out the definition fields' transformation
\[ x \in Q_{l_j}(x_0) \quad \text{[resp. } Q_{3l_j/(4\sqrt{n})}(x_0), Q_{l_j/(4\sqrt{n})}(x_0) \text{]} \iff \]
\[ y \in Q_{4\sqrt{n}} \quad \text{[resp. } Q_{3}, Q_{1} \text{]} \]
We can check that \( v(y) \in S \quad \text{[resp. } Q_{4\sqrt{n}} \quad \text{[resp. } Q_{3}, Q_{1} \text{]} \]
so we have
\[ |\{v(x) > M_0\} \cap Q_{1}| < dM_0^{-\epsilon}. \]
this inequality and the decay one gives
\[ \left( -\frac{l_j}{4\sqrt{n}} \right)^n \leq dM_0^{-\epsilon}, \]
after simplify the inequality we get contradiction with \( \frac{1}{2} \sigma^n > d2^\epsilon (4\sqrt{n})^n \)

Proof of (4.7). Recall that \( l_j = \sigma M_0^{-\epsilon/n} \nu^{-\epsilon/n}, j = 1, 2, 3, \ldots \) Therefore, there exists a universal integer \( j_0 \geq 1 \) such that
\[ \sum_{j \geq j_0} l_j \leq 1/4 \]
We claim that \( \sup Q_j u \leq \nu^{j_0-1} M_0 \), then we get the "constant" form of Harnack Inequality. Suppose the claim is not true. Then there exists \( x_{j_0} \) such that
\[ |x_{j_0}| \leq 1/8 \quad \text{and} \quad \nu^{j_0 - 1} M_0. \]
We can use the (4.10)to get the existence of a point \( x_{j_0+1} \) such that \( |x_{j_0+1} - x_{j_0}| \leq l_{j_0}/2 \) and \( u(x_{j_0+1}) \geq \nu^{j_0} M_0 \)
We repeat this process and getting a sequence of points \( \{x_j\}_{j \geq j_0} \) such that
\[ |x_{j+1} - x_j| \leq l_j/2 \quad \text{and} \quad \nu^{j} M_0 \quad \forall j \geq j_0, \]
\[ |x_j|_\infty \leq |x_{j_0}|_\infty + \sum_{k=j_0}^{j-1} |x_{k+1} - x_k|_\infty \]
\[ \leq \frac{1}{8} + \sum_{k \geq j_0} \frac{l_k}{2} \leq \frac{1}{4}, \]

Hence we can always apply the (4.10), which contradicts with the fact that \( u \) is continuous so is bounded in \( Q_{1/2} \).

Now we state a corollary of the above result. We only assume \( u \) to be subsolution.

**Corollary 4.11.** Let \( u \in S(\lambda, \Lambda, f) \) in \( Q_1 \), where \( f \) is continuous and bounded in \( Q_1 \). Then, for any \( p > 0 \),
\[
\sup_{Q_{1/2}} u \leq C(p)(\|u^+\|_{L^p(Q_{1/4})} + \|f\|_{L^n(Q_1)}),
\]
where \( C(p) \) is a constant depending only on \( n, \lambda, \Lambda \) and \( p \).

**Proof.** We first assume that \( u \in S(f) \subset S(-|f|) \) in \( Q_{4\sqrt{n}} \), \( \|f\|_{L^n(Q_{4\sqrt{n}})} \leq \epsilon_0 \), \( u^+ \in L'(Q_1) \leq d^{1/\epsilon} \), where \( \epsilon \) is as in (4.8) and \( d \) and \( \epsilon \) are as in (4.9). We then have
\[
|\{u \geq t\} \cap Q_1| \leq t^{-\epsilon} \int_{Q_1} (u^+)^\epsilon \leq dt^{-\epsilon} \quad \forall t > 0
\]
and hence \( u \) satisfies (4.9). It follows that (4.10) holds and then the "constant" Harnack inequality holds. We have
\[
\sup_{Q_{1/4}} u \leq C.
\]
Rescaling \( u \), we get
\[
\sup_{Q_{1/4}} u \leq C\left(\|u^+\|_{L^{\infty}(Q_1)} + \|f\|_{L^n(Q_{4\sqrt{n}})}\right)
\]
We get the result when \( p = \epsilon \), by interpolation we can get result for any \( p \).

Also we state a corollary for the supersolution.

**Corollary 4.12.** Let \( u \in S(\lambda, \Lambda, f) \) in \( Q_1 \), satisfy \( u \geq 0 \) in \( Q_1 \), where \( f \) is continuous and bounded in \( Q_1 \). Then
\[
\|u\|_{L^{p_0}} \leq C\left(\inf_{Q_{1/2}} \|f\|_{L^n(Q_1)}\right)
\]
where \( p_0 > 0 \) and \( C \) are universal constants.

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Proof. we use the formula
\[ \int_{Q_1} u^{p_0} = p_0 \int_0^\infty t^{p_0-1} |\{u \geq t\} \cap Q_1| \, dt \]
then substitute the estimate for $|\{u \geq t\} \cap Q_1|$.

After we getting the Harnack Inequality we can immediately get the $C^\alpha$ regularity of the solution. That is

**Theorem 4.13.** Let $u \in S^*(\lambda, \Lambda, f)$ in $Q_1$. Then
(1) For a universal constant $\mu < 1$
$$\text{osc}_{Q_{1/2}} u \leq \text{osc}_{Q_1} u + \|f\|_{L^p(Q_1)}.$$ (2) $u \in C^\alpha(\overline{Q}_{1/2})$ and
$$\|u\|_{C^\alpha(\overline{Q}_{1/2})} \leq (\|u\|_{L^\infty(Q_1)} + \|f\|_{L^p(Q_1)}),$$
where $0 < \alpha < 1$ and $C > 0$ are universal constants.

We do not give the proof since it’s just the same as in the linear elliptic theory.

Next we state some result of boundary estimates, where we use barrier function for proof just the same as in the Schauder estimate. The only difference is we use Alexandroff Maximum Principle instead of Maximum Principle.

**Lemma 4.14.** Let $u \in S(\lambda, \Lambda, 0)$ in $B_1$. Assume that $0 < \beta < 1$, $u \in C(\overline{B}_1)$ and $u|_{\partial B_1} = \varphi$, where $\varphi \in C^\beta(\partial B_1)$.
Then, for any $x_0 \in \partial B_1$, $u$ is $C^{\beta/2}$ Hölder continuous at $x_0$, and
$$\sup_{x \in B_1} \frac{|u(x_0) - u(x)|}{|x - x_0|^{\beta/2}} \leq 2^{\beta/2} \sup_{x \in \partial B_1} \frac{\varphi(x) - \varphi(x_0)}{|x - x_0|^\beta}.$$ We refer the readers to [HL] or just [CC]. I want to point out that for linear elliptic equation we may improve global $C^{\beta/2}$ to global $C^\beta$ by potential integral. I don’t know if it’s true for this problem.

We combine with the interior estimate getting following results.

**Lemma 4.15.** Let $u \in S(\lambda, \Lambda, 0)$ in $B_1$. Assume that $0 < \beta < 1$, $u \in C(\overline{B}_1)$ and $u|_{\partial B_1} = \varphi$, where $\varphi \in C^\beta(\partial B_1)$. Then $u \in C^\gamma(B_1)$ and
$$\|u\|_{C^\gamma(B_1)} \leq C \|\varphi\|_{C^\beta(\partial B_1)}$$
where $C$ is a universal constant and $\gamma = \min(\alpha, \beta/2)$, with $\alpha$ (universal) as in (4.13).
Theorem 4.16. Let $u$ be continuous in $B_1$ satisfy $u \in S(\lambda, \Lambda, f)$ in $B_1$, where $f$ is a continuous function. Let $\varphi := u|\partial B_1$ and let $\rho(|x - y|)$ be a modulus of continuity of $\varphi$; that is $\rho$ is a nondecreasing function in $(0, \infty)$ with $\lim_{\delta \to 0}\rho(\delta) = 0$ such that

$$|\varphi(x) - \varphi(y)| \leq \rho(|x - y|) \quad \forall x, y \in \partial B_1.$$ 

Assume finally that $K$ is a positive constant such that $\|\varphi\|_{L^n(\partial B_1)} \leq K$ and $\|f\|_{L^n(B_1)} \leq K$. Then there exists a modulus of continuity $\rho^*$ of $u$ in $B_1$, i.e., $\rho^*$ is nondecreasing,

$$\lim_{\delta \to 0}\rho^*(|x - y|) \quad \forall x, y \in \overline{B_1},$$

and $\rho^*$ depends only on $n, \lambda, \Lambda, f$ and $\rho$.

5 Negative Combination of Solutions and Regularity Improvement

Before going on towards Evans-Krylov Theorem, we introduce two theorems. They all based on Jesen’s Approximate Solutions.

5.1 Jesen’s Approximate Solutions

Let $u$ be a continuous function in $\Omega$ and let $H$ be an open set such that $H \subset \Omega$. We define, for $\epsilon \geq 0$, the upper $\epsilon$-envelop of $u$ (with respect to $H$):

$$u^\epsilon(x_0) = \sup_{x \in H} \{u(x) + \epsilon - \frac{1}{\epsilon} |x - x_0|^2\}, \quad \text{for } x_0 \in H$$

Theorem 5.1. (a) $u^\epsilon \in C(H)$ and $u^\epsilon \downarrow u$ uniformly in $H$ as $\epsilon \to 0$

(b) For any $x_0 \in H$ there is a concave paraboloid of opening $2/\epsilon$ (the coefficient of quadratic term) that touches $u^\epsilon$ by below at $x_0$ in $H$. Hence $u^\epsilon$ is $C^{1,1}$ by below in $H$. In particular, $u^\epsilon$ is punctually second order differentiable at almost every point of $H$

(c) Suppose that $u$ is a viscosity subsolution of $F(D^2u) = 0$ in $\Omega$ and that $H_1$ is an open set such that $H_1 \subset H$. We then have that for $\epsilon \leq \epsilon_0$ (where $\epsilon_0$ depends only on $u$, $H$ and $H_1$) $u^\epsilon$ is a viscosity subsolution of $F(D^2u) = 0$ in $H_1$, in particular, $F(D^2u^\epsilon(x_0)) \geq 0$ a.e $x \in H_1$

Remark: We say that a continuous function $u$ in $\Omega$ is punctually second order differentiable at $x_\in \Omega$ if there exists a paraboloid $P$ such that

$$u(x) = P(x) + o(|x - x_0|^2) \quad \text{as } x \to x_0$$

Remark: We also can define $u^\epsilon$ the lower $\epsilon$-envelop of $u$

Before we prove this theorem we need some concept which help analysis a function locally.

Let us say that $P$ is a paraboloid of opening $M$ whenever

$$P(x) = l_0 + l(x) \pm \frac{M}{2}|x|^2,$$
where M is a positive constant, $l_0$ is a constant and l is a linear function. P is convex when we have + and concave when we have −.

Given two continuous functions $u$ and $v$ defined in an open set $A$ and a point $x_0 \in A$, we say that $v$ touches $u$ by above at $x_0$ in $A$ whenever

$$u(x) \leq v(x) \quad \forall x \in A$$

$$u(x_0) = v(x_0).$$

Let $u$ be a continuous function defined in $\Omega$ and $A$ be an open subset of $\Omega$. For $x_0 \in A$, we define

$$\Theta(u,A)(x_0)$$

to be the infimum of all positive constants $M$ for which there is a convex paraboloid of opening $M$ that touches $u$ by above at $x_0$ in $A$. If no such constant $M$ exists, we define it to be $\infty$. Hence $\Theta(u,A)$ is a measurable function in $A$.

Similarly we define

$$\Theta(u,A)(x_0) \in [0, \infty]$$

and

$$\Theta(u,A)(x_0) = \sup \{ \Theta(u,A)(x_0), \Theta(u,A)(x_0) \} \leq \infty$$

Given $x_0 \in \Omega$, we say that $u$ is $C^{1,1}$ by above at $x_0$ if $\Theta(u,A)(x_0) < \infty$ for some neighborhood $A$ of $x_0$.

Let us consider the second differential quotients of $u$ at $x_0$

$$\Delta^2_h u(x_0) = \frac{u(x_0 + h) + u(x_0 - h) - 2u(x_0)}{|h|^2};$$

here $h \in \mathbb{R}^n$ and we assume that $x_0 + h$ and $x_0 - h$ belong to $\Omega$. Note that $\Delta^2_h P \equiv M$ when $P$ is convex paraboloid of opening $M$. It follows that, for any $x_0 \in \Omega$,

$$-\Theta(u,B_{|h|}(x_0))(x_0) \leq \Delta^2_h u(x_0) \leq \Theta(u,B_{|h|}(x_0))(x_0),$$

if $B_{|h|}(x_0) \subset \Omega$.

**Lemma 5.2.** Let $1 < p \leq \infty$ and $u$ be a continuous function in $\Omega$. Let $\epsilon$ be a positive constant and define

$$\Theta(u,\epsilon)(x) := \Theta(u,\Omega \cap B_{\epsilon}(x))(x), \quad x \in \Omega$$

Assume that $\Theta(u,\epsilon) \in L^p(\Omega)$. Then $D^2 u \in L^p(\Omega)$ and

$$\|D^2 u\|_{L^p(\Omega)} \leq 2\|\Theta(u,\epsilon)\|_{L^p(\Omega)}.$$
Proof. Since $1 < p \leq \infty$ we only need to prove that
$$
\left| \int_{\Omega} u \varphi_{ij} \right| \leq 2 \| \Theta(u, \epsilon) \|_{L^p(\Omega)} \| \varphi \|_{L^{p'}(\Omega)}
$$
for any $C^\infty$ function $\varphi$ with compact support in $\Omega$ and any indices $i, j$. Here $p'$ denotes the conjugate exponent of $p$.

To show it, we have
$$
\partial_{ij} \varphi = \frac{1}{2} (\partial_{ei+ej} \varphi - \partial_{ii} \varphi - \partial_{jj} \varphi)
= \frac{1}{2} (2 \partial_{ee} \varphi - \partial_{ii} \varphi - \partial_{jj} \varphi),
$$
where $v = (e_i + e_j)/\sqrt{2}$ and $\{e_i\}$ is the canonical basis of $\mathbb{R}^n$. Hence it suffices to prove
$$
\left| \int_{\Omega} u \varphi_{ii} \right| \leq \| \Theta(u, \epsilon) \|_{L^p(\Omega)} \| \varphi \|_{L^{p'}(\Omega)}
$$

Let $K \subset \subset \Omega$ be the support of $\varphi$. We have that
$$
\hat{\Theta}(\varphi) = \lim_{\delta \to 0} \hat{\Theta}(\Delta_{\delta/e_i} \varphi) = \lim_{\delta \to 0} \hat{\Theta}(\Delta_{\delta/e_i} u \varphi)
$$
Take $\delta < \epsilon$ and $\delta < \text{dist}(K, \mathbb{R}^n \setminus \Omega)$ we have
$$
|\Delta_{\delta/e_i} u| \leq \Delta(u, \epsilon)
$$

\begin{lemma}
Let $u$ be continuous in $\Omega$ and $B$ be a convex domain such that $\overline{B} \subset \Omega$. Let $\epsilon > 0$ and define
$$
\Theta(u, \epsilon)(x) := \Theta(u, \Omega \cap B_\epsilon(x))(x), \quad x \in \overline{B}
$$
Assume that, for some constant $K$, $\Theta(u, \epsilon)(x) \leq K$ for any $x \in \overline{B}$. Then $u \in C^{1,1}(\overline{B})$ and
$$
|Du(x) - Du(y)| \leq 2n \| \Theta(u, \epsilon) \|_{L^\infty(B)} |x - y| \quad \forall x, y \in \overline{B}.
$$
\end{lemma}

\begin{proof}
We know (since $\Theta(u, \epsilon) < \infty$ for any $x \in \overline{B}$) that $u$ is differentiable at any point of $\overline{B}$. By the above lemma we know that $D^2 u \in L^\infty(B)$ and $\|D^2 u\|_{L^\infty(B)} \leq 2 \| \Theta(u, \epsilon) \|_{L^\infty(B)}$.

Since $u_i \in W^{1,\infty}$ and $B$ is convex, we have that $u_i$ is continuous and
$$
u_i(x) - u_i(y) = \int_0^1 \frac{d}{dt} u_i(tx + (1 - t)y) dt
= \sum_i \int_0^1 \partial_{ij} u(tx + (1 - t)y) dt (x_j - y_j),
$$
for any $x, y \in \overline{B}$. Using that $\|D^2 u\|_{L^\infty(B)} \leq 2 \| \Theta(u, \epsilon) \|_{L^\infty(B)}$, we get the result.
\end{proof}
Remark: we also can prove directly. Using the Alexandroff-Buselman-Feller theorem we get the following theorem:

**Lemma 5.4.** Let \( u \) be a continuous function in a convex domain \( \Omega \) such that
\[
\Theta(u, \Omega)(x) \leq K \quad \forall x \in \Omega,
\]
for some positive constant \( K \). Then
\[
u(x) + \frac{K}{2}|x|^2
\]
is convex in \( \Omega \). In particular, \( u \) is punctually second order differentiable at almost every point \( x \in \Omega \).

We turn back to Jesen’s Approximate Solutions.

**Lemma 5.5.** Let \( x_0, x_1 \in H \). Then
1. \( \exists x^*_0 \in \overline{H} \) such that \( u^\epsilon(x_0) = u(x^*_0) + \epsilon - \frac{|x^*_0 - x_0|^2}{\epsilon} \)
2. \( u^\epsilon(x_0) \geq u(x_0) + \epsilon \)
3. \( |u^\epsilon(x_0) - u^\epsilon(x_1)| \leq (3/\epsilon) \text{diam}(H) |x_0 - x_1| \).
4. \( 0 < \epsilon < \epsilon' \Rightarrow u^\epsilon(x_0) \leq u^{\epsilon'}(x_0) \)
5. \( |x_0 - x_0|^2 \leq \epsilon \text{osc}_H u \).
6. \( 0 < u^\epsilon(x_0) - u(x_0) \leq u(x^*_0) - u(x_0) + \epsilon \)

**Proof.** (1)(2)(4)(6) are easy. For (3), let \( x \in \overline{H} \) and note that
\[
\begin{align*}
u^\epsilon(x_0) &\geq u(x) + \epsilon - \frac{1}{\epsilon} |x - x_0|^2 \\
&\geq u(x) + \epsilon - \frac{1}{\epsilon} |x - x_0|^2 - \frac{1}{\epsilon} |x_1 - x_0|^2 - \frac{2}{\epsilon} |x - x_1||x_1 - x_0| \\
&\geq u(x) + \epsilon - \frac{1}{\epsilon} |x - x_1|^2 - \frac{3}{\epsilon} \text{diam}(H) |x_1 - x_0|.
\end{align*}
\]
Taking the supremum over \( x \in \overline{H} \), we get (3).

For (5), we note by (1)(2)
\[
\frac{1}{\epsilon} |x^*_0 - x_0|^2 = u(x^*_0) + \epsilon - u^\epsilon(x_0) \leq u(x^*_0) - u(x_0).
\]

We can give the proof of properties of Jesen’s Approximate Solutions:

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Proof. (3) in (5.5) gives the continuity of \( u^\epsilon \). (4)(5)(6) imply the other assertion in (a). For (b),
\[
P_0(x) = u(x_0^*) + \epsilon - \frac{1}{\epsilon} |x - x_0^*|^2 \leq u^\epsilon(x) \quad \forall x \in H
\]
and that equality holds at \( x_0 \). That is, \( P_0 \) touches \( u^\epsilon \) by below at \( x_0 \) in \( H \). Then we use the (5.4) we get the (b).
For (c). Let \( x_0 \in H_1 \) and let \( P(x) \) be a paraboloid that touches \( u^\epsilon \) by above at \( x_0 \). Consider the paraboloid
\[
Q(x) = P(x + x_0 - x_0^*) + \frac{1}{\epsilon} |x_0 - x_0^*|^2 - \epsilon
\]
We have
\[
u(x) \leq u^\epsilon(x + x_0 - x_0^*) + \frac{1}{\epsilon} |x_0 - x_0^*|^2 - \epsilon
\]
Therefore
\[
u(x) \leq P(x + x_0 - x_0^*) + \frac{1}{\epsilon} |x_0 - x_0^*|^2 - \epsilon = Q(x)
\]
and \( u(x_0) = Q(x_0^*) \). Hence \( Q \) touches \( u \) by above at \( x_0^* \) we have
\[
0 \leq F(D^2 Q) = F(D^2 P)
\]
\( \square \)

Now we prove a theorem allowing a negative combination of solutions in some sense.

**Theorem 5.6.** Let \( u \) be a viscosity subsolution of \( F(D^2 u) = 0 \) in \( \Omega \) and \( v \) be a viscosity supersolution of \( F(D^2 v) = 0 \) in \( \Omega \). Then
\[
u - v \in S(\frac{\lambda}{n}, \Lambda) \quad \text{in} \quad \Omega
\]

*Proof.* We fix \( H \) and \( H_1 \) such that \( \overline{H}_1 \subset H \) \( \overline{H} \subset \Omega \); we will prove that for \( \epsilon \) small enough, \( u^\epsilon - v_\epsilon \in S(\lambda/n, \Lambda) \) in \( H_1 \). It follows that \( u - v \in S(\lambda/n, \Lambda) \) in \( \Omega \).

Let \( P \) be a paraboloid such that \( u^\epsilon - v_\epsilon \leq P \) in \( \overline{B}_{r}(x_0) \subset H_1 \) and \( u^\epsilon(x_0) - v_\epsilon(x_0) = P(x_0) \).

We need to see that \( M^+(D^2 P, \lambda/n, \Lambda) \geq 0 \). We may assume that \( \overline{B}_{2r}(x_0) \subset H \). Take \( \delta \geq 0 \) and define
\[
\omega(x) = v_\epsilon(x) - u^\epsilon(x) + P(x) + \delta |x - x_0|^2 - \delta \nu^2
\]
Using (5.5) (b), we know that for any \( x \in \overline{B}_{r}(x_0) \) there exists a convex paraboloid \( P^x \) of opening \( K \) (independent of \( x \)) that touches \( \omega \) by above at \( x \) in \( B_r(x) \). And we have \( \omega > 0 \) on \( \partial B_{r}(x_0) \) and \( \omega(x_0) \leq 0 \), we have
\[
0 < \int_{B_{r}(x_0) \cap \{\omega = \Gamma_{\omega}\}} \text{det} D^2 \Gamma _\omega
\]
By (5.4) we know that almost everywhere in \( B_{r}x_0 \) we have \( u^\epsilon \) and \( v_\epsilon \) are punctually second order differentiable in \( A \). Hence by (5.5)(c), we have
\[
F(D^2 v_\epsilon(x)) \leq 0 \quad \text{and} \quad F(D^2 u^\epsilon(x)) \geq 0 \quad \text{for} \quad x \in A
\]
Since \( \Gamma \omega \) is convex and \( \Gamma \omega \leq \omega \), we have that
\[
D^2 \omega(x) \text{ is nonnegative definite, for } x \in A \cap \{ \omega = \Gamma \omega \}
\]
We choose one point \( x_1 \in \{ \omega = \Gamma \omega \} \cap A \). At this point we have
\[
0 \leq F(D^2 u^t(x_1)) = F(D^2 v_t(x_1) - D^2 \omega(x_1) + D^2 P + 2\delta I)
\]
\[
\leq F(D^2 v_t(x_1) + D^2 P + 2\delta I) \leq F(D^2 v_t(x_1) + D^2 P) + 2\lambda
\]
\[
\leq F(D^2 v_t(x_1)) + \lambda \|(D^2 P)^+\| - \lambda \|(D^2 P)^-\| + 2\lambda
\]
\[
\leq \lambda \|(D^2 P)^+\| - \lambda \|(D^2 P)^-\| + 2\lambda \leq M^+(D^2 P, \lambda/n, \Lambda) + 2\lambda.
\]
Letting \( \delta \to 0 \), we get \( M^+(D^2 P, \lambda/n, \Lambda) \geq 0 \) \( \square \).

Now we turn to regularity improvement and get \( C^{1,\alpha} \) regularity for \( F(D^2 u) = 0 \). We denote, for \( h > 0 \), \( \Omega_h = \{ x \in \Omega : d(x, \partial \Omega) > h \} \).

**Lemma 5.7.** Let \( u \) be a viscosity solution of \( F(D^2 u) = 0 \) in \( \Omega \). Let \( h > 0 \) and \( e \in \mathbb{R}^n \) with \( |e| = 1 \). Then
\[
u(x + h e) - u(x) \in S(\lambda h, \Lambda) \quad \text{in } \Omega_h
\]

**Lemma 5.8.** Let \( 0 < \alpha < 1, 0 < \beta \leq 1 \) and \( K > 0 \) be constants. Let \( u \in L^\infty([-1,1]) \) satisfy \( \|u\|_{L^\infty([-1,1])} \leq K \). Define, for \( h \in \mathbb{R} \) with \( 0 < |h| \leq 1 \),
\[
v_{\beta,h}(x) = \frac{u(x + h) - u(x)}{|h|^\beta}, \quad x \in I_h,
\]
where \( I_h = [-1, 1 - h] \) if \( h > 0 \) and \( I_h = [-1 - h, 1] \) if \( h < 0 \). Assume that \( v_{\beta,h} \in C^\alpha(I_h) \) and \( \|v_{\beta,h}\|_{C^\alpha(I_h)} \leq K \), for any \( 0 < |h| \leq 1 \). We then have

1. If \( \alpha + \beta < 1 \) then \( u \in C^{\alpha + \beta}([-1,1]) \) and \( \|u\|_{C^{\alpha + \beta}([-1,1])} \leq CK \);
2. If \( \alpha + \beta \geq 1 \) then \( u \in C^{0,1}([-1,1]) \) and \( \|u\|_{C^{0,1}([-1,1])} \leq CK \);

where the constants \( C \) in (1)(2) depends only on \( \alpha + \beta \).

**Proof.** By symmetry of the problem with respect to the change \( x \to -x \), it is enough to bound \( |u(x + e) - u(x)| \) for
\[
-1 \leq x \leq 0 \quad \text{and} \quad x + e \leq 1
\]
Let \( i \leq 0 \) be the integer such that \( x + 2^i \epsilon \leq 1 < x + 2^{i+1} \epsilon \), and define \( \tau_0 = 2^i \epsilon \). Then \( -1 \leq x < x + \tau_0 \leq 1 \) and
\[
1/2 \leq \tau_0 \leq 2
\]
Define
\[
\omega(\tau) = u(x + \tau) - u(x), \quad 0 < \tau \leq \tau_0
\]
We have that
\[ |\omega(\tau) - 2\omega(\tau/2)| = |u(x+\tau) - 2u(x+\tau/2) + u(x)| = \left(\frac{\tau}{2}\right)^\beta |v_{\beta,\tau/2}(x+\tau/2) - v_{\beta,\tau/2}(x)| \leq \cdots \]
for some constant \( C \) which depends (as all \( C \)'s in the rest of this proof) only on \( \alpha + \beta \).

Adding all the inequalities, we get
\[ |\omega(\tau) - 2^i \omega(\epsilon)| = |\omega(\tau_0) - 2^i \omega(\tau_0/2^i)| \leq CK\tau_0^{\alpha+\beta} \sum_{j=0}^{i-1} 2^j(1-(\alpha+\beta)). \]

Since \( 2^{-i} = \tau_0^{-1}\epsilon \leq 2\epsilon \) and \( \|u\|_{L^\infty([-1,1])} \leq K \), we have
\[
|\omega(\epsilon)| \leq 2^{-i}|\omega(\tau_0)| + CK2^{-i}\tau_0^{\alpha+\beta} \sum_{j=0}^{i-1} 2^j(1-(\alpha+\beta)) \\
\leq 4K\epsilon + CK\epsilon\tau_0^{\alpha+\beta} \sum_{j=0}^{i-1} 2^j(1-(\alpha+\beta)) 
\]
If \( \alpha + \beta < 1 \) we get \( |\omega(\epsilon)| \leq 4K\epsilon + CK\epsilon\tau_0^{\alpha+\beta-1}2^{i(1-(\alpha+\beta))} = 4K\epsilon + CK\epsilon^{\alpha+\beta} \leq CK\epsilon \).

If \( \alpha + \beta \geq 1 \), we get \( |\omega(\epsilon)| \leq 4K\epsilon + CK\epsilon\tau_0^{\alpha+\beta+1} \leq CK\epsilon \).

**Lemma 5.9.** Let \( u \) be a viscosity solution of \( F(D^2u) = 0 \) in \( B_1 \). Then \( u \in C^{4,\alpha}(\overline{B}_{1/2}) \) and
\[
\|u\|_{C^{4,\alpha}(\overline{B}_{1/2})} \leq C(\|u\|_{L^\infty}(B_1) + |F(0)|), 
\]
where \( 0 < \alpha < 1 \) and \( C \) are universal constants.

**Proof.** We fix \( \epsilon \in \mathbb{R}^n \) with \( |\epsilon| = 1 \) and \( 0 < h < 1/8 \). By the above lemma, we know that for \( 0 < \beta < 1 \)
\[
v_\beta(x) = \frac{1}{h^\beta}(u(x+he) - u(x)) \in S(\lambda/n, \Lambda) \quad \in \quad B_{7/8} \tag{15}
\]

Hence by \( C^\alpha \) interior estimates we have that
\[
\|v_\beta\|_{C^\alpha(\overline{B}_r)} \leq C(r, s)\|v_\beta\|_{L^\infty(B_{r+s/2})} \leq C(r, s)\|u\|_{C^{0,\alpha}(\overline{B}_r)},
\]
where \( 0 < r < s \leq 7/8 \), \( 0 < h < (s-r)/2 \), \( \alpha \) is universal and \( C(r, s) \) depends on \( n, \lambda, \Lambda, r, s \).

By making \( \alpha \) slightly smaller, we can assume that there is a universal integer \( i \) such that
\[
i\alpha < 1 \quad \text{and} \quad (i+1)\alpha > 1.
\]
We have \( u \in S(\lambda/n, \Lambda, -F(0)) \) in \( B_1 \). Hence
\[
\|u\|_{L^\infty(B_{7/8})} \leq C(\|u\|_{L^\infty(B_1)} + |F(0)|) =: CK
\]
We define \( K = \|u\|_{L^\infty} + |F(0)| \). We apply (15) with \( \beta = \alpha \) and \( r = r_1 < s = 7/8 \) to get
\[
\|v_\alpha\|_{C^\alpha(B_{r_1})} \leq C(r_1)\|u\|_{C^\alpha(B_{7/8})} \leq \ldots \leq 0
\]
where \( 0 < h < (7/8 - r_1)/2 \), and \( C(r_1) \) depends only on \( n, \lambda, \Lambda, r_1 \). We can apply above lemma get that
\[
\|u\|_{C^{2,\alpha}(\overline{B}_{r_2})} \leq C(r_1, r_2)K \quad \text{for} \quad r_2 < r_1
\]
We now apply (15) and above lemma with \( \beta = 2\alpha \), after finite steps we get
\[
\|u\|_{C^{0,\alpha}(\overline{B}_{3/4})} \leq CK
\]
We finally apply(15) with \( \beta = 1 \) and get
\[
\|v_1\|_{C^\alpha(\overline{B}_{1/2})} \leq C\|u\|_{C^{0,\alpha}(\overline{B}_{3/4})} \leq CK \quad \forall |e| = 1 \quad \forall 0 < h < 1/8
\]
We conclude that \( u \in C^{1,\alpha}(\overline{B}_{1/2}) \) and \( \|u\|_{C^{1,\alpha}(\overline{B}_{1/2})} \leq CK \)

We now state some easy application of the above lemma.

**Theorem 5.10.** Let \( F \) be concave and let \( u \) and \( v \) be viscosity subsolutions of \( F(D^2\omega) = 0 \) in \( \omega \). Then \( \frac{1}{2}(u + v) \) is a viscosity subsolution of \( F(D^2\omega) = 0 \) in \( \omega \)

**Corollary 5.11.** Let \( f \) be concave and suppose that \( F(D^2u) = 0 \) in \( \Omega \) in the viscosity sense. Let \( e \in \mathbb{R}^n \) with \( |e| = 1 \) and \( h > 0 \). Then
\[
\frac{u(x + he) + u(x - he) - 2u(x)}{h^2} \in \mathcal{L}(\lambda/n, \Lambda) \quad \text{in} \ \Omega_h
\]

**Corollary 5.12.** Let \( F \) be concave and \( u \in C^2\Omega \) be a solution of \( F(D^2u) = 0 \) in \( \Omega \). Then, for any \( e \in \mathbb{R}^n \) with \( |e| = 1 \),
\[
u_{ee} = \frac{\partial^2 u}{\partial e \partial e} \in \mathcal{L}(\lambda/n, \Lambda) \quad \text{in} \ \Omega
\]

**Proof of** (5.10). It’s enough to see that \( \frac{1}{2}(u^e + v^e) \) is viscosity subsolution of \( F(D^2\omega) = 0 \). Let \( P \) be a paraboloid that touches \( \frac{1}{2}(u^e + v^e) \) by above at \( x_0 \). We need to show \( F(D^2P) \geq 0 \).

Define
\[
\omega(x) = P(x) + \delta|x - x_0|^2 - \delta r^2 - \frac{1}{2}(u^e + v^e).
\]
We have \( x_1 \in \Omega \) such that \( u^e, v^e \) and \( \omega \) are punctually second order differentiable at \( x_1 \), and
\[
D^2P + \delta|x - x_0|^2 - \frac{1}{2}(u^e + v^e)(x_1) \geq 0
\]
We also have that \( F(D^2u^e) \geq 0 \) and \( F(D^2v^e(x_1)) \geq 0 \)

Since \( F \) is concave we get \( F(D^2\frac{1}{2}(u^e + v^e(x_1))) \geq 0 \), then
\[
F(D^2P + 2\delta I) \geq 0
\]
Letting \( \delta \to 0 \), we get \( F(D^2P) \geq 0 \)
6 Evans-Krylov Theorem

Now we turn to prove Evans-Krylov Theorem. This section is concerned with equations

$$F(D^2u) = 0$$

for a concave operator $F$. We prove $C^{2,\alpha}$ interior regularity for viscosity solutions of concave equations of the form (16), for some universal $\alpha \in (0, 1)$.

**Theorem 6.1.** Let $F$ be concave and $u$ be a viscosity solution of $F(D^2u) = 0$ in $B_1$. Then $u \in C^{2,\alpha}(\overline{B}_{1/2})$ and

$$\|u\|_{C^{2,\alpha}(\overline{B}_{1/2})} \leq C(\|u\|_{L^\infty(B_1)} + |F(0)|),$$

where $0 < \alpha < 1$ and $C$ are universal constants.

Remark: using the ellipticity condition for $F$, it is easy to see that there exists one $t \in \mathbb{R}$ such that $F(tI) = 0$ and $t \leq |F(0)|/\lambda$. Define for any $x_0 \in \mathbb{R}^n$, the paraboloid

$$P(x) = \frac{t}{2}|x - x_0|^2.$$ Then

$$F(D^2u) = F(D^2(u - P) + tI) =: G(D^2(u - P)).$$

We have now that $G(0) = 0$; $G$ is uniformly elliptic.

6.1 $C^{1,1}$ Regularity

**Theorem 6.2.** Let $F$ be concave and $u$ be a viscosity solution of $F(D^2u) = 0$ in $B_1$. Then $u \in C^{1,1}(\overline{B}_{1/11})$ and

$$\|u\|_{C^{1,1}(\overline{B}_{1/11})} \leq C(\|u\|_{L^\infty(B_1)} + |F(0)|),$$

where $C$ are universal constants.

**Proof.** We assume that $F(0) = 0$. By considering the functional $t^{-1}F(tD^2\omega)$, with $t = \|u\|_{L^\infty(B_1)}$, we may assume that $\|u\|_{L^\infty(B_1)} \leq 1$. We need to prove that $\|u\|_{C^{1,1}} \leq C$. Where $C$ denotes a universal constant.

Recall that $F$ is a concave function on the space $S$ of symmetric matrices; it follows that there is a supporting hyperplane(by above) to the graph of $F$ at $0 \in S$. That is there is a linear functional $L$ on the space of symmetric matrices such that $L(0) = 0$ and $L(M) \geq F(M)$ for any $M \in S$.

Therefore, $L$ is of the form $L(M) = \sum a_{ij}m_{ij} = tr(AM)$ for some $A \in S$. Then $L$ is uniformly linear operator with eigenvalues of $A$ belong to $[\lambda, \Lambda]$. That is because for any $\xi \in \mathbb{R}^n$, we have

$$a_{ij}\xi_i\xi_j = L(\xi\xi^t) \geq F(\xi\xi^t) \geq F(0) + \lambda\|\xi\xi^t\| = \lambda|\xi|^2 \quad \text{and},$$

$$a_{ij}\xi_i\xi_j = -L(-\xi\xi^t) \leq -F(-\xi\xi^t) \leq -F(0) + \Lambda\|\xi\xi^t\| = \Lambda|\xi|^2.$$
Hence, we can make a linear change of space variables such that in the new variables we have \( L(D^2 u) = \Delta u \). Since \( L \)'s ellipticity constants is in \([\lambda, \Lambda]\), we just need to prove the situation \( L(D^2 u) = \Delta u \).

Since \( F(D^2 u) = 0 \) in the viscosity sense, and \( \Delta \varphi = L(D^2 \varphi) \leq F(D^2 \varphi) \) for any \( \varphi \in C^2 \), we immediately see that \( u \) also satisfies \( \Delta u \geq 0 \) in the viscosity sense in \( B_1 \). From this, we get that

\[
  u(x_0) \leq \int_{S_h(x_0)} u
\]

for any \( x_0 \in B_{1/2} \) and \( 0 < h < 1/2 \); here \( S_h(x_0) = \partial B_h(x_0) \) and \( \int \) denote average. To see this we considers the harmonic function \( \omega \) in \( B_h(x_0) \) equal to \( u \) on \( S_h(x_0) \). Then apply the maximum principle.

We therefore have that the function

\[
  u_h^*(x) = \frac{1}{h^2} \left( \int_{S_h(x)} u - u(x) \right)
\]

is continuous and nonnegative in \( B_{1/2} \), for any \( 0 < h < 1/2 \). Note that \( u_h^* \) is an approximation of \( \frac{1}{2n} \Delta u \), in the following sense. Using Taylor’s formula, we see that if \( \varphi \in C^\infty \) then \( \varphi_h^* \) converges as \( h \to 0 \) uniformly in compact sets to \( \frac{1}{2n} \Delta \varphi \); more over, for a constant \( C(n) \) depending only on \( n \), \( \| \varphi_h^* \|_{L^\infty(B_1)} \leq C(n) \| D^2 \varphi \|_{L^\infty(B_2)} \).

Using that \( u^* \leq 0 \), we now bound the \( L^1 \)– norm of \( u_h^* \) as follows. Let \( \varphi \geq 0 \) be a \( C^\infty \) function with compact support in \( B_{1/2} \) and \( \varphi \equiv 1 \) on \( B_{1/3} \). Then

\[
  \int_{B_{1/3}} |u_h^*| = \int_{B_{1/3}} u_h^* \leq \int_{B_{1/2}} u \varphi_h^* \leq C(n) \| D^2 \varphi \|_{L^\infty} \leq C(n).
\]

Recall that

\[
  \int_{S_h(x)} u = \int_{S_h(0)} u + u(x+y)dy
\]

Because any positive linear combination of viscosity subsolution of \( F(D^2 u) = 0 \) is a subsolution, viscosity subsolutions is closed under uniform limits and \( u \) is a supersolution, we have \( u_h^* \in \mathcal{S}(\lambda/n, \Lambda) \) in \( B_{1/2} \). Please notice the definition of \( u_h^* \)

\[
  u_h^*(x) = \frac{1}{h^2} \left( \int_{S_h(x)} u - u(x) \right)
\]

We now apply (4.11)(stated in balls and applied with \( p=1 \)) to \( u_h^* \). We have that \( 0 < u_h^* \in \mathcal{S}(\lambda/n, \Lambda) \) and \( \| u_h^* \|_{L^1(B_{1/3})} \leq C(n) \). It follows that

\[
  \| u_h^* \|_{L^\infty(B_{1/4})} \leq C,
\]

for a universal constant \( C \)(independent of \( h \)).

Let \( \psi \) be any \( C^\infty \) function with compact support in \( B_{1/4} \). Since \( 2n\psi_h^* \to \Delta \psi \) as \( h \to 0 \),
uniformly in $B_{1/4}$, we have that
\[
\int_{B_{1/4}} u \Delta \psi = 2n \lim_{h \to 0} \int_{B_{1/4}} u^h \psi \quad \text{and combined with } \|u^h\|_{L^\infty(B_{1/4})} \leq C, \text{ we have}
\]
\[
\|u_\infty - \psi\|_{L^1(B_{1/4})} \leq C \|\psi\|_{L^1(B_{1/4})}
\]
and combined with $\|u^h\|_{L^\infty(B_{1/4})} \leq C$, we have
\[
\|u_\infty - \psi\|_{L^1(B_{1/4})} \leq C \|\psi\|_{L^1(B_{1/4})}
\]
\[
\Delta u \in L^\infty(B_{1/4}) \text{ and } \|\Delta u\|_{L^\infty(B_{1/4})} \leq C.
\]
In particular, $\|\Delta u\|_{L^2(B_{1/4})} \leq C$ and then by $L^2$ regularity theory, $u \in W^{2,2}(B_{1/5})$ and
\[
\|D^2 u\|_{L^2(B_{1/5})} \leq C \quad (19)
\]
By (5.11) we know that the pure second order incremental quotients of $u$,
\[
\Delta_{he}^2 u(x) = \frac{1}{h^2} [u(x + he) + u(x - he) - 2u(x)],
\]
belong to $\mathcal{S}(\lambda/n, \Lambda)$ in $B_{1/2}$; here $e$ is any vector with $|e| = 1$. (19) implies that $\|\Delta_{he}^2 u\|_{L^\infty(B_{1/10})} \leq C$, for a universal constant $C$ independent of $h \in (0, 1/10)$. (4.11) gives
\[
\sup_{B_{1/11}} \leq C \quad \forall 0 < h < 1/10, \quad \forall |e| = 1
\]
(20)

Then $v(x) := u(x) - \frac{C}{2} |x|^2$ is a concave function in $B_{1/11}$. The Alexandroff-Buselman-Feller theorem implies that $v$ and therefore $u$, is punctually second order differentiable at every point in a set $A$ with $|B_{1/11} \setminus A| = 0$. (20) implies that
\[
u_{ee}(x) \leq C \quad \forall x \in A \quad \forall |e| = 1
\]
We also know that the weak derivative $D^2 u$ of $D u$ equals the punctually derivative a.e., so we can see weak derivative $D^2 u$ as the punctually derivative. Hence $u_{ee}(x)$ is a linear combination of $D^2 u$, then $u_{ee}$ is an $L^2$ function in $B_{1/11}$. Because $u$ is second punctually differentiable, we have $F(D^2 u) = 0$ for any $x \in A$. Then
\[
\|D^2 u\| \leq C \sup_{|e| = 1} u_{ee}(x) \quad \forall x \in A
\]
Hence $D^2 u \in L^\infty(B_{1/11})$ and $\|D^2 u\|_{L^\infty(B_{1/11})} \leq C$. We also know that $W^{2,\infty}(B_{1/11}) = C^{1,1}(\overline{B}_{1/11})$, so
\[
\|u\|_{C^{1,1}(\overline{B}_{1/11})} \leq C
\]
Remark: We have $C^{1,\alpha}$ regularity of $u$ for free. Now we assume $F$ is concave, then get the $C^{1,1}$. Without the concave condition, we still have a long way to go.
6.2 From $C^{1,1}$ to $C^{2,\alpha}$

We have proved the $C^{1,1}$ regularity for $u$, but next we need to assume $u$ is $C^2$. It seems there is a gap, however we can do the same argument for $C^{1,1}$ solution $u$ except we have the almost everywhere $C^{2,\alpha}$ regularity: there exists $A \subset B_{1/2}$ and $|B \setminus A| = 0$ satisfies

1. $D^2u$ exists $\forall x \in A$;
2. $|D^2u(x) - D^2u(y)| \leq C|x - y|^\alpha$ $\forall x, y \in A$;

Where $C$ is a constant independent of $x, y$. Then we can find a $C^\alpha$ function $\tilde{D}^2u$, such that $D^2u = \tilde{D}^2u$ in $A$. We integrate $\tilde{D}^2u$ back to get $\tilde{D}u$. Since $\tilde{D}u$ is Lipschitz, we can have $\tilde{D}u = Du$ that is $D(Du) = \tilde{D}^2u$.

Hence we just need to prove the almost everywhere $C^{2,\alpha}$, without lost of generality, we assume $u$ is $C^2$.

**Lemma 6.3.** Under the assumption that $F$ is concave, $u \in C^2(B_1)$ satisfy $F(D^2u) = 0$ in $B_1$, there exists a universal constant $0 \leq \delta_0 \leq 1$ such that $\text{diam } D^2u(B_1) = 2$ implies $\text{diam } D^2u(B_{\delta_0}) \leq 1$.

Remark: we can state a version for $u \in C^{1,1}$, then you may believe there will be no big difference between the proofs.

**Lemma 6.4.** Under the assumption that $F$ is concave, $u \in C^{1,1}(B_1)$ (thus $D^2u$ exists a.e. in $B_1$) satisfy $F(D^2u) = 0$ in $B_1$, there exists a universal constant $0 \leq \delta_0 \leq 1$ such that $\text{diam } \{D^2u(B_1 \setminus A)\} = 2$ implies $\text{diam } D^2u(B_{\delta_0} \setminus A) \leq 1$, where $A$ has measure zero.

Before going on towards the proof, we state several lemmas.

**Lemma 6.5.** Let $v \in \bar{S}(\lambda, \Lambda, 0)$ in $B_1$ satisfy $v \geq 0$ in $B_1$. Then

$$\inf_{B_{1/2}} v \geq C|\{v \geq 1\} \cap B_{1/4}|^\delta$$

where $C$ and $\delta$ are positive universal constants.

Now we prove the main lemma towards Evans-Krylov Theorem

**Lemma 6.6.** Let $F$ be concave and let

$$u \in C^2(B_1) \text{ satisfy } F(D^2u) = 0 \text{ in } B_1$$

Assume that

$$1 \leq \text{diam } D^2u(B_1) \leq 2$$

and that $D^2u(B_1)$ is covered by $N$ balls $B^1, \ldots, B^N$ of radius $\epsilon$ (in the space $S$ of symmetric matrices) with $N \geq 1$ and $\epsilon \leq \epsilon_0$, where $\epsilon_0 > 0$ is a universal constants.

Then $D^2(B_{1/2})$ is covered by $N$-1 balls among $B^1, \ldots, B^N$. 

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Proof. Take $c_0$ universal constants such that

$$\text{If } F(M_1) = F(M_2) = 0 \quad \text{then } c_0\|M_2 - M_1\| \leq \|(M_2 - M_1)^+\|$$

For $i = 1, 2, \ldots, N$ we take $x_i \in B_1$ such that $B_i \subset B_{2\epsilon}(M_i)$, where

$$M_i = D^2u(x_i)$$

Hence, taking $\epsilon_0$ such that $2\epsilon \leq 2\epsilon_0 \leq c_0/16$, we have that

$$B_{c_0/16}(M_1), \ldots, B_{c_0/16}(M_N)$$

cover $D^2u(B_1)$. Since $D^2u(B_1)$ has diameter at most 2, every $M_i$ belongs to one closed ball $\overline{B}$ of radius 2 in $\mathbb{S}$. Let $N'$ be the maximum number of points in the ball $\overline{B}$ such that the distance between any two of them is at least $c_0/16$. We have that $N'$ depends only on $n$ and $c_0$. Therefore, we can assume that $\{B_{c_0/8}(M_i)\}_{i=1}^{N'}$ cover $B_1$ and therefore, there exists one $M_i$, say $M_1$, such that

$$|D^2u^{-1}(B_{c_0/8}(M_1)) \cap B_{1/4}| \geq \eta > 0$$

where $\eta$ is a universal. Since $\text{diam}D^2u(B_1) > 1$ and take $2\epsilon \leq \epsilon_0 \leq 1/4$. Hence we have $M_2$ such that $\|M_2 - M_1\| \geq 1/4$. Hence there is an $e \in \mathbb{R}^n$ and $|e| = 1$ such that

$$u_{ee}(x_2) \geq u_{ee}(x_1) + c_0/4$$

Define

$$K = \sup_{B_1} u_{ee} \quad \text{and} \quad v = K - u$$

We have that $0 \leq v \in \overline{S(\lambda/n, \Lambda, 0)}$ in $B_1$, by (5.12).

Also we from above have $|v(\geq c_0/8) \cap B_{1/4}| \geq \eta$. We can apply (6.5) and get for a universal constant $C_1$

$$\inf_{B_{1/2}} (K - u_{ee}) \geq C_1 > 0$$

By definition of $K$, there is one $j$, $1 \leq j \leq N$, such that

$$K - u_{ee}(x_j) < 3\epsilon$$

If we finally take $5\epsilon \leq 5\epsilon_0 \leq C_1$, then $D^2u(B_{1/2}) \cap B_{2\epsilon}(M_{2\epsilon}(M_j))$.

Proof of (6.3). We remember the transformation

$$\omega(x) = 4u(x/2) \quad \text{for } x \in B_1$$

We can normalize $u$ in $B_{1/2}$ to $\omega$ in $B_1$, then continue the above process and get $N - 2$ balls. When we get only one ball, we get the final result.
7 $C^{2,\alpha}$ Perturbation

We skip the $W^{2,p}$ estimate for the perturbation of fully nonlinear equation. The reason is that this part maybe not so important as the $C^{2,\alpha}$ perturbation in the application to the geometry. Recently, Krylov has some new result maybe enlarge the $W^{2,p}$ estimate, we refer the reader to those papers. While $C^{2,\alpha}$ is more complicated, since we cannot hope to get that right for all functions. We don’t know if the concave condition is the best or can we beyond that. Anyway, Caffarelli’s work says if we have that large class of functions which has “good solution”, then the small perturbation is also good.

7.1 Approximation lemma

Before we going on directly to the $C^{2,\alpha}$ estimate we prove an approximation lemma which is also a key point in the proof of $W^{2,p}$ estimate.

**Theorem 7.1.** Let $0 < \epsilon < 1$ and $u$ be a viscosity solution of $F(D^2u, x) = f(x)$ in $B_{8\sqrt{n}}$ such that $\|u\|_{L^{\infty}(B_{8\sqrt{n}})} \leq 1$. Assume that $\|\beta\|_{L^{n}(B_{7\sqrt{n}})} \leq \epsilon$ and that $F(D^2\omega, 0) = 0$ has $C^{1,1}$ interior estimates (with constants $c_e$). Then there exists $h \in C^2(\overline{B_{6\sqrt{n}}})$ and $\varphi \in C(B_{6\sqrt{n}})$ such that

$$\|h\|_{C^{1,1}(B_{6\sqrt{n}})} \leq c(n) c_e \quad \text{for a constant } c(n) \text{ depending only on } n,$$

and

$$\|u - h\|_{L^{\infty}(B_{6\sqrt{n}})} + \|\varphi\|_{L^{n}(B_{6\sqrt{n}})} \leq C(\epsilon^\gamma + \|f\|_{L^{n}(B_{8\sqrt{n}})}),$$

where $C$ is a positive constant depending only on $n, \Lambda, \lambda$ and $c_e$, and $0 < \gamma < 1$ is universal.

**Proof.** Let $h \in C^2(B_{7\sqrt{n}})$ be the solution of

$$\begin{cases}
F(D^2h, 0) = 0 & \text{in } B_{7\sqrt{n}} \\
h = u & \text{on } \partial B_{7\sqrt{n}}
\end{cases} \quad (21)$$

Using the $C^{1,1}$ interior estimates and a covering argument, we have that

$$\|h\|_{C^{1,1}(B_{6\sqrt{n}})} \leq c(n) c_e \|u\|_{L^{\infty}(B_{7\sqrt{n}})} \leq c(n) c_e.$$

By interior Hölder estimates and a covering argument, we have that $u \in C^{\alpha}(\overline{B_{7\sqrt{n}}})$ and

$$\|u\|_{C^{\alpha}(B_{7\sqrt{n}})} \leq C(1 + \|f\|_{L^{n}(B_{8\sqrt{n}})}),$$

with $0 < \alpha < 1$ and $C > 0$ universal constants. We use the boundary estimate by lowering the Hölder index.

$$\|h\|_{C^{\alpha/2}(B_{7\sqrt{n}})} \leq C\|u\|_{C^{\alpha}(\partial B_{7\sqrt{n}})} \leq C(1 + \|f\|_{L^{n}(B_{8\sqrt{n}})}).$$
Let $0 < \delta < 1$ and $x_0 \in B_{7\sqrt{n}-\delta}$. Then $B_\delta(x_0) \subset B_{7\sqrt{n}}$ and we can apply interior $C^{1,1}$ estimates in $B_\delta(x_0)$ to $h - h(x_1)$ where $x_1 \in \partial B_\delta(x_0)$, we get

$$\|D^2h(x_0)\| \leq c\delta^{-2}\delta^{\alpha/2}C(1 + \|f\|_{L^\infty(B_{8\sqrt{n}})})$$

These implies that

$$|F(D^2h(x_0), x_0)| \leq C\delta^{\alpha/2}\beta(x_0)(1 + \|f\|_{L^\infty(B_{8\sqrt{n}})}) \quad \text{for } x_0 \in B_{7\sqrt{n}-\delta}$$

Since $u - h = 0$ on $\partial B_{7\sqrt{n}}$ we have,

$$\|u - h\|_{L^\infty(\partial B_{7\sqrt{n}-\delta})} \leq \delta^{\alpha/2}(1 + \|f\|_{L^\infty(B_{8\sqrt{n}})}).$$

Since $h$ is $C^2$ we have $u - h \in S(\lambda/n, \Lambda, \varphi)$ in $B_{7\sqrt{n}}$, with $\varphi(x) = f(x) - F(D^2h(x), x) \in C(B_{7\sqrt{n}})$. By the maximum principle, we have

$$\|u - h\|_{L^\infty(B_{7\sqrt{n}-\delta})} + \|\varphi\|_{L^\infty(B_{7\sqrt{n}-\delta})} \leq \|u - h\|_{L^\infty(\partial B_{7\sqrt{n}-\delta})} + C\|\varphi\|_{L^\infty(B_{7\sqrt{n}-\delta})}$$

$$C[\delta^{\alpha/2} + \delta^{\alpha/2-2}\|\beta\|_{L^\infty(B_{7\sqrt{n}})}(1 + \|f\|_{L^\infty(B_{8\sqrt{n}})}) + C\|f\|_{L^\infty(B_{8\sqrt{n}})}]$$

Recall that $\|\beta\|_{L^\infty(B_{7\sqrt{n}})} \leq \epsilon < 1$. Take $\delta = \epsilon^{1/2}$. We have $\gamma = \alpha/4$.

### 7.2 $C^{2,\alpha}$ Estimates

In this part we study the $C^{2,\alpha}$ regularity of viscosity solutions of

$$F(D^2u, x) = f(x), \quad x \in B_1$$

We first define the oscillation of $F$ in $x$ near the origin

$$\tilde{\beta}(x) = \beta_F(x) = \sup_{M \in \delta} \frac{|F(M, x) - F(M, 0)|}{\|M\| + 1},$$

Since we consider the regularity we can add a paraboloid such that we can assume $F(0,0)=f(0)=0$. And we denotes the adimensional $C^{2,\alpha}(B_d)$ norm:

$$\|\omega\|_{C^{2,\alpha}(B_d)}^* = \|\omega\|_{L^\infty(B_d)} + d\|D\omega\|_{L^\infty(B_d)} + d^2\|D^2\omega\|_{L^\infty(B_d)} + d^{2+\alpha} \sup_{x,y \in B_d} \frac{\|D^2\omega(x) - D^2\omega(y)\|}{|x - y|^\alpha}.$$

**Theorem 7.2.** Assume that $F$ is uniformly elliptic with ellipticity constants $\lambda$ and $\Lambda$, that $F$ and $f$ are continuous in $x$ and that $F(0,0)=f(0)=0$. Suppose that there are constants $0 < \alpha < 1$ and $c_\alpha > 0$ such that for any symmetric matrix $M$ with $F(M, 0) = 0$ and any $\omega_0 \in C(\partial B_1)$, there exists a omega $\in C^2(B_1) \cap C(B_1) \cap C^{2,\alpha}(B_1/2)$ which satisfies

$$\left\{
\begin{array}{ll}
F(D^2\omega(x) + M, 0) = 0 & \text{in } B_1, \\
\omega = \omega_0 & \text{on } \partial B_1
\end{array}
\right.$$

(25)
And
\[ \|\omega\|_{C^{2,\alpha}(B_{1/2})} \leq c_\varepsilon \|\omega\|_{L^\infty(B_1)}. \]

Assume that \( 0 < \alpha < \bar{\alpha}, \ r_0 > 0, \ C_1 > 0, C_2 > 0, \)
\[
\left( \int_{B_{r_0}(0)} \beta^n \right)^{1/n} \leq C_1 r_0^{-\alpha} r^\alpha, \quad \left( \int_{B_r(0)} |f^n| \right)^{1/n} \leq C_2 r_0^{-\alpha} r^\alpha, \quad \forall r \leq r_0 \]  
(26)

and that \( u \) is a viscosity solution of \( F(D^2u, x) = f(x) \) in \( B_{r_0}(0) \). Then \( u \) is \( C^{2,\alpha} \) at the origin; that is there is a polynomial \( P \) of degree 2 such that
\[
\|u - P\|_{L^\infty(B_r)} \leq C r^{2+\alpha} \quad \forall r \leq 1 \]  
(27)

\[
r_0 |DP(0)| + r_0^2 \|D^2P\| \leq C_3 \]  
(28)

\[
C_3 \leq C(\|u\|_{L^\infty(B_{r_0}(0))} + r_0^2(C_2 + 1)) \]  
(29)

and
\[
r_1 = C^{-1} r_0 \]  
(30)

where \( C > 1 \) depends only on \( n, \lambda, \Lambda, c_\varepsilon, \bar{\alpha}, \alpha \) and \( C_1 \).

**Proof.** We will prove that there is a constant \( \delta > 0 \) (small enough) depending only on \( n, \lambda, \Lambda, c_\varepsilon, \bar{\alpha}, \alpha \) such that if \( u \) is viscosity solution of \( F(D^2u, x) = f(x) \) in \( B_1 = B_1(0) \) with \( \|u\|_{L^\infty(B_1)} \leq 1, \)
\[
\left( \int_{B_r(0)} \beta^n \right)^{1/n} \leq \delta r^\alpha \quad \text{and} \quad \left( \int_{B_r(0)} |f^n| \right)^{1/n} \leq \delta r^\alpha \quad \forall r \leq 1, \]  
(31)

then there is a polynomial \( P \) of degree 2 such that
\[
\|u - P\|_{L^\infty(B_r)} \leq Cr^{2+\alpha} \quad \forall r \leq 1 \]  
(32)

and
\[
|DP(0)| + \|D^2P\| \leq C \]  
(33)

for some constant \( C > 0 \) depending only on \( n, \lambda, \Lambda, c_\varepsilon, \bar{\alpha}, \alpha \).

Then the theorem follows from the scaling. Consider
\[
\tilde{u}(y) = \frac{r_1^{-2} u(r_1 y)}{r_1^{-2} \|u\|_{L^\infty(B_{r_0})} + \delta^{-1} C_2} = \frac{r_1^{-2} u(r_1 y)}{K}, \quad y \in B_1 \]
if \( K \geq 1 \)(otherwise consider \( \tilde{u}(y) = r_1^{-2} u(r_1 y) \)).

Claim: There exists \( 0 < \mu < 1 \) depending only on \( n, \lambda, \Lambda, c_\varepsilon, \bar{\alpha}, \alpha \) and a sequence of polynomials
\[
P_k(x) = a_k + \langle b_k, x \rangle + \frac{1}{2} x^t C_k x \]

such that
\[ F(C_k,0) = 0 \quad \forall k \geq 0 \] (34)

\[ \|u - P_k\|_{L^\infty(B_{\mu_k})} \leq \mu_k(2+\alpha) \quad \forall k \geq 0 \] (35)

and

\[ |a_k - a_{k-1}| + \mu_k^{k-1}|b_k - b_{k-1}| + \mu^{2k-1}\|C_k - C_{k-1}\| \leq 13c_\varepsilon \mu_k^{(k-1)(2+\alpha)} \quad \forall k \geq 0, \] (36)

where \( P_0 = P_{-1} = 0 \). Then since \( \{P_k\} \) converges uniformly in \( B_1 \) to a polynomial \( P \) of degree 2 which clearly satisfies (33) If we have proved the claim is true then we as follows to show the (32)

\[ \|u - P\|_{L^\infty(B_{\mu_k})} \leq \|u - P_k\|_{L^\infty(B_{\mu_k})} + \sum_{i=k+1}^{\infty} \|P_i - P_{i-1}\|_{L^\infty(B_{\mu_k})} \]

\[ \leq \mu_k^{(2+\alpha)} + \sum_{i=k+1}^{\infty} \left[ |a_i - a_{i-1}| + \mu_k^{i}|b_i - b_{i-1}| + \frac{1}{2} \mu^{2k}\|C_i - C_{i-1}\| \right] \]

\[ \leq \mu_k^{(2+\alpha)} + 13c_\varepsilon \sum_{i=k+1}^{\infty} \left[ \mu^{(i-1)(2+\alpha)} + \mu^{k}\mu_i^{(i-1)(1+\alpha)} + \mu^{2k}\mu_i^{(i-1)\alpha} \right] \]

\[ \leq C\mu_k^{(2+\alpha)} \] (40)

We turn to the claim now. We need take the \( \mu \) carefully such that we can have the quantities related to \( \mu \) is small enough to allow the solution have some property (remember that we are doing perturbation). We take \( \mu \) small enough (depending on \( c_\varepsilon, \alpha, \bar{\alpha} \)), such that

\[ \mu^{\alpha} \leq \frac{1}{2}, \mu \leq \frac{7}{16}, 28c_\varepsilon\mu^{\bar{\alpha}} \leq \mu^{\alpha}. \]

We take \( \varepsilon \in (0,1) \) such that

\[ 2C\varepsilon^\gamma \leq c_\varepsilon\mu^{2+\bar{\alpha}} \]

where \( C \) and \( \gamma \) is the constants in the (7.1). And we take \( \delta \) (for which (31)) such that

\[ 2(1 + 26c_\varepsilon\delta(c(n))\delta = \varepsilon \]

(34)(35)(36) hold for \( k = 0 \), since \( P_0 = P_{-1} = 0 \), \( F(0,0) = 0 \) and \( \|u\|_{L^\infty(B_1)} \leq 1 \). We now assume that they hold for \( k \leq i \) and we prove them for \( k = i + 1 \), where \( i \geq 0 \). Consider

\[ v(y) = \frac{(u - P_i)(\mu^i y)}{\mu^{(i+2)\alpha}}, \quad y \in B_1 \]

Which satisfies \( \mu^{-\alpha}F(\mu^{i\alpha}D^2v(y) + C_i, \mu^i y) = \mu^{-\alpha}f(\mu^i y) \), and hence \( F_i(D^2v, y) = f_i(y) \) in \( B_1 \), where

\[ F_i(M, y) = \frac{1}{\mu^{i\alpha}}[F(\mu^{i\alpha}M + C_i) - F(C_i, \mu^i y)], \] (41)

\[ f_i(y) = \frac{1}{\mu^{i\alpha}}[f(\mu^i y) - F(C_i, \mu^i y)]. \] (42)
We apply the rescaled (7.1) to v. We point out that $F_i(D^2\omega,0) = 0$ a $C^{2,\alpha}$ interior estimates (with constant $c_e$).
Also we have
\[
\tilde{\beta}_F(y) = \sup_M \left| \frac{F(\mu^i M + C_i, \mu^i y) - F(\mu^i M + C_i, 0)}{\mu^i (\|M\| + 1)} - \frac{F(C_i, \mu^i y) - F(C_i, 0)}{\mu^i (\|M\| + 1)} \right|
\]
\[
\leq \sup_M \left( \frac{\|\mu^i M + C_i\| + 1 + \|C_i\| + 1}{\|M\| + 1} \right) \tilde{\beta}(\mu^i y)
\]
\[
\leq 2(1 + 26c_e)\mu^{-\alpha}\tilde{\beta}(\mu^i y),
\]
Since $\|C_i\| \leq \sum_{k=1}^i \|C_k - C(k - 1)\| \leq 13c_e \sum_{k=1}^\infty \mu^{(k-1)\alpha} = 13c_e(1 - \mu^\alpha)^{-1} \leq 26c_e$.
\[
\frac{\tilde{\beta}_F}{\mu^i} \|L^n(B_1) \leq 2(1 + 26c_e)\mu^{-\alpha} \mu^{-\epsilon} \|\tilde{\beta}\|_{L^n(B_1)} \leq 2(1 + 26c_e)c(n)\delta = \epsilon;
\]
\[
\frac{\tilde{\beta}_F}{\mu^i} \|L^n(B_1) \leq \mu^{-\alpha} \mu^{-\epsilon} \end{equation}
\begin{equation}
\|f\|_{L^n(B_0)} \leq 1 + 26c_e \|\tilde{\beta}\|_{L^n(B_1)} \leq c(n)\delta(1 + 1 + 26c_e) \leq \epsilon
\end{equation}
The (7.1) gives the existence of $h \in C^2 B_{3/4}$ such that
\[
\|v - h\|_{L^\infty(B_{1/2})} \leq C(\epsilon^\gamma + \epsilon) \leq 2C\epsilon^7
\]
And h satisfies
\[
\begin{cases}
F_i(D^2h, 0) = 0 & \text{in } B_{7/8}, \\
h = v & \text{on } \partial B_{7/8}
\end{cases}
\]
Hence
\[
F(\mu^i D^2 h, C_i, 0) \text{ in } B_{7/8},
\]
\[
\|h\|_{C^{2,\alpha}(\tilde{B}_{7/16})} \leq c_e \|v\|_{L^\infty(\partial B_{7/8})} \leq c_e.
\]
Hence, since $\mu < 7/16$,
\[
\|h - \bar{P}\|_{L^\infty(B_\mu)} \leq c_e \left( \frac{16}{7} \right)^{2+\alpha} \leq 27c_e \mu^{2+\alpha},
\]
where
\[
\bar{P}(y) = h(0) + \langle \nabla h(0), y \rangle + \frac{1}{2} y^t D^2 h(0) y
\]
It follows that
\[
\|v - \bar{P}\|_{L^\infty(B_\mu)} \leq \mu^{2+\alpha}
\]
rescaling back
\[
|u(x) - P_t(x) - \mu^{(2+\alpha)} \bar{P}(\mu^{-i} x)| \leq \mu^{(1+1)(2+\alpha)} \forall x \in B_{\mu^{i+1}}
\]
also $C_{i+1} = C_i + \mu^i D^2 h(0)$ and hence $F(C_{i+1}, 0) = 0$. Finally
\[
|a_{i+1} - a_i| + \mu^i |b_{i+1} - b_i| + \mu^{2i} |C_{i+1} - C_i|
\]
\[
= \mu^{(2+\alpha)} (\|h(0)\| + \|\nabla h(0)\| + \|D^2 h(0)\|)
\]
\[
\leq 13c_e \mu^{(2+\alpha)}
\]
8 Conclusion

This thesis is mainly based on [CC], even can be viewed as a copy of that book. What is elliptic theory? In my view just one sentence: knowing the boundary and try to interpret the interior information. What is the regularity theory? In my view is that the oscillation is being controlled. What is the ABP maximal principle's advantage? In my view is that the oscillation is controlled by integral quantity instead of the pointwise quantity.

Why we focus on elliptic theory? I don’t know. In a short talk with Caffarelli I know that elliptic has some sense stability as the critical point is a minimum. That is to say we can try to get regularity. This property isn’t being shared by other types.

How to develop the elliptic theory? I don’t know. But I think we still need to find new problem and observe new things. The fully nonlinear and the degenerate maybe are the finally fields in elliptic theory. But after elliptic we may want to know the saddle critical’s PDE aspects.

References


Mirzakhani’s Volume Recursion and Witten-Kontsevich Theorem

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Abstract

This note is an exposition of M. Mirzakhani’s work [11] and [10] which gave a new proof of Witten-Kontsevich theorem on the intersection numbers of the $\psi$-classes on the moduli spaces of stable curves.

Keywords: moduli space of curves; Weil-Petersson Form; Witten’s Conjecture

1 Introduction

In his famous 1991 paper [12] E. Witten conjectured that a generating function of the intersection numbers of the $\psi$-classes on the Deligne-Mumford compactification of moduli space of curves should be related to the Korteweg-de Vries hierarchy (KdV hierarchy), namely, it is the $\tau$-function of the KdV hierarchy with initial value $(d/dx)^2 + 2x$. The first proof was given by M. Kontsevich in [6], using the interpretation of the moduli space concerned, in rough terms, as a moduli space of ribbon graphs (or fat graphs). An exposition of these concepts and Kontsevich’s proof can be found in [7]. In this note we will follow [10] to state the Witten conjecture in such a form that the concepts of KdV hierarchy and $\tau$-function will not be involved. See Sect. 3.1.

The theme of this note is M. Mirzakhani’s work [11, 10], which gave a new proof of Witten-Kontsevich Theorem. The key idea of Mirzakhani was to cut open a hyperbolic surface along a closed geodesic to decompose the surface into simpler ones. Thus it is natural to consider the moduli space $\mathcal{M}_{g,n}(L)$ of hyperbolic surfaces with geodesic boundary components and prescribed boundary lengths, which generalizes the (open) moduli space of nonsingular curves $\mathcal{M}_{g,n}$. The definitions of these concepts will be given in Sect. 2. By a classification of simple geodesics on a hyperbolic surface with geodesic boundaries, Mirzakhani obtained a generalization of MacShane’s identity [9] satisfied by the lengths of certain geodesics on such a surface, which will be discussed in Sect. 5. Using this identity Mirzakhani managed to get recursion formulae for the Weil-Petersson volumes $V_{g,n}(L) := \text{Vol} \mathcal{M}_{g,n}(L)$ of the various moduli spaces. See Sect. 3.2. On the other hand, by an application of Duistermaat-Heckman Theorem Mirzakhani showed that $V_{g,n}(L)$ is in fact a polynomial in $L$ with coefficients equal to the intersection numbers of certain
tautological classes over $\mathcal{M}_{g,n}$. Thus the recursion formulae obtained give rise to relations satisfied by these intersection numbers, and thus imply Witten-Kontsevich Theorem.

2 Moduli Spaces of Bordered Riemann Surfaces and Weil-Petersson Geometry

2.1 $\mathcal{M}_{g,n}$ and its Weil-Petersson Form

Let $\mathcal{M}_{g,n}$ be the moduli space of nonsingular complex algebraic curves. $\mathcal{M}_{g,n}$ is endowed with a natural symplectic structure, given by the Weil-Petersson form $\omega_{WP}$ [5]. As the moduli space $\mathcal{M}_{g,n}$ can be obtained as a quotient $\mathcal{T}_{g,n}/\text{Mod}_{g,n}$ of the Teichmüller space $\mathcal{T}_{g,n}$ by the mapping class group $\text{Mod}_{g,n}$. $\omega_{WP}$ can be expressed in the Fenchel-Nielsen coordinates of $\mathcal{T}_{g,n}$. The following simple formula was proved by Wolpert in [13].

$$\omega_{WP} = \sum dl_j \wedge d\tau_j,$$

where $(l_j, \tau_j)$ are the length and twist parameters with respect to a pair-of-pants decomposition of the marking surface. These concepts will be explained in detail in the next subsection. Note that it is not obvious that the expression on the right-hand-side should be invariant under $\text{Mod}_{g,n}$, which is generated by all the Dehn twists, not necessarily along the circles in the chosen pair-of-pants decomposition. This formula is fundamental in studying the Hamiltonian geometry of $\mathcal{M}_{g,n}$, as it shows that the length of a closed geodesic is a Hamiltonian potential for the vector field induced by the infinitesimal Fenchel-Nielsen twist along that geodesic. These terms will be made more precise in what follows. Moreover, in [14] Wolpert showed that $\omega_{WP}$ can be extended to $\overline{\mathcal{M}}_{g,n}$ so that $\omega_{WP}/2\pi^2$ represents a cohomology class in $H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$. (This is indeed the first kappa class $\kappa_1$.)

2.2 Generalization to the Bordered Case

Now we generalize the setting to include hyperbolic surfaces with geodesic boundaries and give precise definitions. Let $S$ be a smooth manifold that is a finite disjoint union of closed surfaces, and $P = \{p_i\}$ a finite set of points on $S$, such that each connected component of $S - P$ has negative Euler characteristic. Define a bordered Riemann surface of type $(S, P)$ to be a complete hyperbolic surface with $n = \# P$ boundary components, each of which is a closed geodesic, such that the interior of the surface is diffeomorphic to $S - P$. Here by abuse of language we count a cusp of a hyperbolic surface as a geodesic boundary component of length 0. Let $X$ be a bordered Riemann surface of type $(S, P)$, define a marking of $X$ to be a map $\phi : S - P \rightarrow X$ that is a diffeomorphism onto the interior $X - \partial X$. By abuse of notation we denote by $\phi(p_i)$ the boundary component (or cusp) of $X$ which $\phi(q)$ approaches as $q$ approaches $p_i$ on $S$. For $L = (L_{p_1}, \ldots, L_{p_n}) \in \mathbb{R}^P$, consider the set

$$S(S, P; L) = \{[\phi : S - P \rightarrow X] \mid X \text{ is a bordered Riemann surface of type } (S, P), \phi \text{ is a marking for } X, \text{ such that } \phi(p_i) \text{ has length } L_i, \}.$$
where we define the equivalence relation \([\phi : S - P \to X] = [\psi : S - P \to Y]\) if and only if there exists an isometry \(f : X \to Y\) such that \(\psi = f \circ \phi\). The group \(\text{Diff}_+(S, P)\) of orientation-preserving auto-diffeomorphisms of \(S\) that fix each \(p_i \in P\) acts on \(S(S, P; L)\) by precomposing with the markings. Define the Teichmüller space of bordered Riemann surfaces of type \((S, P)\) with prescribed boundary lengths \(L \in \mathbb{R}_{\geq 0}^n\) to be

\[
\mathcal{T}(S, P; L) := S(S, P; L) / \text{Diff}^0(S, P),
\]

where \(\text{Diff}^0(S, P) \subset \text{Diff}_+(S, P)\) is the component of identity. Define the moduli space of bordered Riemann surfaces of type \((S, P)\) with prescribed boundary lengths \(L \in \mathbb{R}_{\geq 0}^n\) to be

\[
\mathcal{M}(S, P; L) := S(S, P; L) / \text{Diff}_+(S, P).
\]

Thus we have \(\mathcal{M}(S, P; L) = \mathcal{T}(S, P; L) / \text{Mod}(S, P)\) where

\[
\text{Mod}(S, P) := \text{Diff}_+(S, P) / \text{Diff}^0(S, P) = \pi_0(\text{Diff}_+(S, P))
\]

is the mapping class group.

Obviously if \((S, P) = \bigsqcup_{j=1}^k (S^{(j)}, P^{(j)})\), and \(L = (L^{(j)})_{1 \leq j \leq k} \in \mathbb{R}^P = \bigsqcup_{j=1}^k \mathbb{R}^{P^{(j)}}\), then

\[
\mathcal{T}(S, P; L) = \prod_{j=1}^k \mathcal{T}(S^{(j)}, P^{(j)}; L^{(j)}),
\]

\[
\mathcal{M}(S, P; L) = \prod_{j=1}^k \mathcal{M}(S^{(j)}, P^{(j)}; L^{(j)}).
\]

We observe that the spaces \(\mathcal{T}(S, P; L)\) and \(\mathcal{M}(S, P; L)\) only depend on the diffeomorphic class of \(S - P\), thus it makes sense to introduce the following notations. For \(2g - 2 + n > 0\), \(S\) a genus \(g\) closed surface, and \(P\) a set of \(n\) distinct points on \(S\), let

\[
\mathcal{T}_{g,n}(L) := \mathcal{T}(S, P; L),
\]

\[
\mathcal{M}_{g,n}(L) := \mathcal{M}(S, P; L),
\]

and also

\[
\text{Mod}_{g,n} := \text{Mod}(S, P).
\]

In such a case we will also use the term bordered Riemann surface of type \((g, n)\) in place of bordered Riemann surface of type \((S, P)\) and use the symbol \(S_{g,n}\) to denote \(S - P\).

**Remark 2.1.** In view of (2.1) and (2.2) it suffices to study \(\mathcal{T}_{g,n}(L)\) and \(\mathcal{M}_{g,n}(L)\), however it is convenient to treat more arbitrary pairs \((S, P)\).

**Remark 2.2.** When \(L = 0 \in \mathbb{R}^n\), \(\mathcal{T}_{g,n}(0) = \mathcal{T}_{g,n}, \mathcal{M}_{g,n}(0) = \mathcal{M}_{g,n}\), by the equivalence of complex structure and hyperbolic structure on surfaces of negative Euler characteristic.
### 2.3 Weil-Petersson Geometry

The Teichmüller space $\mathcal{T}(S,P; L)$ also has Fenchel-Nielsen coordinates. Let $P = \{\gamma_i\}_{1 \leq i \leq m}$ be a pair-of-pants decomposition of $S - P$, that is, $\gamma_i$ are mutually disjoint embedded circles in $S - P$ such that $S - P - P$ has each connected component a pair of pants (i.e., a 2-sphere minus three points). For any embedded circle $\gamma$ in $S - P$, we define the function

$$l_\gamma : \mathcal{T}(S,P; L) \rightarrow \mathbb{R}_{>0}$$

(2.3)

$$[\phi : S - P \rightarrow X] \mapsto \text{the length of the unique geodesic in the isotopy class of } \phi(\gamma) \text{ on } X.$$ 

Thus we have a map

$$l_P = (l_{\gamma_1}, \ldots, l_{\gamma_m}) : \mathcal{T}(S,P; L) \rightarrow \mathbb{R}_{>0}^m.$$ 

(2.4)

A standard result is that this map is an $\mathbb{R}^m$-principal bundle, where the action of $\mathbb{R}^m$ on each fibre is given by the Fenchel-Nielsen shearing along the $\gamma_i$'s. In particular we have a non-canonical isomorphism of real analytic manifolds

$$(l_i, \tau_i)_{1 \leq i \leq m} : \mathcal{T}_g(L) \rightarrow \mathbb{R}_{>0}^m \times \mathbb{R}^m.$$ 

(2.5)

$(l_i, \tau_i)_{1 \leq i \leq m}$ are called the Fenchel-Nielsen coordinates of $\mathcal{T}(S,P; L)$ with respect to the pair-of-pants decomposition $\mathcal{P}$, although $\tau_i$ cannot be chosen in a canonical way. Thus $\tau$ measures the shearing in terms of arc length.

**Remark 2.3.** By counting the Euler characteristic it is easy to compute the cardinal $m$ of a pair-of-pants decomposition $\mathcal{P}$ of $S - P$: Let $n = \# P$. The number of connected components of $S - (\mathcal{P} \cup P)$ is $(2m + n)/3$. Thus the Euler characteristic of $S - P$ is

$$\chi(S - P) = -\frac{2m + n}{3} = \chi(S) - n.$$ 

Thus $m = n - \frac{3}{2}\chi(S)$. In particular if $S$ is a genus $g$ closed surface then $m = 3g - 3 + n$.

Now over $\mathcal{T}(S,P; L)$ is defined a symplectic form $\omega_{WP}$ invariant under $\text{Mod}(S,P)$, called the Weil-Petersson form, which has the form [11]

$$\omega_{WP} = \sum dl_j \wedge d\tau_j$$

(2.6)

under the Fenchel-Nielsen coordinates $(l_j, \tau_j)$. By the invariance under $\text{Mod}(S,P)$, $\omega_{WP}$ descends to a symplectic form in the orbifold sense over $\mathcal{M}(S,P; L)$, which will still be denoted by $\omega_{WP}$ and called the Weil-Petersson Form. Note that (2.6) implies that the products (2.1) and (2.2) are actually compatible with the Weil-Petersson forms defined on the various spaces, since a pair-of-pants decomposition for $S$ always comes from those for the connected components of $S$. We define the **Weil-Petersson volume form** to be $\omega_{WP}^m/m!$ and the **Weil-Petersson volume** of $\mathcal{M}(S,P; L)$ to be

$$V(S,P; L) = \text{Vol}(\mathcal{M}(S,P; L)) := \frac{1}{m!} \int_{\mathcal{M}(S,P; L)} \omega_{WP}^m.$$
We also use the notations $V_{g,n}(L) := \text{Vol}(\mathcal{M}_{g,n}(L))$, and $V_{g,n} := \text{Vol}(\mathcal{M}_{g,n})$. By what has been said and the following remark, the volume is multiplicative under the product (2.2). Here we set by convention $V_{0,3}(L) = 1$ since $\mathcal{M}_{0,3}(L)$ consists of only one point.

Remark 2.4. The factor $1/m!$ is a convention in symplectic geometry. It has the effect that if a symplectic form $\omega = \sum_{i=1}^{m} dp_i \wedge dq_i$ on a $2m$-dimensional manifold is in its Darboux form, then $\omega^m/m! = \prod_{i=1}^{m} dp_i \wedge dq_i$.

Remark 2.5. A remark on the finiteness of $V_{g,n}(L)$. Bers observed that there exists a constant $b_{g,n} > 0$ for each $g, n \in \mathbb{Z}_{\geq 0}, 2g - 2 + n > 0$, such that every bordered Riemann surface of type $(g, n)$ has a pair of pants decomposition each constituent of which is a geodesic of length $\leq b_{g,n}$. Consider the action of $\text{Mod}_{g,n}$ on $\{ P | P$ is an isotopy class of pair of pants decompositions of $S_{g,n} \}$. The orbits $\mathcal{O}_i$ are indexed by the isomorphism classes of trivalent graphs with $n$ half edges of genus $g$, hence are finite in number. Pick $P_i \in \mathcal{O}_i$ for each $i$ and let $(l^{(i)}, \tau^{(i)}_j)$ be the Fenchel-Nielsen coordinate of $T_{g,n}(L)$ with respect to $P_i$. Then each $\text{Mod}_{g,n}$-orbit in $T_{g,n}(L)$ intersects with the set $\Delta = \bigcup_i \left\{ l^{(i)}_j \leq b_{g,n}, 0 \leq \tau^{(i)}_j < l^{(i)}_j \forall j \right\}$ for a positive number of times. This observation together with formula (2.6) implies that $V_{g,n} < +\infty$.

3 Mirzakhani’s Main Results and Witten-Kontsevich Theorem

In this section we will first state Witten-Kontsevich Theorem. We will also state Mirzakhani’s main results (Theorem 1.1 and Section 5 in [11], Corollary 3.3 in [10]) and show that they imply Witten-Kontsevich, while the proof of these results will be presented in the subsequent sections.

3.1 Statement of Witten-Kontsevich Theorem

Let $\overline{\mathcal{M}}_{g,n}$ be the Deligne-Mumford compactification of $\mathcal{M}_{g,n}$, i.e. it is the moduli space of $n$-pointed genus $g$ stable curves [4]. Let $\mathcal{L}_i, 1 \leq i \leq n$ be the tautological line bundle over $\overline{\mathcal{M}}_{g,n}$, whose fibre at $X \in \overline{\mathcal{M}}_{g,n}$ is the cotangent space to $X$ at the $i$th marked point. Let $\psi_i$ be its first Chern class. For any set $\{ d_1, \ldots, d_n \}$ of integers define the symbol

$$\langle \tau_{d_1}, \ldots, \tau_{d_n} \rangle_{g} := \int_{\overline{\mathcal{M}}_{g,n}} \psi_{1}^{d_1} \cdots \psi_{n}^{d_n}. \quad (3.1)$$

This is understood to be zero if $\sum d_i \neq 3g - 3 + n$. Define a generating function of these intersection numbers

$$F(\lambda, t_0, t_1, \ldots) := \sum_{g=0}^{\infty} \lambda^{2g-2} \sum_{(d_1,d_2,\ldots) \in \mathbb{Z}_{\geq 0}^{\infty}} \langle \prod \tau_{d_j} \rangle_{g} \prod_{r\geq0} t_{r}^{n_r}/n_r!.$$
where $\mathbb{Z}_0^\infty$ is the set of sequences of integers with all but finitely many coefficients zero, and $n_r := \# \{ j : d_j = r \}$. Define a sequence of differential operators $L_{-1}, L_0, L_1, \ldots, L_n, \ldots$ as follows,

$$L_{-1} = \frac{\partial}{\partial t_0} + \frac{\lambda - 2}{2} t_0^2 + \sum_{i=1}^\infty t_{i+1} \frac{\partial}{\partial t_i},$$

$$L_0 = \frac{3}{2} \frac{\partial}{\partial t_1} + \sum_{i=1}^\infty \frac{2i + 1}{2} \frac{\partial}{\partial t_i} + \frac{1}{16},$$

and for $n \geq 1$,

$$L_n = -\left( \frac{2n + 3}{2^{n+1}} \right) \frac{\partial}{\partial t_{n+1}} + \sum_{i=0}^\infty \frac{(2i + 2n + 1)!}{(2i - 1)! 2^{n+1} i!} \frac{\partial}{\partial t_{i+n}}
+ \frac{\lambda^2}{2} \sum_{i=0}^{n-1} \frac{(2i + 1)!(2n - 2i - 1)!}{2^{n+1}} \frac{\partial^2}{\partial t_i \partial t_{n-i-1}}.$$  

Now we can state Witten-Kontsevich Theorem as follows.

**Theorem 3.1** (Witten-Kontsevich [10]). For $n \geq -1$, $L_n(e^F) = 0$, i.e.

$$0 = -\left( \frac{2n + 3}{2^{n+1}} \right) \frac{\partial F}{\partial t_{n+1}} + \sum_{i=0}^\infty \frac{(2i + 2n + 1)!}{(2i - 1)! 2^{n+1} i!} \frac{\partial F}{\partial t_{i+n}}
+ \frac{\lambda^2}{2} \sum_{i=0}^{n-1} \frac{(2i + 1)!(2n - 2i - 1)!}{2^{n+1}} \left( \frac{\partial^2 F}{\partial t_i \partial t_{n-i-1}} + \frac{\partial F}{\partial t_i} \frac{\partial F}{\partial t_{n-i-1}} \right).$$  

**(Remark 3.2.)** The original version of Witten’s conjecture involves KdV hierarchy and its $\tau$ function, and Theorem 3.1 is a consequence of that. However, as it can be shown that the relations in Theorem 3.1 actually determine all the intersection numbers $\langle \tau_{d_1}, \ldots, \tau_{d_n} \rangle_g$, Theorem 3.1 is actually equivalent to the original version.

### 3.2 Mirzakhani’s Main Results

**Theorem 3.3.** For $2g - 2 + n > 0$, $L = (L_1, \ldots, L_n) \in \mathbb{R}_{\geq 0}^n$, $V_{g,n}(L)$ has the form

$$V_{g,n}(L) = \sum_{|\alpha| \leq 3g - 3 + n} C_\alpha^g L^{2\alpha},$$  

(3.3)

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi index and $0 < C_\alpha \in \pi^{6g-6+2n-2|\alpha|} \mathbb{Q}$. Moreover,

$$C_\alpha^g = \frac{2^{m(g,n)}}{2^{n|\alpha|}(3g - 3 + n - |\alpha|)!} \int_{\mathcal{M}_{g,n}} \psi^\alpha \omega^{3g-3+n-|\alpha|},$$  

(3.4)

where $\omega = \omega_{WP}$ is the Weil-Petersson form, $\psi = (\psi_1, \ldots, \psi_n)$ are the $\psi$-classes as introduced in Section 3.1 and $m(i,j) := \delta_{1,i} \delta_{1,j}$ is equal to 1 if and only if $i = j = 1$ and 0 otherwise.
Remark 3.4. Making use of the transcendence of $\pi$ over $\mathbb{Q}$ it is convenient to express the fact $0 < C_\alpha \in \pi^{6g-6+2n-2|\alpha|} \mathbb{Q}$ in saying that $V_{g,n}(L)$ is a homogeneous polynomial of degree $3g - 3 + n$ with positive rational coefficients in the variables $L_i^2$ and $\pi^2$, where we view $\pi$ as having the same homogeneous degree as $L_i$ in $\mathbb{Q}[L, \pi]$.

Theorem 3.5 (Mirzakhani’s Recursion Formulae, Section 5 in [11]). $V_{0,3}(L) = 1$ by convention, and $V_{1,1}(L) = \frac{L^2}{24} + \frac{x^2}{6}$. Let $g \in \mathbb{Z}_{\geq 0}, n \in \mathbb{N}, 2g - 2 + n > 0, (g, n) \neq (0, 3)$ or $(1, 1), L = (L_1, \ldots, L_n) \in \mathbb{R}_{\geq 0}^n, \hat{L} := (L_2, \ldots, L_n)$. Then $V_{g,n}(L)$ satisfies the following relation.

$$\frac{\partial}{\partial L_1} L_1 V_{g,n}(L) = A_{g,n}^{con}(L_1, \hat{L}) + A_{g,n}^{dcon}(L_1, \hat{L}) + B_{g,n}(L_1, \hat{L}),$$

where

$$A_{g,n}^{con}(L_1, \hat{L}) := \frac{1}{2} \int_0^{+\infty} \int_0^{+\infty} xy \hat{A}_{g,n}^{con}(x, y, L_1, \hat{L}) dx dy,$$

$$A_{g,n}^{dcon}(L_1, \hat{L}) := \frac{1}{2} \int_0^{+\infty} \int_0^{+\infty} xy \hat{A}_{g,n}^{dcon}(x, y, L_1, \hat{L}) dx dy,$$

$$B_{g,n}(L_1, \hat{L}) := \int_0^{+\infty} x \hat{B}_{g,n}(x, L_1, \hat{L}) dx,$$

$$\hat{A}_{g,n}^{con}(x, y, L_1, \hat{L}) := V_{g-1,n+1}(x, y, \hat{L})H(x + y, L_1),$$

$$\hat{A}_{g,n}^{dcon}(x, y, L_1, \hat{L}) := \sum_{a \in I_{g,n}} V(a, x, y, \hat{L})H(x + y, L_1),$$

$$I_{g,n} := \{ ((g_1, I_1), (g_2, I_2)) | g_1 + g_2 = g, \{2, \ldots, n\} = I_1 \sqcup I_2, 2g_j - 2 + \#I_j \geq 0 \},$$

$$V(a, x, y, \hat{L}) := \frac{V_{g_1,n_1+1}(x, L_{i_1})}{2^{m(g_1, n_1+1)}} \times \frac{V_{g_2,n_2+1}(x, L_{j_1})}{2^{m(g_2, n_2+1)}}$$

for $a = ((g_1, I_1), (g_2, I_2)) \in I_{g,n}$,

$$L_{\{i_1, \ldots, i_k\}} := (L_{i_1}, \ldots, L_{i_k})$$

$$\hat{B}_{g,n}(x, L_1, \hat{L}) := \frac{1}{2^{m(g, n-1)}} \sum_{i=2}^{n} \left( H(x, L_1 + L_j) + H(x, L_1 - L_j) \right) V_{g,n-1}(x, L_2, \ldots, \hat{L}_j, \ldots, L_n)$$

$$H(x, y) := \frac{1}{1 + e^{\frac{xy}{2}}} + \frac{1}{1 + e^{\frac{-xy}{2}}}.$$
Remark 3.6. The formulae (3.5) actually determine $V_{g,n}(L)$ for all permitted $(g,n)$. Since all the $V_{g',n'}(L')$ appearing on the right hand side of (3.5) have $2g' - 2 + n' < 2g - 2 + n$, and only $(g,n) = (0,3)$ or $(1,1)$ will produce $2g - 2 + n = 1$.

3.3 Computational Aspects of the Recursion Formulae

We develop some formulae to facilitate computations in the recursion formulae in Theorem 3.5. In particular we show that Theorem 3.5 already implies the polynomial nature of $V_{g,n}(L)$ (3.3) by induction on $2g - 2 + n$. Note that $V_{0,3}(L)$ and $V_{1,1}(L)$ are of the form (3.3). For the induction step, we are interested in integrals of the forms

$$F_i(t) = \int_0^{+\infty} x^i H(x,t) dx$$

and

$$F_{i,j}(t) = \int_0^{+\infty} \int_0^{+\infty} x^i y^j H(x+y,t) dx dy,$$

for $i,j \geq 0$. We compute

$$F_{i,j}(t) = \int_0^{+\infty} \int_0^{+\infty} x^i y^j H(x+y,t) dx dy$$

$$= \int_0^{+\infty} du \int_0^{+\infty} dy (u-y)^i y^j H(u,t)$$

$$= B(i+1,j+1) \int_0^{+\infty} u(i+j+1) H(u,t) du$$

$$= B(i+1,j+1) F_{i+j+1}(t),$$

where $B(x,y)$ is Euler’s beta function and

$$B(i+1,j+1) = \frac{i!j!}{(i+j+1)!}.$$
Hence it suffices to compute $F_i(t)$.

\[
F_i(t) = \int_0^{+\infty} x^i H(x, t) dx.
\]

\[
= \int_0^{+\infty} x^i \left( \frac{1}{1 + e^{x+t}} + \frac{1}{1 + e^{x-t}} \right) dx
\]

\[
= \int_0^{+\infty} \frac{(x+t)^i + (x-t)^i}{1 + e^x} dx + \int_0^{t} \frac{(t-x)^i}{1 + e^x} + \frac{(t-x)^i}{1 + e^{-x}} dx
\]

\[
= \int_0^{+\infty} \frac{(x+t)^i + (x-t)^i}{1 + e^x} dx + \int_0^{t} (t-x)^i dx
\]

\[
= \int_0^{+\infty} \frac{(x+t)^i + (x-t)^i}{1 + e^x} dx + \frac{t^{i+1}}{i+1}
\]

\[
= \frac{t^{i+1}}{i+1} + \sum_{0 \leq k \leq i, i-k \text{ even}} 2 \binom{i}{k} t^{i-k} \int_0^{+\infty} \frac{x^k}{1 + e^x} dx. \tag{3.9}
\]

We have

\[
\int_0^{+\infty} \frac{x^k}{1 + e^x} dx = \int_0^{+\infty} \sum_{l=1}^{+\infty} (-1)^{l+1} x^k e^{-lx} dx
\]

\[
= \sum_{l=1}^{+\infty} (-1)^{l+1} k! l^{-k-1}
\]

\[
= (1 - 2^{-k}) k! \zeta(k+1). \tag{3.10}
\]

Hence

\[
F_i(t) = \frac{t^{i+1}}{i+1} + \sum_{0 \leq k \leq i, i-k \text{ even}} 2 \binom{i}{k} t^{i-k} (1 - 2^{-k}) k! \zeta(k+1). \tag{3.11}
\]

Note that for $k \geq 1$ odd, $\zeta(k+1) \in \pi^{k+1} \mathbb{Q}$. Therefore for $i \geq 1$ odd, $F_i(t)$ is a rational coefficient polynomial homogeneous in $t$ and $\pi$ of degree $i+1$. In the induction step of proving (3.3), actually only the $F_i(t)$ with odd $t$ are involved. By Theorem 3.5, (3.11) and (3.8), we have proved (3.3) as well as the assertion that $0 < C_\alpha \in \pi^{6g-6+2n-2|\alpha|} \mathbb{Q}$. 

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3.4 Mirzakhani’s Main Results Imply Witten-Kontsevich Theorem

In this subsection we use Theorem 3.3 and Theorem 3.5 to prove Theorem 3.1. For \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_{\geq 0}^n \), define

\[
\Delta^g_\alpha := \begin{cases} 
1, & \text{if } 3g - 3 + n = |\alpha| \\
0, & \text{otherwise}
\end{cases}
\]

Thus the intersection numbers (3.1) and the coefficients in the volume polynomials are related by (3.4) in the following form.

\[
\langle \tau_\alpha \rangle_g := \langle \tau_{\alpha_1} \cdots \tau_{\alpha_n} \rangle_g = \Delta^g_\alpha C^g_{\alpha} \frac{2^{|\alpha|} \alpha! (3g - 3 + n - |\alpha|)!}{2^{m(g,n)}}
\]

(3.12)

By virtue of (3.8) and (3.11), we know that for each \( V_{g',n'}(L') \) appearing on the right hand side of (3.5), only its top coefficients (i.e. \( C^g_{\alpha'} \), with \( \Delta^g_{\alpha'} = 1 \)) will contribute to the top coefficients of \( V_{g,n}(L) \). Explicitly,

\[
(2\alpha_1 + 1)!! \langle \tau_\alpha \rangle = \frac{1}{2} \sum_{i+j=\alpha_1-2} (2i+1)!!(2j+1)!! \sum_{I \subset \{2, \ldots, n\}} \langle \tau_i \tau_{\alpha_I} \rangle \langle \tau_j \tau_{\alpha_{I^c}} \rangle
\]

\[
+ \frac{1}{2} \sum_{i+j=\alpha_1-2} (2i+1)!!(2j+1)!! \langle \tau_i \tau_{\alpha_1} \cdots \tau_{\alpha_n} \rangle
\]

\[
+ \sum_{j=2}^n \frac{(2\alpha_1 + 2\alpha_j - 1)!!}{(2\alpha_j - 1)!!} \langle \tau_{\alpha_2} \cdots \tau_{\alpha_{j-1}} \tau_{\alpha_1 + \alpha_j - 1} \tau_{\alpha_{j+1}} \cdots \tau_{\alpha_n} \rangle
\]

(3.13)

where

\[
\tau_{\alpha_I} := \prod_{i \in I} \tau_{\alpha_i},
\]

and

\[
\langle \cdots \rangle := \sum_{g=0}^{\infty} \langle \cdots \rangle_g.
\]

Now (3.13) implies (3.2).

4 A Computation for \( V_{1,1}(L) \) as a Motivation for the Ideas

In this section we compute \( V_{1,1}(L) \). The result is used as the starting point of the recursion of volume and the computation also illustrates the ideas. The computation is based on the following lemma, which is a special case of Theorem 5.3.

**Lemma 4.1.** Let \( X \) be a bordered Riemann surface of type \((1,1)\), \( L \geq 0 \) be the length of the boundary geodesic. The following holds.

\[
\sum_{\gamma \in \mathcal{F}} D(L, l(\gamma), l(\gamma)) = L
\]

(4.1)
where $F$ is the set of all simple closed geodesics $\gamma$ on $X$ (countably many), $l(\gamma)$ means the length of $\gamma$, and $D(x,y,z): \mathbb{R}^3 \to \mathbb{R}_{\geq 0}$ is an explicitly defined function satisfying
\[ \frac{\partial}{\partial x} D(L,x,x) = \frac{1}{1+e^{-\frac{L}{2}}} + \frac{1}{1+e^{\frac{L}{2}}}. \] (4.2)

Obviously for any simple closed geodesic $\gamma$ on $X$, $X - \gamma$ is of the same diffeomorphic class. Take any marking $\phi: S_1 \to X$, as $\gamma$ runs through $F$, the isotopy class of $\phi^{-1}(\gamma)$ on $S_{1,1}$ exhausts an $\text{Mod}_{1,1}$-orbit without repetition, where $\text{Mod}_{1,1}$ permutes the isotopy classes of closed curves on $S_{1,1}$ naturally. Denote this orbit by $O$.

(4.1) can be interpreted as an equality between two functions defined on $T_{1,1}$:
\[ \sum_{[\gamma] \in O} D(L,l_\gamma,l_\gamma) = L. \] (4.3)

where $l_\gamma: T_{1,1} \to \mathbb{R}_{>0}$ is defined as in (2.3), which only depends on the isotopy class of $\gamma$.

Take any embedded circle $\gamma$ in $S_{1,1}$ such that $S_{1,1} - \gamma$ is connected, then $[\gamma] \in O$. Denote the stabilizer of $[\gamma]$ in $\text{Mod}_{1,1}$ by $\text{Stab}(\gamma)$, then (4.3) becomes
\[ \sum_{\sigma \in \text{Mod}_{1,1}/\text{Stab}(\gamma)} \sigma^* D(L,l_\gamma,l_\gamma) = L. \] (4.4)

Now let
\[ M_{1,1}^\gamma(L) := T_{1,1}(L)/\text{Stab} \gamma, \]
then
\[ M_{1,1}(L) = \frac{M_{1,1}^\gamma(L)}{\text{Mod}_{1,1}/\text{Stab}(\gamma)}. \]

Denote by $\pi^\gamma$ the projection $M_{1,1}^\gamma(L) \to M_{1,1}(L)$. For any function $g: M_{1,1}(L) \to \mathbb{R}$, define its pushforward
\[ \pi^\gamma_* g := \sum_{\sigma \in \text{Mod}_{1,1}/\text{Stab} \gamma} \sigma^* g, \] (4.5)
which is a well-defined function on $M_{1,1}(L)$. We have
\[ \int_{M_{1,1}(L)} \pi_* g = \int_{M_{1,1}^\gamma(L)} g, \] (4.6)

where on both sides we are integrating with respect to the Weil-Petersson volumes $\omega_{WP}^m/m!$ inherited from $T_{1,1}(L)$, where $m = 3g - 3 + n = 1$. In particular, let $g$ be the left-hand-side of (4.4), which is well defined on $M_{1,1}^\gamma(L)$, combining (4.4), (4.5) and (4.6) we have
\[ LV_{1,1}(L) = \int_{M_{1,1}^\gamma(L)} D(L,l_\gamma,l_\gamma) \] (4.7)
and hence by (4.2)
\[
\frac{\partial}{\partial L} LV_{1,1}(L) = \int_{\mathcal{M}_{1,1}(L)} \frac{1}{1 + e^{l - \frac{L}{2}}} + \frac{1}{1 + e^{l + \frac{L}{2}}}. \tag{4.8}
\]

Note that \(\mathcal{P} = \{\gamma\}\) is a pair-of-pants decomposition of \(S_{1,1}\). Let \((l, \tau) = (l_\gamma, \tau_\gamma)\) be the Fenchel-Nielsen coordinates as in (2.5). A fundamental domain of the action of \(\text{Stab}(\gamma)\) on \(\mathcal{T}_{1,1}(L)\) is \(\{(l, \tau) \in \mathbb{R}_+ \times \mathbb{R} \mid 0 \leq \tau < l\}\). Using the \(dl \wedge d\tau\) formula (2.6), we compute

\[
\frac{\partial}{\partial L} LV_{1,1}(L) = \int_{0}^{+\infty} dl \int_{0}^{l} d\tau \frac{1}{1 + e^{l - \frac{L}{2}}} + \frac{1}{1 + e^{l + \frac{L}{2}}}
= \int_{0}^{+\infty} \frac{l}{1 + e^{l - \frac{L}{2}}} + \frac{l}{1 + e^{l + \frac{L}{2}}} dl
= \int_{-\frac{L}{2}}^{+\infty} x + \frac{L}{2} dx + \int_{0}^{+\infty} x - \frac{L}{2} dx
= 2 \int_{0}^{+\infty} \frac{x}{1 + e^{x}} dx + \int_{-\frac{L}{2}}^{0} \frac{x + \frac{L}{2}}{1 + e^{x}} dx - \int_{0}^{+\infty} \frac{x - \frac{L}{2}}{1 + e^{x}} dx
= 2 \int_{0}^{+\infty} \frac{x}{1 + e^{x}} dx + \int_{0}^{\frac{L}{2}} (\frac{L}{2} - x) \left[\frac{1}{1 + e^{x}} + \frac{1}{1 + e^{-x}}\right] dx
= 2 \int_{0}^{+\infty} \frac{x}{1 + e^{x}} dx + \int_{0}^{\frac{L}{2}} (\frac{L}{2} - x) dx
= 2 \int_{0}^{+\infty} \frac{xe^{-x}}{1 + e^{-x}} dx + \frac{L^2}{8} = 2 \int_{0}^{+\infty} x \sum_{k \geq 1} (-1)^{k+1} e^{-kx} dx + \frac{L^2}{8}
= 2 \sum_{k \geq 1} \frac{(-1)^{k+1}}{k^2} \Gamma(2) + \frac{L^2}{8} = 2(\zeta(2) - \frac{1}{2} \zeta(2)) + \frac{L^2}{8}
= \frac{L^2}{8} + \frac{\pi^2}{6}
\]

Hence

\[
V_{1,1}(L) = \frac{L^2}{24} + \frac{\pi^2}{6}. \tag{4.9}
\]

In the next section we will prove Mirzakhani’s identity Theorem 5.9 (Theorem 4.2 in [11]), which will play the same role as (4.1) when computing general \(V_{g,n}(L)\). However, unlike the case here, for large \((g, n)\) it is not easy to find fundamental domains for subgroups
of Mod$_g,n$. A new idea, that is, symplectic reduction, is used to avoid such difficulties as it allows us to reduce the computation for $V_{g,n}$ to the computation of $V_{g',n'}$ for some $(g',n')$ with $2g' - 2 + n' < 2g - 2 + n$, thus the recursion.

5 Mirzakhanı’s Identity

In what follows a geodesic means a complete geodesic.

5.1 Geodesics on a Hyperbolic Pair of Pants

Our object of study is a hyperbolic pair of pants $X$ (a bordered Riemann surface of type $(0,3)$) with geodesic boundary components $a_1,a_2,a_3$. There is a unique geodesic $b_1$ on $X$ joining $a_2$ and $a_3$, which is orthogonal at the intersections. Similarly we have $b_2$ joining $a_1$ and $a_3$ and $b_3$ joining $a_1$ and $a_2$. $X - b_i$ is the union of two hyperbolic rectangular hexagons $X_1$ and $X_2$. By an elementary theorem we know that the a hyperbolic rectangular hexagon is uniquely determined up to isometry by the lengths of its three mutually nonadjacent edges, hence $X_1$ and $X_2$ are isometric, having each $b_i$ as a common edge. In particular each $a_i$ is divided into two arcs of equal length by its intersection points with $b_j$ and $X$ has a canonical isometric involution $j$, mapping $X_1$ isometrically onto $X_2$.

Now we construct the universal cover of $X$. First we embed $X$ into the upper half plane $\mathbb{H}$. Denote the image of $X$ by $Y_1$, and the images of $b_i$ by $B_i$, images of $a_1 \cap X$ by $A_1$. Consider the group $G = \langle g_1,g_2,g_3 | g_1^2 = g_2^2 = g_3^3 = 1, (\mathbb{Z}/2\mathbb{Z}) \ast (\mathbb{Z}/2\mathbb{Z}) \ast (\mathbb{Z}/2\mathbb{Z}) \rangle$. For each $g \in G$, the length of $g$, denoted by $l(g)$, is the minimal integer $k$ for which $g = g_1 g_2 \cdots g_k$, for some $j_i \in \{1,2,3\}, 1 \leq k \leq 3$. By induction on $l(g)$, we define for each $g \in G$ a hexagon $Y_g$ in $\mathbb{H}$ with tree marked edges $B_i^g, 1 \leq i \leq 3$, and $\tau_g \in Aut(\mathbb{H})$.

For $l(g) = 0$, we have $g = 1$, and we define $Y_g = Y_1$, $B_i^g = B_i$, and an element $\tau_g = id$. For general $g \in G$ with $l(g) > 0$, let $g = g_1 g_1'$ where $l(g') < l(g)$. Let $\sigma$ be the M"obius inversion with respect to the circle arc $B_i^g$. We define $Y_g = \sigma Y_{g'}, B_i^g = \sigma B_i^{g'}, 1 \leq j \leq 3$, and $\tau_g = \sigma \circ \tau_{g'}$. Thus we have $Y_g = \tau_g(Y_1)$ and $B_i^g = \tau_g(B_i)$. Define $A_1^g = \tau_g(A_i)$. Let $\tilde{X} = \cup_{g \in G} Y_g$. We define a map $\pi : \tilde{X} \to X$. Define $\pi|_{Y_1}$ to be the inverse of the embedding $X_1 \to Y_1$ and define $\pi|_{Y_g} := j^{l(g)} \circ \pi|_{Y_1} \circ \tau_g^{-1}$, where $j$ is the involution of $Y_g$.

Thus we have defined $\pi : \tilde{X} \to X$, which is easily seen to be a covering map. Since $G$ is a free product, $\tilde{X}$ has the homotopy type of a trivalent tree, and is therefore simply connected. Hence $(\tilde{X}, \pi)$ is the universal cover of $X$.

Remark 5.1. For each $g \in G$ with even length we can define a deck transformation $t_g$ by defining $t_g|_{Y_g} := \tau_{g'} \circ \tau_{g^{-1}}$ for each $g' \in G$. Since these transformations are already transitive on the fibres of $\pi$, we know that $Deck(\tilde{X}/X) = \{ t_g | l(g) \text{ is even} \} \cong \{ g \in G | l(g) \text{ is even} \}$. The fundamental group $\pi_1(X)$ of $X$ is a free group of two generators $\{ h_1, h_2 \}$, which is indeed isomorphic to $Deck(\tilde{X}/X)$ by the map sending $h_1$ to $g_2 g_3$ and sending $h_2$ to $g_3 g_1$.

By the transitive property of $Aut(\mathbb{H})$, we may arrange that

$$A_1 = \left\{ y \sqrt{-1} | 1 \leq y \leq e^{l(a_1)/2} \right\},$$

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intersecting with \(B_3\) and \(B_2\) at \(\tilde{p}_3 = \sqrt{-1} \) and \(\tilde{p}_2 = e^{i(a_1)/2} \sqrt{-1} \) respectively, and that \(Y_1\) lies to the left of the imaginary axis. We will assume this is the case in what follows.

For any \(x \in a_1 \cap X_1 (\bar{x} \in A_1)\), let \(\gamma_x (\gamma_{\bar{x}})\) be the geodesic ray emanating from \(x\) (from \(\bar{x}\) and to the left), orthogonal to \(a_1 \) \((A_1)\). We study its behavior.

Let \(\bar{y}_2 \in A_1\) be such that \(\gamma_{\bar{y}_2}\) is asymptotic to the half circle supporting \(A_2\). Let \(y_2 = \pi(\bar{y}_2)\). We claim that for any \(\bar{x}\) in the open interval \(|\tilde{p}_3, \bar{y}_2|, x = \pi(\bar{x}), \gamma_x \) is simple and will finally touch \(a_2\) nonorthogonally, and \(\gamma_{y_2}\) is simple and asymptotic to \(a_2\). Similarly, we find \(\bar{y}_3 \in A_1\) such that \(\gamma_{\bar{y}_3}\) is asymptotic to the half circle supporting \(A_3\). Let \(p_i = \pi(\bar{p}_i), y_i = \pi(\bar{y}_i), z_i = j(y_i), i = 2, 3,\) then we know that for any \(x\) in the open arc \(y_2p_3z_2\) and \(x \neq p_3, \gamma_x\) is simple and will touch \(a_2\) nonorthogonally, while \(\gamma_{y_2}\) and \(\gamma_{z_2}\) are asymptotic to \(a_2\), spiraling around \(a_2\) in opposite directions, and \(\gamma_{p_3}\) is simple and orthogonal to \(a_2\). Similarly for the arc \(y_3p_2z_3\).

**Proof of Claim.** Denote by \(C(A_1)\) the half circle in \(\mathbb{H}\) circle supporting \(A_1\), etc. We may apply a transformation of \(\mathbb{H}\) to arrange that \(C(B_i) = \{ z \in \mathbb{H} | |z| = r_i \}, i = 2, 3, r_2 < r_3\) and \(C(A_2) = \sqrt{-1} \mathbb{R}_{>0}\). We may also arrange that \(A_1\) lies in the region \(\{ r_2 < |z| < r_3 \}\), that \(\gamma_{\bar{y}_2} = x_0 + \sqrt{-1} \mathbb{R}_{>0}, -r_3 < x_0 < -r_2\) and that the direction of \(\gamma_{\bar{y}_2}\) points upward. Define a new curve \(\gamma\) in the region \(\{ r_2 \leq |z| \leq r_3 \}\) by reflecting \(\gamma_{\bar{x}}\) along \(C(B_i)\) every time it touches it. \(i = 2, 3\). It is obvious that \(\gamma\) has consecutive portions as a portion of a vertical line and a portion of a circle arc whose supporting circle passes through the origin. In particular \(\gamma\) is simple and is asymptotic to \(C(A_2)\). This shows the desired property of \(\gamma_{\bar{y}_2}\).

The behavior of \(\gamma_x\) for \(x = \pi(\bar{x}), \bar{x} \in [\tilde{p}_3, \bar{y}_2]\) is similar.

We next find \(\bar{w}_1 \in A_1\) such that \(\gamma_{\bar{w}_1}\) is orthogonal to \(\gamma_{g_1}A_1\), and let \(w_1 = \pi(\bar{w}_1)\), then \(\gamma_{w_1}\) is simple and touches \(a_1\) orthogonally at \(j(w_1)\). We claim that for any \(\bar{x}\) in the open interval \([\bar{y}_2, \bar{y}_3]\) and \(\bar{x} \neq \bar{w}_1, x = \pi(\bar{x}), \gamma_x\) is either non-simple or finally touches \(a_1\) nonorthogonally.

**Proof of Claim.** Suppose \(\gamma_x\) is simple, and does not touch \(a_1\) nonorthogonally (i.e. either does not touch \(a_1\), or touches \(a_1\) orthogonally). \(\gamma_{\bar{x}}\) touches \(B_1\) before leaving \(Y_1\). When it goes into \(Y_{g_1}\), it either touches some \(B_{g_1}^i, i = 2, 3\) or touches \(A_{g_1}^i\). The latter case would imply that \(\gamma_{\bar{x}}\) touches \(a_1\) nonorthogonally. Suppose \(\gamma_{\bar{x}}\) touches \(B_{g_1}^2\) (the case for \(B_3^1\) is the same) at \(y\), then it enters \(Y_{g_2g_1}\). Suppose it touches \(A_{g_1}^{2g_2}\). Then the intersection must be nonorthogonal, since otherwise we would have on \(X\) two different geodesic rays emanating from \(\pi(y)\) that are orthogonal to \(a_1\), namely a portion of \(\gamma_{\bar{x}}\) and a portion of \(b_2\), a contradiction. But this means that \(\gamma_{\bar{x}}\) touches \(a_1\) nonorthogonally. Hence \(\gamma_{\bar{x}}\) must touch \(B_{g_2g_1}^1\) or \(B_{g_2g_1}^3\). Supposing \(\gamma_{\bar{x}}\) first touches \(b_1\), then \(b_2\), then \(b_1\) or \(b_3\). Since \(\gamma_{\bar{x}}\) is simple, the alternative must be \(b_2\), and we know that after these three crossings \(\gamma_{\bar{x}}\) can never come close to \(a_1, a_2\) or \(b_3\) again. But this means that \(\gamma_{\bar{x}}\) will stay in \(Y_{g_1(g_2g_1)^n}\) and \(Y_{g_2g_1^n}, n \geq 0\) before its possible intersection with some \(A_{g_1}^{2g_2}(g_2g_1)^n\) or \(A_{g_1}(g_2g_1)^n\), \(n \geq 0\).

Each \(A_{g_1}^{2g_2}(g_2g_1)^n\) and \(A_{g_1(g_2g_1)^n}, n \geq 0\) is supported on the half circle \(C\) supporting \(A_2\), not intersecting with \(\gamma\), hence \(\gamma_{\bar{x}}\) will touch \(B_{g_1}^{2g_2}(g_2g_1)^n\) and \(B_{g_1}^{g_2}(g_2g_1)^n\) for each \(n \geq 0\). But

\[
\lim_{n \to \infty} \sup_{s \in B_{g_1}^{2g_2}(g_2g_1)^n} d_e(s, C) = \lim_{n \to \infty} \sup_{s \in B_{g_1}^{g_2}(g_2g_1)^n} d_e(s, C) = 0,
\]

where \(d_e\) means the Euclidean distance in \(\mathbb{H}\), implying that \(\gamma_{\bar{x}}\) is asymptotic to the circle.
C, a contradiction.

Let $w_1 = \pi(\tilde{w}_1), w_2 = jw_1$, from the above we know that for any $x$ in the two open arcs $y_2w_1y_3$ and $z_2w_2z_3$, $\gamma_x$ is either nonsimple or finally touches $a_1$ again, and the second intersection with $a_1$ is orthogonal if and only if $x = w_1$ or $w_2$. We will call the open arcs $y_2w_1y_3$ and $z_2w_2z_3$ the main gaps on $a_1$ for $X$ and call the closed arcs $y_2p_3z_2$ and $y_3p_2z_3$ the complementary arcs on $a_1$ for $X$ near $a_2$ and $a_3$ respectively.

We can actually compute the lengths of the complementary arcs as follows. Let $l_i = l(a_i), 1 \leq i \leq 3$. First note that $[\tilde{p}_3, \tilde{y}_2]$, $B_3$, a portion of the half circle supporting $A_2$, and $\gamma_2$ are the four edges of a hyperbolic quadrangle with one angle zero and two infinite edges. Hence we have

$$\sinh d(\tilde{p}_3, \tilde{y}_2) = \frac{1}{\sinh l(B_3)}.$$  

or equivalently

$$\sinh d(p_3, y_2) = \frac{1}{\sinh l(b_3)}.$$

Since the hexagon $X_1$ is determined by $l_i/2, 1 \leq i \leq 3$, so is $l(b_3)$, and we can write down an explicit formula

$$\cosh l(b_3) = \frac{\cosh \frac{1}{2} + \cosh \frac{1}{2} \cosh \frac{l_3}{2}}{\sinh \frac{1}{2} \sinh \frac{l_3}{2}}.$$

Therefore the length of the complementary arc $y_2p_3z_2$ is

$$l(y_2p_3z_2) = 2d(p_3, y_2) = \log \frac{\cosh \frac{l_3}{2} + \cosh \frac{l_1 + l_2}{2}}{\cosh \frac{l_3}{2} + \cosh \frac{l_1 - l_2}{2}}.$$  

For later convenience let $\mathcal{R}(l_1, l_3, l_2)$ be the length of the arc $y_2p_2z_2$ and $\mathcal{D}(l_1, l_2, l_3)$ be the total length of the main gaps on $a_1$ for $X$. We have

$$\mathcal{R}(l_1, l_3, l_2) = l_1 - l(y_2p_3z_2) = l_1 - \log \frac{\cosh \frac{l_3}{2} + \cosh \frac{l_1 + l_2}{2}}{\cosh \frac{l_3}{2} + \cosh \frac{l_1 - l_2}{2}} \tag{5.1}$$

$$\mathcal{D}(l_1, l_2, l_3) = \mathcal{R}(l_1, l_3, l_2) + \mathcal{R}(l_1, l_2, l_3) - l_1 = 2 \log e^{\frac{l_3}{2}} + e^{\frac{l_1 + l_2}{2}} \tag{5.2}.$$  

We will view $\mathcal{R}$ and $\mathcal{D}$ as functions defined on $\mathbb{R}^3$ by the explicit formulæ above. Note that $\mathcal{D}(x, y, z)$ is a function of $y + z$. By explicit computation, we have

$$\mathcal{D}(x, y, z) + \mathcal{D}(x, -y, z) = 2\mathcal{R}(x, y, z) \tag{5.3}$$

$$\frac{\partial}{\partial x} \mathcal{D}(x, y, z) = H(y + z, x) \tag{5.4}$$

$$\frac{\partial}{\partial x} \mathcal{R}(x, y, z) = \frac{1}{2} \left( H(z, x + y) + H(z, x - y) \right) \tag{5.5}$$

where

$$H(x, y) = \frac{1}{1 + e^{\frac{x+y}{2}}} + \frac{1}{1 + e^{\frac{x-y}{2}}} \tag{5.6}.$$  

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As a consequence of (5.4) and (5.5), we have, as $x \to 0$,
\[ D(x, y, z) \sim xH(y + z, x) = 2x \frac{1}{1 + e^{-y}} \quad (5.7) \]
\[ R(x, y, z) \sim x^2 \left( H(z, y) + H(z, -y) \right) = x \left( \frac{1}{1 + e^{-y}} + \frac{1}{1 + e^{-z}} \right) \quad (5.8) \]

5.2 Geodesics on a Bordered Riemann Surface

Let $X$ be a bordered Riemann surface, with geodesic boundary components $\beta_i, 1 \leq i \leq n$. Let $B = \{ \gamma | \gamma \text{ is a simple geodesic on } X \text{ with any possible intersection with } \beta_i \text{ orthogonal} \}$. For any $x \in \beta_1$ let $\gamma_x$ be the geodesic ray emanating from $x$ orthogonal to $\beta_1$. We investigate cases when $\gamma_x \notin B$.

Case 1. $\gamma_x$ is simple and intersects with $\beta_1$ nonorthogonally at $y$. $x$ and $y$ divide $\beta_1$ into two portions $a$ and $b$. In the homotopy class of $\overline{a} \cup \gamma_x (\overline{b} \cup \gamma_x)$ on $X$ there is a unique simple closed geodesic $A$ ($B$). $\beta_1, A$ and $B$ are the boundary components of a unique embedded hyperbolic pair of pants $Y$ containing $\gamma_x$. Moreover, if $Y'$ is an embedded pair of pants with geodesic boundaries such that $\beta_1$ is one of its boundaries and $\gamma_x$ is contained in $Y'$, then $Y' = Y$. $x$ is in the main gaps on $\beta_1$ for $Y$.

Case 2. $\gamma_x$ is simple and intersects with some $\beta_i, i > 1$ nonorthogonally at $y$. Consider the boundary $a$ of a small neighborhood of $\beta_1 \cup \beta_i \cup \gamma_x$ in $X$. In the homotopy class of $a$ on $X$ there is a unique simple closed geodesic $A$. $\beta_1, \beta_i$ and $A$ are the boundary components of a unique embedded hyperbolic pair of pants $Y$. Moreover, if $Y'$ is an embedded pair of pants with geodesic boundaries such that $\beta_1$ and $\beta_i$ are two of its boundaries and $\gamma_x$ is contained in $Y'$, then $Y' = Y$. $x$ is in the complementary arc on $\beta_1$ for $Y$ near $\beta_i$.

Case 3. $\gamma_x$ is nonsimple. Let $y$ be the first self intersection of $\gamma_x$. Let $\gamma$ be the portion of $\gamma_x$ from $x$ to $y$. Thus $\gamma$ has the shape of a lasso. The boundary of a small neighborhood of $\gamma$ in $X$ has two connected components, in which one is an embedded circle in $X - \partial X$ and the other is a portion of $\beta_1$ union an embedded segment $a$ in $X$ with two end points on $\beta_1$. Similar to Case 1, $a$ and $\beta_1$ determines a unique embedded pair pants $Y$ with geodesic boundaries including $\beta_1$, and $Y$ contains $\gamma$. Hence $x$ is in the main gap on $\beta_1$ for $Y$.

5.3 Mirzakhani’s Identity

Lemma 5.2. The set of $x \in \beta_1$ for which $\gamma_x \in B$ is of measure zero (with respect to the 1-dimensional measure on $\beta_1$).

Proof. By a theorem of Birman-Series [2], the union of all the simple geodesics on a complete hyperbolic surface without boundary is of Hausdorff measure 1. We double $X$ to get such a complete hyperbolic surface without boundary $Y$, and every geodesic in $B$ is a portion of a simple geodesic on $Y$. Hence we know that the union $E = \cup_{\gamma \in \mathcal{B}}$ is of Hausdorff dimension 1 and is of measure zero in particular. Consider a half collar neighborhood $U$ of $\beta_1$ in $X$, we have $\mu_2(E \cap U) = \mu_1(E \cap \beta_1) \sinh r = 0$, where $\mu_i$ means the $i$-dimensional measure and $r$ is the width of the collar neighborhood. Hence $\mu_1(E \cap \beta_1) = 0$.

By the previous lemma and the analysis in the previous subsection, we have proved the following theorem.
Theorem 5.3 (Mirzakhani’s Identity). Let $X$ be a bordered Riemann surface with geodesic boundaries $\beta_i, 1 \leq i \leq n$. Let $L_i$ be the length of $\beta_i$, then the following holds

$$\sum_{\{\gamma_1, \gamma_2\} \in F_1} D(L_1, l(\gamma_1), l(\gamma_2)) + \sum_{i=2}^{n} \sum_{\gamma \in F_{1,i}} R(L_1, L_i, l(\gamma)) = L_1$$

where the functions $R$ and $D$ are defined by (5.1) and (5.2), $F_1$ is the set of all unordered pairs $\{\gamma_1, \gamma_2\}$ such that there is an embedded pair of pants in $X$ with $\beta_1$, $\gamma_1$ and $\gamma_2$ as its geodesic boundaries, and $F_{1,i}$ is the set of all $\gamma$ such that there is an embedded pair of pants in $X$ with $\beta_1$, $\beta_2$ and $\gamma$ as its geodesic boundaries.

Remark 5.4. Lemma 4.1 is a consequence of Theorem 5.3 and (5.4) (5.6).

Remark 5.5. In what follows we will interpret (5.9) as an identity between functions defined on $T_{g,n}(L)$. Then $F_1$ is the set of all unordered pairs $\{\gamma_1, \gamma_2\}$ of non-peripheral isotopy classes of embedded circles on $S_{g,n}$ such that there is an embedded topological pair of pants $T$ in $S_{g,n}$, with $\gamma_1$, $\gamma_2$ as its boundaries, and such that $T$ is a punctured neighborhood of $p_1$; and $F_{1,i}$ is the set of all non-peripheral isotopy classes $\gamma$ such that there is an embedded topological pair of pants $T$ in $S_{g,n}$, with $\gamma$ as its boundary, and such that $T$ is a punctured neighborhood of $p_1$ and $p_i$. Here an embedded circle in $S_{g,n}$ is called nonperipheral if it does not bound a disc or a once punctured disc. We interpret $l(\gamma) = l_\gamma$ and $l(\gamma_i) = l_{\gamma_i}$ as in (2.3).

6 Integration over the Moduli Spaces and Symplectic Reduction

6.1 Basic Definitions

We generalize the constructions in Sect. 4. Let $\gamma_1, \ldots, \gamma_k$ be mutually disjoint non-peripheral isotopic classes of embedded circles in $S_{g,n}, 2g - 2 + n > 0$. Let

$$\Gamma := (\gamma_1, \ldots, \gamma_k) \quad (6.1)$$

$$|\Gamma| := \{\gamma_1, \ldots, \gamma_k\} \quad (6.2)$$

$$\text{Stab}(\Gamma) := \bigcap_{i=1}^{k} \text{Stab}(\gamma_i) \subset \text{Mod}_{g,n}, \quad (6.3)$$

where Mod$_{g,n}$ permutes the isotopy classes of embedded circles in $S_{g,n}$ naturally. Let

$$\mathcal{M}_{g,n}^\Gamma (L) := T_{g,n}(L)/\text{Stab}(\Gamma).$$

Define the function

$$l_\Gamma = (l_{\gamma_1}, \ldots, l_{\gamma_k}) : T_{g,n}(L) \rightarrow \mathbb{R}_{>0}^k$$

where $l_{\gamma_i}$ is defined by (2.3). $l_\Gamma$ descends to a function $\mathcal{M}_{g,n}^\Gamma (L) \rightarrow \mathbb{R}_{>0}^k$, which will be denoted by $\mathcal{L}_\Gamma$. Denote by $\pi^\Gamma$ the projection $\mathcal{M}_{g,n}^\Gamma (L) \rightarrow \mathcal{M}_{g,n}(L)$.
For any measurable function $F : \mathbb{R}^k_{>0} \to \mathbb{R}_{>0}$, let $F_\Gamma = F \circ L_\Gamma$. Integration of $F_\Gamma$ against the volume form $\omega_{3g-3+n}/(3g-3+n)!$ over $\mathcal{M}_{g,n}^k(L)$ is meaningful. We omit the volume form in the notation when convenient. We introduce a general principle of integration.

**Lemma 6.1.** Let $\pi : X \to Y$ be a locally trivial fibration of manifolds. Let $H$ be a group that acts on $X$ and $Y$ properly discontinuously, such that the actions are equivariant with respect to $\pi$, i.e. $\pi(hx) = g\pi(x), \forall x \in X, h \in H$. Suppose $dV$ is a volume form on $Y$ invariant under $H$, $\omega$ is a differential form on $X$ such that $dW = \omega \wedge \pi^*dV$ is a volume form on $X$ invariant under $H$. Let $\hat{\pi} : X/H \to Y/H$ be the induced fibration. Then for any $y \in Y$, $\omega$ is $H$-invariant on $H\pi^{-1}(y)$, and we have

$$\int_{X/H} dW = \int_{\bar{\gamma} \in Y/H} \int_{\hat{\pi}^{-1}(\bar{\gamma}) = H\pi^{-1}(y)/H} \omega. \quad (6.4)$$

**Proof** Since locally $H\pi^{-1}(y_0)$ is a disjoint union of open sets of $\pi^{-1}(y), y \in H y_0$, it suffices to prove that

$$i_y^*(h^*\omega - \omega) = 0, \quad (6.5)$$

where $i_y : \pi^{-1}(y) \to X$ is inclusion. Now both $dV$ and $dW$ are invariant, hence $(h^*\omega - \omega) \wedge \pi^*dV = 0$. For $x_0 \in \pi^{-1}(y_0)$, we find local coordinates $(x, y) : U \to \mathbb{R}^{M+N}$ near $x_0$ for $X$, local coordinates $y : V \to \mathbb{R}^N$ near $y_0$ for $Y$ such that $\pi(x, y) = y$, $dV = dy = dy_1 \cdots dy_n$ locally. Now locally $(h^*\omega - \omega) \wedge dy = 0$, hence $h^*\omega - \omega$ has the form $\sum dy_i \wedge \omega_i$. Hence $i_{y_0}^*(h^*\omega - \omega) = 0$. (6.4) is just integration along fibres.

**Remark 6.2.** Since the mapping class group $\text{Mod}_{g,n}$ has a normal subgroup of finite index that acts on $\mathcal{T}_{g,n}(L)$ without fixed point, any quotient of $\mathcal{T}_{g,n}(L)$ by a subgroup of $\text{Mod}_{g,n}$ has a finite Galois covering that is a manifold. Since any integral over a space is $1/r$ of the corresponding integral over its $r$-fold covering, we may proceed as if the various quotients of $\mathcal{T}_{g,n}(L)$ were manifolds obtained by a good group action when studying integrals on them.

We extend $|\Gamma|$ to a pair of pants decomposition $\mathcal{P} = \{\gamma_1, \ldots, \gamma_m\}$ of $S_{g,n}$, where $m = 3g-3+n \geq k$. Let $(t_i, \tau_i)_{1 \leq i \leq m}$ be the Fenchel-Nielsen coordinates. Applying Lemma 6.1 to the fibration $l_\Gamma : \mathcal{T}_{g,n}(L) \to \mathbb{R}^k_{>0}$ with $H = \text{Stab}(\Gamma)$ acting on $\mathcal{T}_{g,n}(L)$ naturally and acting on $\mathbb{R}^k_{>0}$ trivially, we have

$$\int_{\mathcal{M}_{g,n}^k(L)} F_\Gamma = \int_{\mathbb{R}^k_{>0}} F(t) \text{Vol}(L^{-1}_\Gamma(t)) dt, \quad (6.6)$$

where $\text{Vol}(L^{-1}_\Gamma(t)) = \int_{L^{-1}_\Gamma(t)} d\tau_1 \cdots d\tau_k dl_{k+1} \cdots dl_m d\tau_m$.

We will compute $\text{Vol}(L^{-1}_\Gamma(t))$ in the next subsection.

For convenience we introduce the following definitions. Let $S$ be a disjoint union of closed surfaces and $P = \{p_i | 1 \leq i \leq n+2k\}$ a set of $n+2k$ distinct points on $S$ such that
$S - P \cong S_{g,n} - |\Gamma|$, with $p_i$ corresponding to the $i^{th}$ marked point for $S_{g,n}$, $1 \leq i \leq n$, and $p_{n+j}$ and $p_{n+k+j}$ corresponding to $\gamma_j$, $1 \leq j \leq k$. We define

$$\mathcal{T}(S_{g,n} - |\Gamma|; L, l(\Gamma) = t) := \mathcal{T}(S, P; L, t, t),$$

$$\mathcal{M}(S_{g,n} - |\Gamma|; L, l(\Gamma) = t) := \mathcal{M}(S, P; L, t, t),$$

and

$$\text{Mod}(S_{g,n} - |\Gamma|) := \text{Mod}(S, P).$$

### 6.2 Symplectic Reduction

We recall the theory of symplectic reduction. Suppose $(M, \omega)$ is a symplectic manifold, and $G$ is a Lie group acting smoothly on $M$ preserving $\omega$. Let $\mathfrak{g}$ be the Lie algebra of $G$ and $\mathfrak{g}^*$ its dual space. A moment map is a map $\Phi : M \rightarrow \mathfrak{g}^*$ such that for any $X \in \mathfrak{g}$, the map $\Phi_X : M \rightarrow \mathbb{R}, p \mapsto \Phi(p)(X)$ is a Hamilton potential for the vector field $X^\#$, i.e. $\omega(X^\#, \cdot) = d\Phi_X$, where $X^\#$ is the vector field on $M$ induced by $X$ through the action. Such $(M, \omega, G, \Phi)$ is called a Hamiltonian $G$-space. Assume $\Phi$ is ad*-equivariant, that is $\Phi(gp) = ad^*_g \Phi(p)$, for any $g \in G$, $p \in M$. For $a \in \mathfrak{g}^*$ let $G_a = \{g \in G|ad^*_g a = a\}$. $G_a$ is a closed subgroup of $G$, hence a Lie subgroup. By the ad*-equivariance of $\Phi$, $G_a$ acts on $\Phi^{-1}(a)$. We have the following theorem.

**Theorem 6.3 (Symplectic Reduction).** Let $\Phi$ be an ad*-equivariant moment map for $(M, \omega, G)$, $a \in \mathfrak{g}^*$ a regular value of $\Phi$, and suppose that $G_a$ acts on $\Phi^{-1}(a)$ properly and without fixed points, so that $M_a := \Phi^{-1}(a)/G_a$ is a smooth manifold. On $M_a$ there is a unique symplectic form $\omega_a$ such that $\pi_a^* \omega_a = i_a^* \omega$, where $\pi_a : \Phi^{-1}(a) \rightarrow M_a$ is the projection and $i_a : \Phi^{-1}(a) \rightarrow M$ is the inclusion.

For a proof see [1] Chapter 4.

Now let $M = \mathcal{T}_{g,n}(L)$, $\omega = \omega_{WP}$, $G = \mathbb{R}^k$. Let $\Gamma = (\gamma_1, \ldots, \gamma_k)$ as in the previous subsection. We define a $G$-action on $(M, \omega)$. We extend $|\Gamma|$ to a pair of pants decomposition $P = \{\gamma_1, \ldots, \gamma_m\}$ and let $(l_i, \tau_i)_{1 \leq i \leq m}$ be the Fenchel-Nielsen coordinates. For $g = (t_1, \ldots, t_k) \in G$, let

$$g(l_p, \tau_p) = (l_p, \tau_1 - l_1 t_1, \ldots, \tau_k - l_k t_k, \tau_{k+1}, \ldots, \tau_m).$$

Thus for example $t = (1, 0, \ldots, 0)$ gives a Dehn twist along $\gamma_1$. We will call this action the normalized shearing along $\Gamma$. By (2.6) this action preserves $\omega$. Define

$$\Phi = \frac{l_1^2}{2} = \left(\frac{l_2^2}{2}, \ldots, \frac{l_k^2}{2}\right).$$

By (2.6) $\Phi$ is an ad*-equivariant moment map for the action. Moreover any $a \in \mathbb{R}^k_0 \subset \mathfrak{g}^*$ is a regular value of $\Phi$. The action of $G_a = G$ on $\Phi^{-1}(a)$ is proper and without fixed points. Let $a = t^2/2 = (t_1^2/2, \ldots, t_k^2/2) \in \mathbb{R}^k_0$, $t \in \mathbb{R}^k_0$, then $\Phi^{-1}(a) = l_1^{-1}(t)$. The space
\[ \frac{l_{\Gamma}^{-1}(t)}{G} \text{ has coordinates } (l_i, \tau_i)_{k+1 \leq i \leq m} \text{ and the symplectic form on it defined by Theorem 6.3 is } \sum_{i=k+1}^{m} d l_i \wedge d \tau_i. \]

We recognize that
\[ \frac{l_{\Gamma}^{-1}(t)}{G} \cong \mathcal{T}(S_{g,n} - |\Gamma|), L, l(\Gamma) = t) \]  
(6.7)
as symplectic manifolds.

Our aim is to compute the volume of
\[ \mathcal{L}_{\Gamma}^{-1}(t) \subset \mathcal{M}_{g,n}^\Gamma(L) = \mathcal{T}_{g,n}(L)/\text{Stab}(\Gamma), t \in \mathbb{R}_{\geq 0}^k. \]
The action of \( G \) on \( M \) and the moment map \( \Phi \) are both equivariant with respect to the action of \( \text{Stab}(\Gamma) \) on \( M \). In particular we know that
\[ (M', \omega', G, \Phi') = (\mathcal{M}_g^\Gamma(L), \omega_{WP}, G, \frac{L_{\Gamma}^2}{2}) \]
satisfy the hypotheses of Theorem 6.3 (where we view \( M' \) as if it were a manifold by Remark 6.2). By (6.7)
\[ \mathcal{L}_{\Gamma}^{-1}(t)/G \cong \mathcal{T}(S_{g,n} - |\Gamma|), L, l(\Gamma) = t)/\text{Stab}(\Gamma) = \mathcal{M}(S_{g,n} - |\Gamma|), L, l(\Gamma) = t) \]
as symplectic manifolds, where \( \text{Stab}(\Gamma) \) acts on \( \mathcal{T}(S_{g,n} - |\Gamma|), L, l(\Gamma) = t) \) via the surjective map
\[ \text{Stab}(\Gamma) \rightarrow \text{Mod}(S_{g,n} - |\Gamma|). \]  
(6.8)
Let \( \pi: l_{\Gamma}^{-1}(t) \rightarrow l_{\Gamma}^{-1}(t)/G \) and \( \bar{\pi}: \mathcal{L}_{\Gamma}^{-1}(t) \rightarrow \mathcal{L}_{\Gamma}^{-1}(t)/G \) be the projections. Applying Lemma 6.1 again to \( \pi \) and \( H = \text{Stab}(\Gamma) \) we have
\[ \text{Vol}(\mathcal{L}_{\Gamma}^{-1}(t)) = \int_{\mathcal{M}(S_{g,n} - |\Gamma|)} \left[ \int_{\bar{\pi}^{-1}(X)} d\tau_1 \cdots d\tau_k \right] dV \]  
(6.9)
where \( dV \) is the volume form on \( \mathcal{M}(S_{g,n} - |\Gamma|) \) and we omit the prescribed lengths in the notation. For a generic \( X \in \mathcal{M}(S_{g,n} - |\Gamma|) \), we compute \( \int_{\bar{\pi}^{-1}(X)} d\tau_1 \cdots d\tau_k \). Let \( Y \in \mathcal{T}_{g,n}(L) \) represent \( X \). \( \bar{\pi}^{-1}(X) = \text{Stab}(\Gamma)\bar{\pi}^{-1}(Y)/\text{Stab}(\Gamma) \). We find a fundamental domain for the \( \text{Stab}(\Gamma) \)-action on \( \text{Stab}(\Gamma)\bar{\pi}^{-1}(Y) \) to be \( \prod_{i=1}^{k}[0, m_i t_i] \times \{ Y \} \), where \( m_i \) is equal to \( 1/2 \) if at least one connected component of \( S_{g,n} - \gamma_i \) is of type \((1,1)\) and equal to \( 1 \) otherwise. Let \( M(\Gamma) = \# \{ i | m_i = 1/2 \} \), we have
\[ \int_{\bar{\pi}^{-1}(X)} d\tau_1 \cdots d\tau_k = 2^{-M(\Gamma)} t_1 \cdots t_k. \]  
(6.10)
By (6.9)(6.10) we have
\[ \text{Vol}\mathcal{L}_{\Gamma}^{-1}(t) = 2^{-M(\Gamma)} t_1 \cdots t_k V(S_{g,n} - |\Gamma|), L, l(\Gamma) = t) \]  
(6.11)
where \( V(S_{g,n} - |\Gamma|), L, l(\Gamma) = t) = \text{Vol}(\mathcal{M}(S_{g,n} - |\Gamma|), L, l(\Gamma) = t)). \) Combining (6.6) and (6.11) we have proved the following.
Theorem 6.4.

\[
\int_{M_{g,n}(L)} F_{\Gamma} = 2^{-M(\Gamma)} \int_{t \in \mathbb{R}^k_+} F(t) t_1 \cdots t_k dt_1 \cdots dt_k \quad (6.12)
\]

6.3 The Covolume Formula

We use Theorem 6.4 to derive a fundamental formula. Let \( \Gamma = (\gamma_1, \ldots, \gamma_k), \) \( \gamma_i \) mutually disjoint non-peripheral isotopic classes of embedded circles in \( S_{g,n}, 2 - 2g + n > 0. \) Consider a formal linear combination of the \( \gamma_i, \)

\[
\gamma = c_1 \gamma_1 + \cdots + c_k \gamma_k, c_i \in \mathbb{R}
\]
called a multicurve. The mapping class group \( \text{Mod}_{g,n} \) acts on such multicurves linearly. Let \( \text{Stab}(\gamma) \) be the stabilizer of \( \gamma \) in \( \text{Mod}_{g,n}. \) Thus \( \text{Stab}(\Gamma) \) is a normal subgroup of \( \text{Stab}(\gamma) \) and these are equal when all the \( c_i \) are distinct. Define

\[
\text{Sym}(\gamma) := \text{Stab}(\gamma)/\text{Stab}(\Gamma),
\]

\[
l_\gamma := c_1 l_{\gamma_1} + \cdots + c_k l_{\gamma_k} : T_{g,n}(L) \to \mathbb{R}.
\]

Let

\[
f : \mathbb{R} \to \mathbb{R}_{>0}
\]
be a measurable function. Define

\[
F(t_1, \ldots, t_k) := f(c_1 t_1 + \cdots + c_k t_k), \text{ for } (t_1, \ldots, t_k) \in \mathbb{R}^k
\]

\( F_{\Gamma} = F \circ L_{\Gamma} \) is a function on \( M_{g,n}(L). \) We use the projection \( \pi : M_{g,n}(L) \to M_{g,n}(L) \) to define a function \( \pi_{\Gamma} F_{\Gamma} \) on \( M_{g,n}(L) \) by pushing forward \( F_{\Gamma} \)

\[
\pi_{\Gamma} F_{\Gamma}(X) := \sum_{Y \in \pi^{-1}(X)} F_{\Gamma}(Y), \text{ for } X \in M_{g,n}(L).
\]

We have

\[
\int_{M_{g,n}(L)} \pi_{\Gamma} F_{\Gamma} = \int_{M_{g,n}(L)} F_{\Gamma}. \quad (6.13)
\]

Let \( Y \in \pi_{\Gamma}^{-1}(X) \), we compute

\[
\pi_{\Gamma} F_{\Gamma}(X) = \sum_{\sigma \in \text{Mod}_{g,n}/\text{Stab}(\Gamma)} F_{\Gamma}(\sigma Y)
\]

\[
= \sum_{\sigma \in \text{Mod}_{g,n}/\text{Stab}(\Gamma)} f(c_1 l_{\sigma \gamma_1}(Y) + \cdots + c_k l_{\sigma \gamma_k}(Y))
\]

\[
= |\text{Sym}(\gamma)| \sum_{\sigma \in \text{Mod}_{g,n}} f(l_{\sigma \gamma}(Y))
\]
The function \( Y \mapsto \sum_{\sigma \in \text{Mod}_{g,n}} f(l_{\sigma \gamma}(Y)) \) on \( T_{g,n}(L) \) is \( \text{Mod}_{g,n} \)-invariant and descends to a function \( f_\gamma : M_{g,n}(L) \to \mathbb{R}_{>0} \). Hence

\[
\pi^\Gamma_* F_\Gamma = |\text{Sym}(\gamma)|^{-1} f_\gamma
\]  

(6.14)

By Theorem 6.4, (6.13) and (6.14), we have

**Theorem 6.5** (The Covolume Formula, Theorem 7.1 in [11]).

\[
\int_{M_{g,n}(L)} f_\gamma = \frac{2^{-M(\Gamma)}}{|\text{Sym}(\gamma)|} \int_{t \in \mathbb{R}_{>0}^k} f(c_1 t_1 + \cdots + c_k t_k) V(S_{g,n} - |\Gamma|, L, l(\Gamma) = t) t_1 \cdots t_k dt
\]

7 Proof of Theorem 3.5

We use the interpretation in Remark 5.5 of Mirzakhani’s identity (5.9). For \( 2 \leq i \leq n \), \( F_{I,i} \) is an \( \text{Mod}_{g,n} \)-orbit of (multi)curves on \( S_{g,n} \), since for each \( \gamma \in F_{I,i} \), \( S_{g,n} - \gamma \) is of the same diffeomorphic type. Define \( \mathcal{R}^i(x) := \mathcal{R}(L_1, L_i, x) \), and choose any \( \gamma_i \in F_{I,i} \). Keeping the notation of Section 6.3, we have

\[
\sum_{\gamma \in F_{I,i}} \mathcal{R}(L_1, L_i, l_\gamma) = \mathcal{R}^i_{\gamma_i}.
\]  

(7.1)

The first sum on the left hand side of (5.9) can be rewritten as \( \sum_{\gamma \in F_{I}'} \mathcal{D}(L_1, l_\gamma, 0) \), where \( F_{I}' := \{ \gamma_1 + \gamma_2 \mid (\gamma_1, \gamma_2) \in F_I \} \). We have used the fact that \( \mathcal{D}(x, y, z) = \mathcal{D}(x, y + z, 0) \). \( F' \) is the union of several \( \text{Mod}_{g,n} \)-orbits of multicurves. First there is an orbit \( F_{\text{con}} := \{ \gamma_1 + \gamma_2 \mid (\gamma_1, \gamma_2) \in F_I \} \) contained in \( F' \). Let \( F_{\text{decon}} = F' - F_{\text{con}} \). For any \( \gamma = \gamma_1 + \gamma_2 \in F_{\text{decon}} \), \( S_{g,n} - \gamma_1 \cup \gamma_2 \) is a disjoint union of a pair of pants and two surfaces \( S_1 \) and \( S_2 \) diffeomorphic to \( S_{g_1,n_1+1} \) and \( S_{g_2,n_2+1} \) respectively, such that \( g_1 + g_2 = g, n_1 + n_2 = n - 1, 2g_j - 2 + n_j \geq 0, j = 1, 2 \). Let \( P = \{ p_i \} \) be the set of marked points for \( S_{g,n} \), \( P_j = P \cap S_j, I_j = \{ i \mid p_i \in P_j \}, j = 1, 2 \). Then \( \{ 2, \ldots, n \} = I_1 \cup I_2 \). Let \( J_{g,n} := \{ \{(g_1, I_1), (g_2, I_2)\} \mid g_1 + g_2 = g, \{ 2, \ldots, n \} = I_1 \cup I_2, 2g_j - 2 + \# I_j \geq 0, j = 1, 2 \} \). \( J_{g,n} \) is a disjoint union of \( \text{Mod}_{g,n} \)-orbits indexed by \( J_{g,n} \).

\[
F_{\text{decon}} = \bigcup_{b \in J_{g,n}} O_b.
\]

For each \( b \in J_{g,n} \) choose any \( \gamma_b \in O_b \). Define \( \mathcal{D}(x) := \mathcal{D}(L_1, x, 0) \), we have

\[
\sum_{\{ \gamma_1, \gamma_2 \} \in F_I} \mathcal{D}(L_1, l_{\gamma_1}, l_{\gamma_2}) = \sum_{b \in J_{g,n}} \mathcal{D}_{\gamma_b}.
\]  

(7.2)

Combining (5.9) (7.1) (7.2), we have proved the following identity of functions defined on \( M_{g,n}(L) \).

\[
\sum_{b \in J_{g,n}} \mathcal{D}_{\gamma_b} + \sum_{i=2}^{n} \mathcal{R}_{\gamma_i}^i = L_1.
\]  

(7.3)
We have already computed $V_{1,1}(L)$ in Section 4. Integrate each side of (7.3) over $M_{g,n}(L)$ and apply Theorem 6.5 and properties (5.4) (5.5), we have proved Theorem 3.5, where the only thing to notice is that for $b = \{(g_1, I_1), (g_2, I_2)\} \in J_{g,n}$, $|\text{Sym}(\gamma_b)| = 2$ if and only if $g_1 = g_2$, $\# I_1 = \# I_2 = 0$. In that case, the fibre over $b$ of the forgetful map $\phi : \mathcal{I}_{g,n} \rightarrow J_{g,n}$ consists of one point. Otherwise $|\text{Sym}(\gamma_b)| = 1$ and $\phi^{-1}(b)$ consists of two points.

8 Proof of Theorem 3.3

8.1 Duistermaat-Heckman Theorem

We will use the following version of Duistermaat-Heckman Theorem. Let $(M, \omega, G, \Phi)$ be a Hamiltonian $G$-space, where $G = T^n$ is a torus. Suppose $\Phi$ is ad*-equivariant, proper, and has regular value $0 \in g^* = \mathbb{R}^n$. Thus $G_0 = G$ acts on $\Phi^{-1}(0)$. We suppose this action is free, so that $G$ will also act freely on $\Phi^{-1}(a)$ for $a$ near 0. Ehresmann’s fibration theorem determines for each $a$ near 0, an isotopy class of diffeomorphisms $[f_a : \Phi^{-1}(0) \rightarrow \Phi^{-1}(a)]$. We will call such an $f_a$ a natural in the sense of Ehresmann.

**Theorem 8.1** (Duistermaat-Heckman Theorem). Under the assumptions above, there exists an $\epsilon > 0$ such that $\forall a \in ]-\epsilon_0, \epsilon_0[\mathbb{n}$, $G$ acts freely on $\Phi^{-1}(a)$ and $\Phi^{-1}(a)$ is naturally diffeomorphic to $\Phi^{-1}(0)$ in the sense of Ehresmann. Then $\Phi^{-1}(a)/G$ is also naturally identified with $\Phi^{-1}(0)/G$. Denote by $\omega_a$ the symplectic form on $\Phi^{-1}(a)/G$ defined by symplectic reduction and let $f_a : \Phi^{-1}(0) \rightarrow \Phi^{-1}(a)$ be a natural diffeomorphism, $a \in ]-\epsilon_0, \epsilon_0[\mathbb{n}$. We have

$$[f_a^*\omega_a] = [\omega_0 + \sum_{j=0}^k a_j c_1(L_j)] \in H^1_{dR}(\Phi^{-1}(a)/G, \mathbb{R}),$$

where $L_j$ is the $j^{th}$ principal circle bundle in the principal $T^n$ bundle $\Phi^{-1}(0) \rightarrow \Phi^{-1}(0)/G$, and $c_1(L_j)$ is its first Chern class.

For a proof of this theorem see Chapter 30 in [3].

8.2 Teichmüller Space Obtained by Symplectic Reduction

Let $2g - 2 + n > 0$. We can double $S_{g,n}$ along the boundaries to get a closed surface $S_{g'}$, where $g' = 2g - 1 + n$. More precisely, there exist mutually disjoint embedded circles $\Gamma = (\gamma_1, \ldots, \gamma_n)$ in $S_{g'}$ such that $S_{g'} - |\Gamma| = S_{g,n}^{(1)} \sqcup S_{g,n}^{(2)}$, where $S_{g,n}^{(i)}$ are two copies of $S_{g,n}$. Take a pair of pants decomposition $\mathcal{P}$ for $S_{g,n}$. This gives rise to a pair of pants decomposition $\mathcal{P}^{(i)} = \{\gamma_j^{(i)}\}_{1 \leq j \leq m}$ for $S_{g,n}^{(i)}$, $i = 1, 2$. Then $\mathcal{P}^{(1)} \cup \mathcal{P}^{(2)} \cup |\Gamma|$ is a pair of pants decomposition for $S_{g'}$. Consider the locus $\mathcal{L} \subset T_{g'}$ defined by the equations

$$l_{j_1}^{(i)} = l_{j_2}^{(i)}, \tau_{j_1}^{(i)} = \tau_{j_2}^{(i)}, 1 \leq j \leq m.$$
As in Section 6.2, the normalized shearing along $\Gamma$ give rise to an $\mathbb{R}^\Gamma$ action on $\mathcal{T}_g'$ and $\Phi = l^{-2}/2$ is an $\text{ad}^*$-equivariant, proper moment map. This action preserves the defining equations for $\mathcal{L}$, hence $(l^{-1}_T(L) \cap \mathcal{L})/\mathbb{R}^\Gamma$ is a locus in

$$l^{-1}_T(L)/\mathbb{R}^\Gamma = \mathcal{T}_{g,n}(L) \times \mathcal{T}_{g,n}(L),$$

where $L \in \mathbb{R}^n_{\geq 0}$. This locus is just

$$\{(x,x)\} \in \mathcal{T}_{g,n}(L) \times \mathcal{T}_{g,n}(L),$$

hence we have

$$(l^{-1}_T(L) \cap \mathcal{L})/\mathbb{R}^\Gamma = \mathcal{T}_{g,n}(L).$$

Consider the subgroup of $\mathbb{Z}^\Gamma \subset \mathbb{R}^\Gamma$ generated by the Dehn twists along $\Gamma$. Define

$$\tilde{\mathcal{T}}_{g,n}(L) := (l^{-1}_T(L) \cap \mathcal{L})/\mathbb{Z}^\Gamma.$$

We know that the natural projection

$$\tilde{\mathcal{T}}_{g,n}(L) \longrightarrow (l^{-1}_T(L) \cap \mathcal{L})/\mathbb{R}^\Gamma = \mathcal{T}_{g,n}(L)$$

is a principal $T^n$ bundle.

On the other hand, let $i : \mathcal{L} \to \mathcal{T}_g'$ be the inclusion map. By (2.6) $i^*\omega_{WP}$ is a symplectic form on $\mathcal{L}$. This form is invariant under $\mathbb{Z}^\Gamma$, hence it induces a symplectic structure on $M := \mathcal{L}/\mathbb{Z}^\Gamma$. Now we have $T^n$ acting on $M$ symplectically, induced by the $\mathbb{R}^\Gamma$-action on $\mathcal{L}$. The function $l_T$ descends to be a function $l_T'$ on $M$, and $l_T'^2/2$ is a moment map for the $T^n$ action on $M$. For $L \in \mathbb{R}^n_{\geq 0}$, we have $l^{-1}_T(L) = \tilde{\mathcal{T}}_{g,n}(L)$ and $l^{-1}_T(L)/T^n = \mathcal{T}_{g,n}(L)$ as symplectic manifolds, where the symplectic structure on the left hand side is defined by symplectic reduction.

### 8.3 Extension to the Case $L = 0$

We want to extend the principal $T^n$ bundle

$$\tilde{\mathcal{T}}_{g,n}(L) \longrightarrow (l^{-1}_T(L) \cap \mathcal{L})/\mathbb{R}^\Gamma = \mathcal{T}_{g,n}(L)$$

to the case $L = 0$. Recall the collar and cusp region theorem.

**Lemma 8.2.** Let $X$ be a hyperbolic surface with cusps or geodesic boundary components. For every cusp on $X$, it has a unique neighborhood called the cusp region, which is isometric to

$$\left\{ z \in \mathbb{H} | \text{Im} z \geq \frac{1}{2} \right\}/z \sim z + 1 \cong \left\{ z \in \mathbb{D}^* | |z| \leq e^{-\pi} \right\},$$

where $\mathbb{H}$ is the upper half plane and $\mathbb{D}^*$ is the unit open disc minus origin. For every geodesic boundary of length $l$, it has a unique neighborhood, called a half collar region, isometric to

$$\left\{ z \in \mathbb{H} | d(z, \sqrt{-1}R) \leq w, \text{Re} z \geq 0 \right\}/z \sim e^l z,$$

where $d$ is the distance in $\mathbb{H}$ and $w > 0$ satisfies $\sinh w \sinh \frac{l}{2} = 1$. All the cusp regions and half collar regions are disjoint.
As can be seen from the metric structure, the cusp region of a cusp is foliated by geodesics orthogonal to its boundary, and the half collar region of a geodesic boundary is foliated by geodesics orthogonal to both of the two boundary components of this region. As the length of the geodesic boundary tends to zero, the half collar region Gromov-Hausdorff tends to a cusp region, (i.e. convergence is uniform in any neighborhood with a bounded distance from the geodesic boundary), and its boundaries and geodesic foliation also converge.

Intuitively speaking, the space \( \hat{T}_{g,n}(L), L \in \mathbb{R}_+^0 \) consists of pairs \((X,(q_i)_{1 \le i \le n})\) where \( X \in \mathcal{T}_{g,n}(L) \) and each \( q_i \) lies in the \( i \)th boundary component of \( X \). This is equivalent to the space of all pairs \((X,(q_i)_{1 \le i \le n})\) where \( X \in \mathcal{T}_{g,n}(L) \) and \( q_i \) lies in boundary component of the half collar region of the \( i \)th geodesic boundary other than the \( i \)th geodesic boundary itself. This interpretation still makes sense if we let \( L = 0 \), where we replace the half collar region of the \( i \)th geodesic boundary by the cusp region of the \( i \)th cusps.

To be precise, we consider the Harvey bordification of Teichmüller spaces. We reproduce the exposition in Chapter 9 of [8]. First consider the map

\[ \phi : [-1,1[ \times S^1 \to \mathbb{C}, (t,u) \mapsto tu. \]

Both the restrictions \( \phi|_{[-1,0[ \times S^1} \) and \( \phi|_{0,1[ \times S^1} \) parameterize the punctured unit disc \( \mathbb{D}^* \), so \( \phi \) pulls back the metric

\[ ds^2 = \frac{|dz|^2}{(|z| \log |z|)^2} \]

(induced from the covering \( \mathbb{H} \to \mathbb{H}^*, w \mapsto \exp(2\pi \sqrt{-1}w) \)) to \([-1,1[ - \{0\} \times S^1 \), called the standard degenerate hyperbolic metric on \([-1,1[ \times S^1 \).

Let \( S \) be a closed surface of genus \( g \), \( P = \{p_1, \ldots, p_n\} \) a set of \( n \) points on \( S \), \( 2g - 2 + n > 0 \). Let \( \Gamma = (\gamma_1, \ldots, \gamma_k) \) be mutually disjoint non-peripheral isotopy classes of embedded circles in \( S - P \). A marking for a quasi-hyperbolic surface of type \((S,P,\Gamma)\) is a triple \((X,f,ds)\), where \( X \) is a surface with \( n \) boundary components whose interior is diffeomorphic to \( S - P \), \( f : S - P \to X \) is a diffeomorphism onto the interior of \( X \), and \( ds \) is a hyperbolic metric on each component of \( X - P - f(\{\Gamma\}) \), such that each boundary component of \( X \) is a geodesic and each \( f(\gamma_i) \) has a neighborhood \( U \) in \( X \) parameterizable by some orientation preserving \( \phi : [-r,r[ \times S^1 \to U, 0 < r < 1 \) such that \( \phi(\{0\} \times S^1) = \gamma_i \) and \( \phi|_{[-r,r[ \times \{0\} \times S^1} \) is an isometry onto \( U - \gamma_i \). Define an equivalence relationship \([\cdots]\) between such objects by setting \([X,f,ds] = [X',f',ds']\) if and only if there exists a map \( g : X \to X' \) such that \( f' = g \circ f \) and such that \( g \) induces an isometry from \((X - f(\{\Gamma\}),ds)\) to \((X' - f'(\{\Gamma\}),ds')\). As before, for such an object \((X,f,ds)\) we abuse notation to denote by \( f(p_i) \) the boundary component of \( X \) which \( f(q) \) approaches as \( q \) approaches \( p_i \) in \( S \).

For \( L = (L_1, \ldots, L_n) \in \mathbb{R}_+^n \) define the space

\[ S_T(S,P;L) = \{[X,f,ds] | (X,f,ds) \text{ is a marking for a quasi-hyperbolic surface of type } (S,P,\Gamma) \text{ and } f(p_i) \text{ has length } L_i \} \]

Consider the group \( \text{Diff}_+(S,P \cup |\Gamma|) \) of orientation-preserving auto-diffeomorphisms of \( S \) that fix each point in \( P \cup |\Gamma| \). It acts on \( S_T(S,P;L) \) by precomposing with the markings.
Let $\text{Diff}^0(S, P \cup |\Gamma|)$ be the component of identity in $\text{Diff}_+(S, P \cup |\Gamma|)$. Define

$$T_\Gamma(S, P; L) := S_\Gamma(S, P; L)/\text{Diff}^0(S, P \cup |\Gamma|).$$

We have an $\mathbb{R}^k$-action on $T_\Gamma(S, P; L)$, called the shearing action along $\Gamma$, defined as follows. Let $\tau = (\gamma_1, \ldots, \gamma_k) \in \mathbb{R}^k, [X, f, ds] \in T_\Gamma(S, P; L)$. Define $\tau[X, f, ds] := \{X, \tau(f), ds\}$, where $\tau(f)$ is defined as follows. For each $1 \leq i \leq k$, choose a neighborhood $U_i$ of $f(\gamma_i)$ in $X$ with a parametrization $\phi_i : [-r, r] \times S^1 \rightarrow U_i$ such that $\phi_i([0] \times S^1) = f(\gamma_i)$ and $\phi_i([-r, r] \times S^1)$ is an isometry onto $U - \gamma_i$, where we choose $0 < r < e^{-\pi}$ the same number for each $i$. Then by Lemma 8.2 the $U_i$ are mutually disjoint and their closure does not intersect with the boundary of $X$. Let $\psi_i = f^{-1} \circ \phi_i : [-r, r] \times S^1 \rightarrow S$, and let $\epsilon : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ be a smooth map with the property that

$$\epsilon|_{[0, +\infty]} \equiv 1, \epsilon|_{-\infty, -r/2]} \equiv 0.$$  

Define

$$\alpha_{\tau} : [-r, r] \times S^1 \rightarrow [-r, r] \times S^1, (t, u) \mapsto (t, \epsilon(t)e^{r2\pi\sqrt{-1}u}).$$

Define $\tau(f) : S - P \rightarrow X$ by

$$\tau(f)(x) := \begin{cases} f(x), & x \in S - P - \bigcup_i f^{-1}(U_i) \\ f(\psi_i \circ \alpha \circ \psi_i^{-1}(x)), & x \in f^{-1}(U_i) \end{cases}.$$ 

Now extend $\Gamma$ to be a pair of pants decomposition $\mathcal{P}$ of $S - P$. The standard result is that the map

$$l_{P - \Gamma} : T_\Gamma(S, P; L) \rightarrow \mathbb{R}_{>0}^{P - \Gamma}$$

is a principal $\mathbb{R}^P$-bundle, where the action of $\mathbb{R}^{P - \Gamma}$ is the usual Fenchel-Nielsen shearing and the $\mathbb{R}^\Gamma$-action is defined as above. In particular we have a noncanonical isomorphism of real analytic manifolds

$$(l_{P - \Gamma}, \tau_{\mathcal{P}}) : T_\Gamma(S, P; L) \rightarrow \mathbb{R}_{>0}^{P - \Gamma} \times \mathbb{R}^P.$$ 

Moreover, we can define a cornered manifold structure on $T_\Gamma^+(S, P; L) := T(S, P; L) \cup T_\Gamma(S, P; L)$ independent of the choice of a pair of pants decomposition of $S - P$. The function $l_{\Gamma}$ on $T(S, P; L)$ can be extended to $T_\Gamma^+(S, P; L)$ by setting it to be zero. We know that $l_{P} : T_\Gamma^+(S, P; L) \rightarrow \mathbb{R}_{>0}^{P}$ is a principal $\mathbb{R}^P$ bundle, where the $\mathbb{R}^\Gamma$ action is the normalized Fenchel-Nielsen shearing on $T(S, P; L)$ and is the action defined above on $T_\Gamma(S, P; L)$. We have the following isomorphism of cornered manifolds

$$(l_{P - \Gamma}, l_{\Gamma}, \tau_{\mathcal{P}}) : \mathbb{R}_{>0}^{P - \Gamma} \times \mathbb{R}_{>0}^{\Gamma} \times \mathbb{R}^P,$$  

and $T_\Gamma(S, P; L)$ is recovered as the locus $l_{\Gamma} = 0$.

Warning: The coordinates $(l_{P}, \tau_{\mathcal{P}})$ in (8.1) restricted to $T(S, P; L)$ is different than the Fenchel-Nielsen coordinates since in this case the $\mathbb{R}^\Gamma$ action on $T(S, P; L)$ is defined to be the normalized shearing.

We now replace $T_{g'}$ by $\mathcal{T}_{g'}(S)$ where $S$ is a closed surface of genus $g'$ in the construction in Section 8.2. Understanding that the normalized shearing along $\Gamma$ on $T_\Gamma(S)$ is just the

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shearing defined above, we find that everything still works when extending to the case 
$L = 0$ and we recognize that the principal $T^k$-bundle $\hat{T}_{g,n}(0) \to T_{g,n}(0)$ gives rise to the 
complex line bundle over $M_{g,n}$ which is the pull back of $L_i$ on $\overline{M}_{g,n}$. Here $\text{Mod}_{g,n}$ acts 
on $\hat{T}_{g,n}$ via the homomorphism $\text{Mod}_{g,n} \to \text{Mod}(S, \Gamma)$ where the latter acts on $T^+_1(S)$ 
naturally.

In conclusion we have

**Theorem 8.3.** There exists a Hamiltonian $T^n$-space $(M, \omega, T^k, \Phi)$ satisfying the hypothe-
ses of Theorem 8.1 such that for $L \in \mathbb{R}^n_{\geq 0}$, $\Phi^{-1}(L^2/2)/T^k$ is isomorphic to $T_{g,n}(L)$ as 
symplectic manifolds. In addition, the first Chern class $\hat{\psi}_i \in H^2(T_{g,n}, \mathbb{C})$ of the $i$th 
principal circle bundle in the principal $T^k$-bundle $\Phi^{-1}(0) \to T_{g,n}$ is the pull back of 
$\psi_i \in H^2(\overline{M}_{g,n}, \mathbb{C})$.

Taking coordinates we see that actually there exists an isomorphism $(l_\Gamma, l_P, \tau_P) : M \to \mathbb{R}_{\geq 0}^P \times \mathbb{R}_{\geq 0}^P \times \mathbb{R}^P$ where $P$ is a pair of pants 
decomposition for $S_{g,n}$, such that under these coordinates $\Phi = l_\Gamma^2/2$ and $(l_P, \tau_P)$ restricted to each $\Phi^{-1}(L^2/2) = T_{g,n}(L), L \in \mathbb{R}^n_{\geq 0}$ are the 
Fenchel-Nielsen coordinates with respect to $P$. In particular we see that identifying $T_{g,n}(L)$ with $T_{g,n}$ by identifying their Fenchel-Nielsen coordinates with respect to a same 
pair of pants decomposition of $S_{g,n}$ is natural in the sense of Ehresmann. Under this 
identification a fundamental domain in $T_{g,n}(L)$ for the $\text{Mod}_{g,n}$-action is mapped to a 
fundamental domain in $T_{g,n}$ for the $\text{Mod}_{g,n}$-action. Applying Theorem 8.1, we have for 
$(g, n) \neq (1, 1)$

\[
V_{g,n}(L) = \int_{\mathcal{M}_{g,n}(L)} \frac{\omega_{WP}^{3g-3+n}}{(3g-3+n)!} \\
n = \int_{\mathcal{M}_{g,n}} \frac{(\omega_{WP} + \sum_{j=1}^n \frac{L_j^2}{2} \psi_j)^{3g-3+n}}{(3g-3+n)!} \\
n = \sum_{|\alpha| \leq 3g-3+n} C^g_{\alpha} L^{2\alpha}, \tag{8.2}
\]

where

\[
C^g_{\alpha} = \frac{1}{2^{\alpha} \alpha! (3g-3+n-|\alpha|)!} \int_{\mathcal{M}_{g,n}} \psi^\alpha \omega^{3g-3+n-|\alpha|} \\
n = \frac{1}{2^{\alpha} \alpha! (3g-3+n-|\alpha|)!} \int_{\overline{\mathcal{M}}_{g,n}} \psi^\alpha \omega^{3g-3+n-|\alpha|}. \tag{8.3}
\]

For $(g, n) = (1, 1)$ we already know $V_{1,1}(L)$, and (3.4) is just a convention on the intersection 
numbers over $\overline{\mathcal{M}}_{1,1}$. Since the rest part of Theorem 3.3 has already been proved in 
Section 3.3, we have finished proving Theorem 3.3.
References


Rational Homotopy Theory of Loop Spaces

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Abstract
This article introduces some basic concepts of Sullivan models and how to use Sullivan models to calculate the rational cohomology rings and rational homotopy groups of a topological space.

Keywords: Sullivan models, rational cohomology rings, rational homotopy groups

1 Introduction

Homotopy theory is the study of the invariants and properties of topological spaces $X$ and continuous maps $f$ that depend only on the homotopy type of the space and the homotopy class of the map. The classical examples of such invariants are the singular homology groups $H_i(X)$ and the homotopy groups $\pi_n(X)$.

The groups $H_i(X)$ and $\pi_n(X)$, $n \geq 2$, are abelian and hence can be rationalized to the vector spaces $H_i(X; \mathbb{Q})$ and $\pi_n(X) \otimes \mathbb{Q}$. Rational homotopy theory begins with the discovery by Sullivan in the 1960’s of an underlying geometric construction: simply connected topological spaces and continuous maps between them can themselves be rationalized to topological spaces $X_{\mathbb{Q}}$ and to maps $f_{\mathbb{Q}} : X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$, such that $H_*(X_{\mathbb{Q}}) = H_*(X; \mathbb{Q})$ and $\pi_*(X_{\mathbb{Q}}) = \pi_*(X) \otimes \mathbb{Q}$. The rational homotopy type of a CW complex $X$ is the homotopy type of $X_{\mathbb{Q}}$ and the rational homotopy class of $f : X \to Y$ is the homotopy class of $f_{\mathbb{Q}} : X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$, and rational homotopy theory is then the study of properties that depend only on the rational homotopy type of a space or the rational homotopy class of a map.

Although rational homotopy theory has the disadvantage of discarding a considerable amount of information, it has the advantage of being remarkably computational. The computational power of rational homotopy theory is due to the discovery by Quillen and by Sullivan of an explicit 1-1 correspondence between geometry and algebra:

\[
\begin{align*}
\{ & \text{rational homotopy types of spaces} \} \\
\overset{\cong}{\sim} & \{ & \text{isomorphism classes of minimal Sullivan algebras} \}
\end{align*}
\]

and

\[
\begin{align*}
\{ & \text{homotopy classes of maps between rational spaces} \} \\
\overset{\cong}{\sim} & \{ & \text{homotopy classes of maps between minimal Sullivan algebras} \}
\end{align*}
\]
These models make the rational homology and homotopy of a space transparent.
The main objective of the article is to provide some usable calculation tools and technique in rational homotopy theory. Meanwhile, we will illustrate part the above 1-1 correspondence.

The materials are divided into five chapters, and the following is an overview of the five chapters, indicating some of the highlights and the role of each part within the article.

• In § 3 we construct the Sullivan’s functor from topological spaces $X$ to commutative cochain algebras $A_{PL}(X)$, which satisfies $C^*(X) \simeq A_{PL}(X)$.

• In § 4 we set up the Sullivan model $(\Lambda V, d) \to (A, d)$ for any commutative cochain algebra satisfying $H^0(A, d) = \mathbb{k}$.

• In § 5 Sullivan algebras are extended to relative Sullivan algebras.

• In § 6 we use relative Sullivan algebras to model fibrations and show that the Sullivan fibre of the model is a Sullivan model for the fibre.

• In § 7 we calculate the minimal Sullivan model of $\Omega(SU(n+1)/T^n)$.

2 Conventions and notation

2.1 Fibrations

Let $X$ and $Y$ be topological spaces. $X^Y$ denotes the space of all continuous maps $Y \to X$. A Moore path in $X$ is a pair $(\gamma, \ell)$ in which $\gamma : [0, \infty) \to X$ is a continuous map, $\ell \in [0, \infty)$, and $\gamma(t) = \gamma(\ell)$ for $t \geq \ell$. The path starts at $\gamma(0)$, ends at $\gamma(\ell)$ and has length $\ell$. The space $MX \subset X^{[0,\infty)} \times [0,\infty)$ of all Moore paths is the free Moore path space on $X$. The Moore path space $P(X, x_0) \subset MX$ is the subspace of all Moore paths ending at $x_0$ and the Moore loop space $\Omega(X, x_0)$ is the subspace of all Moore paths starting and ending at $x_0$. We usually write simply $\Omega X$ and $PX$. We may also denote $(\gamma, \ell)$ simply by $\ell$.

Let $f : X \to Y$ be a continuous map between path connected spaces and let $q_0, q_1 : MY \to Y$ denote the maps $(\gamma, \ell) \mapsto \ell(0)$ and $(\gamma, \ell) \mapsto \gamma(\ell)$. Let $c_y$ be the path of length 0 at $y$. Form the commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{j} & X \times_Y MY \\
\downarrow f & & \downarrow q \\
Y & \xleftarrow{\ } & Y
\end{array}
$$

where

• $X \times_Y MY$ is the fibre product with respect to $f$ and $q_0$, 

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• \( j(x) = (x, cfx) \),
• \( q(x, \gamma) = q_1(\gamma) \).

It is easy to verify that \( j \) is a homotopy equivalence and \( q \) is a fibration: we say the diagram converts \( f \) into the fibration \( q \).

### 2.2 Universal G-bundles

Central to the study of principal \( G \)-bundles is Milnor’s universal \( G \)-bundle

\[ p_G : E_G \to B_G \]

constructed as follows.

Let \( G \) be a topological group. Then \( C_G = (G \times I)/G \times \{0\} \) is the cone on \( G \), and the \( n \)th join, \( G^{*n} \), is the subspace of \( C_G \times \ldots \times C_G \) of points \(((g_0, t_0), \ldots, (g_n, t_n))\) such that \( \sum t_i = 1 \). Thus \( G^{*n} \subset G^{*(n+1)} \) (inclusion opposite base point of \( C_G \)). Set \( E_G = \bigcup_n G^{*n} \), equipped with the weak topology determined by the \( G^{*n} \).

A continuous action of \( G \) in \( E_G \) is given by

\[ ((g_0, t_0), \ldots, (g_n, t_n)) \cdot g = ((g_0 g, t_0), \ldots, (g_n g, t_n)). \]

Set \( B_G = E_G/G \) and let \( p_G : E_G \to B_G \) be the quotient map. It is easy to verify that

• \( p_G : E_G \to B_G \) is a principal \( G \)-bundle.
• every continuous map from a compact space to \( E_G \) is homotopic to a constant map.

In particular, \( \pi_*(E_G) = 0 \).

Since \( E_G \) is contractible, \( G \) is weakly homotopic to \( \Omega B_G \) by Proposition 4.46 in [3].

### 2.3 Basic definitions of singular chains

Recall that a singular \( n \)-simplex in a space \( X \) is a continuous map

\[ \sigma : \Delta^n \to X, \]

where \( n \geq 0 \) and \( \Delta^n = \left\{ \sum_0^n t_i e_i \middle| 0 \leq t_i \leq 1, \sum t_i = 1 \right\} \) is the convex hull of the standard basis \( e_0, \ldots, e_n \) of \( \mathbb{R}^{n+1} \). If \( X \subset \mathbb{R}^m \) is a convex subset then any sequence \( x_0, \ldots, x_n \in X \) determines the linear simplex

\[ \langle x_0, \ldots, x_n \rangle : \Delta^n \to X, \sum t_i e_i \mapsto \sum t_i x_i. \]

Two important examples of linear simplices are the \( i \)th face inclusion of \( \Delta^n \),

\[ \lambda_i = \langle e_0 \ldots \hat{e}_i \ldots e_n \rangle : \Delta^{n-1} \to \Delta^n, (\hat{\text{means omit}}) \]
defined for $n \geq 1$ and $0 \leq i \leq n$; and the $j^{th}$ degeneracy of $\Delta^n$,

$$\rho_j = \langle e_0 \ldots e_je_j \ldots e_n \rangle : \Delta^{n+1} \to \Delta^n,$$

defined for $n \geq 0$ and $0 \leq j \leq n$. The image of $\lambda_i$ is called the $i^{th}$ face of $\Delta^n$.

Denote by $S_n(X)$ the set of singular $n$-simplices on a space $X (n \geq 0)$, and by

$$S_n(\varphi) : S_n(X) \to S_n(Y), \quad S_n(\varphi) : \sigma \longmapsto \varphi \circ \sigma,$$

the set map induced from a continuous map $\varphi : X \to Y$. The set maps

$$\partial_i : S_{n+1}(X) \to S_n(X), \sigma \longmapsto \sigma \circ \lambda_i \quad \text{and} \quad s_j : S_n(X) \to S_{n+1}(X), \sigma \longmapsto \sigma \circ \rho_j$$

are called the face and degeneracy maps. A straightforward calculation shows that

$$\partial_i \partial_j = \partial_j \partial_{i-1}, \quad i < j;$$
$$s_is_j = s_{j+1}s_i, \quad i \leq j;$$
$$\partial_is_j = s_{j+1}\partial_i, \quad i < j;$$
$$\partial_is_j = \text{id}, \quad i = j, j + 1;$$
$$\partial_is_j = s_j\partial_{j-1}, \quad i > j + 1. \quad (2.1)$$

### 2.4 Free commutative graded algebras

Let $V$ be a graded vector space over a field $k$ of characteristic zero. Then $\Lambda V$ denotes to be the free commutative graded algebra on $V$. The following notation and basic facts associated to $\Lambda V$ will be used in the article:

- $\Lambda V$ = symmetric algebra $(V^{\text{even}}) \otimes$ exterior algebra $(V^{\text{odd}})$. The subalgebras $\Lambda(V^{\leq p})$, $\Lambda(V^{>q})$, ... are denoted $\Lambda^{V^{\leq p}}, \Lambda^{V^{>q}}, \ldots$.
- If $\{v_\alpha\}$ or $v_1, v_2, \ldots$ is a basis for $V$ we write $\Lambda(\{v_\alpha\})$ or $\Lambda(v_1, v_2, \ldots)$ for $\Lambda V$.
- $\Lambda^qV$ is the linear span of elements of the form $v_1 \wedge \ldots \wedge v_q, v_i \in V$. Elements in $\Lambda^qV$ have wordlength $q$.
- $\Lambda V = \bigoplus_q \Lambda^qV$ and we write $\Lambda^{\geq q}V = \bigoplus_{i \geq q} \Lambda^iV$ and $\Lambda^+V = \Lambda^{\geq 1}V$.
- If $V = \bigoplus_\lambda V_\lambda$ then $\Lambda V = \bigotimes_\lambda \Lambda V_\lambda$.
- Any linear map of degree zero from $V$ to a commutative graded algebra $A$ extends to a unique graded algebra morphism $\Lambda V \to A$. 

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3 Commutative cochain algebras for spaces and simplicial sets

In this chapter the ground ring is a field $k$ of characteristic zero.

Let $X$ be a topological space then $C^*(X; k)$ denotes the cochain algebra of singular cochains on it. In the chapter, we introduce a naturally defined commutative cochain algebra $A_{PL}(X; k)$, and natural cochain algebra quasi-isomorphisms

$$C^*(X; k) \xrightarrow{\sim} D(X) \xleftarrow{\sim} A_{PL}(X; k),$$

where $D(X)$ is a third natural cochain algebra. The construction of $A_{PL}(X; k)$ is due to Sullivan and the functor

$$X \rightsquigarrow A_{PL}(X; k)$$

serves as the fundamental bridge which we use to transfer problems from topology to algebra.

The quasi-isomorphisms above will be constructed in Theorem 3.8. They define a natural isomorphism of graded algebras,

$$H^*(X; k) = H(A_{PL}(X; k)),$$

and we shall always identify these two algebras via this particular isomorphism. Thus for any continuous map, $f$, we identify $H^*(f; k) = H(A_{PL}(f; k))$.

3.1 Simplicial sets and simplicial cochain algebras

Definition 3.1. A simplicial object $K$ with values in a category $\mathcal{C}$ is a sequence \( \{K_n\}_{n \geq 0} \) of objects in $\mathcal{C}$, together with $\mathcal{C}$-morphisms

$$\partial_i : K_{n+1} \to K_n, 0 \leq i \leq n + 1 \quad \text{and} \quad s_j : K_n \to K_{n+1}, 0 \leq j \leq n.$$ 

satisfying the identities

$$\partial_i \partial_j = \partial_{j-1} \partial_i, \quad i < j; \quad s_is_j = s_{j+1}s_i, \quad i \leq j; \quad \partial_i s_j = s_j \partial_i, \quad i < j; \quad \partial_is_j = id, \quad i = j, j + 1; \quad \partial_is_j = s_j \partial_{j-1}, \quad i > j + 1. \quad (3.2)$$

Then the singular complex of a topological space $X$, $S_*(X)$ is a simplicial object with the ordinary face and degeneracy maps.

A simplicial morphism $f : L \to K$ between two such simplicial objects is a sequence of $\mathcal{C}$-morphisms $\varphi_n : L_n \to K_n$ commuting with the $\partial_i$ and $s_j$.

A simplicial set $K$ is a simplicial object in the category of sets. A simplicial cochain algebra $A$ is a simplicial object in the category of cochain algebras. Similarly, we define simplicial cochain complexes, simplicial vector spaces, . . . .
Let $K$ be any simplicial set. By analogy with the singular complex, the elements $\sigma \in K_n$ are called $n$-simplices and $\sigma$ is non-degenerate if it is not of the form $\sigma = s_j \tau$ for some $j$ and some $\tau \in K_{n-1}$. The subset of non-degenerate $n$-simplices is denoted by $NK_n$.

It follows from the relations (3.2) that for each $k \geq 0$ a simplicial set $K(k) \subset K$ is given as follows: for $n \leq k$, $K(k)_n = K_n$ and for $n > k$, $K(k) = \{s_j \tau | 0 \leq j \leq n-1, \tau \in K(k)_{n-1}\}$. The simplicial set $K(k)$ is called the $k$-skeleton of $K$.

Finally, we use the notation of § 2.3 to introduce a sequence of important subsimplicial sets $\Delta[n] \subset S_*(\Delta^n), n \geq 0$. Let $e_0, \ldots, e_n$ be the vertices of $\Delta^n$. Define a subsimplicial set $\Delta[n] \subset S_*(\Delta^n)$ as follows: $\Delta[n]_k \subset S_k(\Delta^n)$ consists of the linear $k$-simplices of the form $\sigma = \langle e_{i_0} \ldots e_{i_k} \rangle$ with $0 \leq i_0 \leq \ldots \leq i_k \leq n$. Thus $\sigma$ is non-degenerate if and only if $i_0 < \ldots < i_k$ and the identity map

$$c_n = \langle e_0 \ldots e_n \rangle : \Delta^n \to \Delta^n$$

is the unique non-degenerate $n$-simplex; $c_n$ is called the fundamental simplex of $\Delta[n]$. The $(n-1)$-skeleton of $\Delta[n]$ is denoted by $\partial \Delta[n]$ and is called the boundary of $\Delta[n]$.

**Lemma 3.2.** If $K$ is any simplicial set then any $\sigma \in K_n$ determines a unique simplicial set map $\sigma_* : \Delta[n] \to K$ such that $\sigma_*(c_n) = \sigma$.

**Proof.** The verification is straightforward using (3.2). \qed

A simplicial object $A$ in a category $C$ is called extendable if for any $n \geq 1$ and any $I \subset \{0, \ldots , n\}$, the following condition holds: given $\Phi_i \in A_{n-1}, i \in I$, and satisfying

$$\partial_i \Phi_j = \partial_{j-1} \Phi_i, i < j,$$

there exists an element $\Phi \in A_n$ such that $\Phi_i = \partial_i \Phi, i \in I$.

### 3.2 The construction $A(K)$

Let $K$ be a simplicial set, and let $A = \{A_n\}_{n \geq 0}$ be a simplicial cochain complex or a simplicial cochain algebra. The “ordinary” cochain complex (or cochain algebra)

$$A(K) = \{A^p(K)\}_{p \geq 0},$$

(the upper script is denoted to be the graded of the cochain algebra) is defined as follows:

- $A^p(K)$ is the set of simplicial set morphisms from $K$ to $A^p = \{A^p_n\}_{n \geq 0}$. Thus an element $\Phi \in A^p(K)$ is a mapping that assigns to each $n$-simplex $\sigma \in K_n (n \geq 0)$ an element $\Phi_\sigma \in A^p_n$, such that $\Phi_{s_j \sigma} = \partial_j \Phi_\sigma$ and $\Phi_{s_j \sigma} = s_j \Phi_\sigma$.

- Addition, scalar multiplication and the differential are given by

  $$(\Phi + \Psi)_\sigma = \Phi_\sigma + \Psi_\sigma, \quad (\lambda \cdot \Psi)_\sigma = \lambda \cdot \Psi_\sigma, \quad \text{and} \quad (d \Psi)_\sigma = d(\Psi_\sigma).$$

- If $A$ is a simplicial cochain algebra multiplication in $A(K)$ is given by

  $$(\Phi \cdot \Psi)_\sigma = \Phi_\sigma \cdot \Psi_\sigma.$$
• If $\varphi : K \to L$ is a morphism of simplicial sets then $A(\varphi) : A(K) \leftarrow A(L)$ is the morphism of cochain complexes (or cochain algebras) defined by
  
  $$(A(\varphi)\Phi)_\sigma = \Phi_{\varphi\sigma}.$$  

• If $\theta : A \to B$ is a morphism of simplicial cochain complexes (or simplicial cochain algebras) then $\theta(K) : A(K) \to B(K)$ is the morphism defined by
  
  $$(\theta(K)\Phi)_\sigma = \theta(\Phi_\sigma).$$

• When $X$ is a topological space we write $A(K)$ for $A(S^*(X)).$

**Remark 3.3.** Note that the construction $A(K)$ is a covariant functor with respect to $A$ and contravariant functor with respect to $K.$

**Proposition 3.4.** Let $A$ be a simplicial cochain algebra.

(i) For $n \geq 0$ an isomorphism $A(\Delta[n]) \xrightarrow{\cong} A_n$ of cochain algebras is given by $\Phi \mapsto \Phi_{c_n},$ where $c_n$ is the fundamental simplex of $\Delta[n].$

(ii) If $A$ is extendable and $L \subset K$ is an inclusion of simplicial sets, then $A(K) \to A(L)$ is surjective.

If $A$ is extendable simplicial cochain algebra and $L \subset K$ is an inclusion of simplicial sets we denote by $A(K, L)$ the kernel of the surjective morphism $A(K) \to A(L).$ Thus, $A(K, L)$ is a differential ideal in $A(K)$ and

$$0 \to A(K, L) \to A(K) \to A(L) \to 0$$

is a short exact sequence of cochain complexes, natural with respect to $(K, L)$ and with respect to the simplicial cochain algebra $A.$

A key step in the proof of Theorem 3.8 is the following

**Proposition 3.5.** Suppose $\theta : D \to E$ is a morphism of simplicial cochain complexes. Assume that

(i) $H(\theta_n) : H(D_n) \to H(E_n)$ is an isomorphism, $n \geq 0.$

(ii) $D$ and $E$ are extendable.

Then for all simplicial sets $K,$

$$H(\theta(K)) : H(D(K)) \to H(E(K))$$

is an isomorphism.

First we establish a preliminary lemma. Define

$$\alpha : A(K(n), K(n-1)) \to \prod_{\sigma \in NK_n} A(\Delta[n], \partial\Delta[n]), n \geq 0,$$

by setting $\alpha : \Phi \mapsto \{A(\sigma_*\Phi)\}_{\sigma \in NK_n},$ where $\sigma_* : \Delta[n] \to K$ is the unique simplicial map (Lemma 3.2) such that $\sigma_*(c_n) = \sigma.$

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Lemma 3.6. If $K$ is a simplicial set and $A$ is extendable simplicial cochain complex then $\alpha$ is an isomorphism, natural in $A$ and in $K$.

Proof of 3.5. The proof is divided into two steps:

• First we show that for any simplicial set $K$, $\theta(K(n))$ is a quasi-isomorphims. Note that for any inclusion $L \subset M$ of simplicial sets we have the row exact commutative diagram:

$$
\begin{array}{c}
0 \\ \\
\downarrow \\
0 \\
\end{array}
\begin{array}{c}
D(M,L) \\
\theta(M,L) \\
D(M) \\
\theta(M) \\
D(L) \\
\theta(L) \\
0
\end{array}
\begin{array}{c}
0 \\
\downarrow \\
0 \\
\end{array}
\begin{array}{c}
E(M,L) \\
E(M) \\
E(L) \\
\end{array}
$$

Hence, if any two of the vertical arrows is a quasi-isomorphism, so is the third. Moreover Proposition 3.4 identifies $\theta(\Delta[n]) : D(\Delta[n]) \to E(\Delta[n])$ with $\theta_n : D_n \to E_n$. Hence $\theta(\Delta[n])$ is a quasi-isomorphism, $\geq 0$. From Lemma 3.6 and the remarks above we show by induction that $\theta(\cdot)$ is a quasi-isomorphism when $\cdot$ is in turn given by:

$$
\partial \Delta[n], \ (\Delta[n], \partial \Delta[n]), \ (K(n), K(n - 1)) \text{ and } K(n).
$$

• By Lemma 3.2 in [1], to show $\theta(K)$ is a quasi-isomorphism, it is enough to prove that for any given $\Phi \in D(K)$ and $\Psi \in E(K)$ such that $d\Phi = 0$ and $\theta(K) = \Phi = d\Psi$, there exist $\Omega \in D(K)$ and $\Gamma \in E(K)$ such that $\Phi = d\Omega$ and $\Psi = \theta(K)\Omega + d\Gamma$.

To find $\Omega$ and $\Gamma$ we construct inductively a sequence $\Omega_n \in D(K, K(n - 1))$ and $\Gamma_n \in E(K, K(n - 1))$ such that

$$
\Phi - \sum_{i \leq n} d\Omega_i \in D(K, K(n))
$$

and

$$
\Psi - \sum_{i \leq n} (\theta(K)\Omega_i + d\Gamma_i) \in E(K, K(n)).
$$

Thus we may define $\Omega \in D(K)$ and $\Gamma \in E(K)$ by $\Omega_\sigma = \sum_n (\Omega_n)_\sigma$ and $\Gamma_\sigma = \sum_n (\Gamma_n)_\sigma$. Clearly

$$
\Phi = d\Omega \quad \text{and} \quad \Psi = \theta(K)\Omega + d\Gamma.
$$

3.3 The simplicial commutative cochain algebra $A_{PL}$, and $A_{PL}(X)$

In this section, we construct the specific simplicial cochain algebra $A_{PL}$ and show it is extendable. Applying the construction above, we obtain $A_{PL}(X)$.

First consider the free graded commutative algebra $\Delta(t_0, \ldots, t_n, y_0, \ldots, y_n)$ in which the basis elements $t_i$ have degree zero and basis elements $y_j$ have degree 1.

Now define $A_{PL} = \{(A_{PL})_n\}_{n \geq 0}$ by:
The cochain algebra \((A_{PL})_n\) is given by
\[
(A_{PL})_n = \Lambda(t_0, \ldots, t_n, y_0, \ldots, y_n); \\
\sum t_i - 1, \sum y_j
\]
\[dt_i = y_i \quad \text{and} \quad dy_j = 0.\]

The face and degeneracy morphisms are the cochain algebra morphisms
\[
\partial_i : (A_{PL})_{n+1} \to (A_{PL})_n \quad \text{and} \quad s_j : (A_{PL})_n \to (A_{PL})_{n+1}
\]
satisfying
\[
\partial_i : t_k \mapsto \begin{cases} t_k, & k < i \\ 0, & k = i \\ t_{k-1}, & k > i \end{cases}
\]
and
\[
s_j : t_k \mapsto \begin{cases} t_k, & k < j \\ t_k + t_{k+1}, & k = j \\ t_{k+1}, & k > j \end{cases}
\]

Notice that the inclusions \(t_i \to (A_{PL})_n\) and \(y_j \to (A_{PL})_n\) extend to an isomorphism of cochain algebras,
\[
(\Lambda(t_1, \ldots, t_n, y_1, \ldots, y_n), d) \xrightarrow{\cong} (A_{PL})_n.
\]

**Lemma 3.7.**  
(i) \((A_{PL})_0 = k \cdot 1.\)

(ii) \(H((A_{PL})_n) = k \cdot 1, n \geq 0.\)

(iii) Each \(A_{PL}^p\) is extendable.

The construction \(A_{PL}(\cdot)\) defined in § 3.2 assigns to each simplicial set \(K\) a commutative cochain algebra, \(A_{PL}(K)\), and to each map \(f\) of simplicial sets a morphism, \(A_{PL}(f)\) of commutative cochain algebras. Since \(A_{PL}\) is extendable, with every pair \(L \subset K\) of simplicial sets is associated
\[
0 \to A_{PL}(K, L) \to A_{PL}(K) \to A_{PL}(L) \to 0.
\]

For topological spaces \(X\) and continuous maps \(f\) we apply this construction to the simplicial set \(S_*(X)\) and to \(S_*(f)\). This defines a contravariant functor from spaces to commutative cochain algebras, which will be denoted
\[
X \leadsto A_{PL}(X) \quad \text{and} \quad f \leadsto A_{PL}(f).
\]

In particular, associated with a subspace \(Y\) is the short exact sequence
\[
0 \to A_{PL}(X, Y) \to A_{PL}(X) \to A_{PL}(Y) \to 0.
\]
3.4 The simplicial cochain algebra $C_{PL}$, and the main theorem

In this section, we construct another simplicial cochain algebra $C_{PL}$ and show that for each simplicial set $K$, there are natural isomorphisms $C_{PL}(K) \xrightarrow{\sim} C^*(K)$, where $C^*(K)$ is the “ordinary” cochain algebra of $K$. In particular, $C_{PL}(X) \xrightarrow{\sim} C^*(X)$ for each topological space $X$. Applying Proposition 3.5, we prove the main theorem by showing the morphisms $C_{PL}(K) \rightarrow (C_{PL} \otimes A_{PL})(K) \leftarrow A_{PL}(K)$ are quasi-isomorphisms.

First recall that every topological space $X$ is associated with its singular cochain algebra, $C^*(X, \mathbb{k})$ ($\mathbb{k}$ is an arbitrary field with character zero). We simplify notation and write $C^*(X)$ for $C^*(X, \mathbb{k})$. The construction can be generalized to any simplicial set $K$ to give the cochain algebra $C^*(K)$, also written $C^*(K, \mathbb{k})$ if we need to emphasize coefficients. More precisely

- $C^*(K) = \{C^p(K)\}_{p \geq 0};$
- $C^p(K)$ consists of the set maps $K_p \rightarrow \mathbb{k}$ vanishing on degenerate simplices.
- For $f \in C^p(K)$, $g \in C^q(K)$ the product is given by
  $$(f \cup g)(\sigma) = (-1)^{pq}f(\partial_{p+1} \ldots \partial_{p+q} \sigma) \cdot g(\partial_0 \partial_0 \ldots \partial_0 \sigma), \sigma \in K_{p+q}.$$
- The differential, $d$, is given by
  $$(df)(\sigma) = \sum_{i=0}^{p+1} (-1)^{p+i+1} f(\partial_i \sigma), \sigma \in K_{p+1}, f \in C^p(K).$$

Observe that, by definition, $C^*(X) = C^*(S_*(X))$ for topological spaces $X$.

Next we define a simplicial cochain algebra $C_{PL}$ using the simplicial sets $\Delta[n] \subset S_*(\Delta^n)$ introduced in § 3.1. Indeed since the face inclusions and degeneracy maps for the $\Delta^n$ have the form

- $\lambda_i = (e_0, \ldots, \hat{e}_i, \ldots, e_{n+1}) : \Delta^n \rightarrow \Delta^{n+1}$
- $\rho_j = (e_0, \ldots, e_j, e_j, \ldots, e_n) : \Delta^{n+1} \rightarrow \Delta^n$,

it follows that $S_*(\lambda_i)$ and $S_*(\rho_j)$ restrict to simplicial maps

$$[\lambda_i] : \Delta[n] \rightarrow \Delta[n + 1] \text{ and } [\rho_j] : \Delta[n + 1] \rightarrow \Delta[n].$$

Thus we define $C_{PL} = \{(C_{PL})_n\}_{n \geq 0}$ by

- $(C_{PL})_n$ is the cochain algebra $C^*(\Delta[n]).$
- The face and degeneracy morphisms are the $C^*([\lambda_i])$ and $C^*([\rho_j]).$
Finally, let
\[ C_{PL} \otimes A_{PL} = \{(C_{PL})_n \otimes (A_{PL})_n; \partial_i \otimes \partial_i; s_j \otimes s_j\} \]
be the tensor product simplicial cochain algebra. Morphisms
\[ C_{PL} \rightarrow (C_{PL} \otimes A_{PL}) \leftarrow A_{PL} \]
are defined by \( \gamma \mapsto \gamma \otimes 1 \) and \( \Phi \mapsto 1 \otimes \Phi \). Thus, for any simplicial set \( K \) they determine the natural cochain algebra morphisms
\[ C_{PL}(K) \rightarrow (C_{PL} \otimes A_{PL})(K) \leftarrow A_{PL}(K). \]

**Theorem 3.8.** Let \( K \) be a simplicial set. Then

(i) There is a natural isomorphism \( C_{PL}(K) \cong C^*(K) \) of cochain algebras.

(ii) The natural morphisms of cochain algebras,
\[ C_{PL}(K) \rightarrow (C_{PL} \otimes A_{PL})(K) \leftarrow A_{PL}(K) \]
are quasi-isomorphisms.

Substituting \( S_*(X) = K \) in Theorem 3.8 we obtain

**Corollary 3.9.** For topological spaces \( X \) there are natural quasi-isomorphisms of cochain algebras.
\[ C^*(X) \cong (C_{PL} \otimes A_{PL})(X) \cong A_{PL}(X) \]

This gives the isomorphisms \( H^*(X) \cong H(A_{PL}(X)) \).

For the proof of Theorem 3.8 we require two lemmas.

**Lemma 3.10.** There are natural isomorphisms \( C_{PL}(K) \cong C^*(K) \).

**Lemma 3.11.**

(i) \( H((C_{PL})_n) = k = H((C_{PL})_n \otimes (A_{PL})_n), n \geq 0 \).

(ii) \( C_{PL} \) is extendable.

(iii) \( C_{PL} \otimes A_{PL} \) is extendable.

Then the first assertion of Theorem 3.8 is Lemma 3.10. The second follows by applying Proposition 3.5 to the morphisms
\[ \theta_C : C_{PL} \rightarrow C_{PL} \otimes A_{PL} \text{ and } \theta_A : A_{PL} \rightarrow C_{PL} \otimes A_{PL} \]
given by \( \gamma \mapsto \gamma \otimes 1 \) and \( \Phi \mapsto 1 \otimes \Phi \).

**4 Sullivan model**

In this chapter the ground ring is an arbitrary field \( k \) of characteristic zero.

In § 4.1, the definitions of Sullivan algebra, Sullivan model and minimal Sullivan model will be presented. And we will show the existence of these models under mild conditions. In § 4.2, the ideal “homotopy” is extended in Sullivan algebras and the homotopy relation between topological spaces will be “preserved” between their Sullivan models.
4.1 Sullivan algebras and models: constructions and examples

**Definition 4.1.** A Sullivan algebra is a commutative cochain algebra of the form $(\Lambda V, d)$, where

- $V = \{V^p\}_{p \geq 1}$ and, as usual, $\Lambda V$ denotes the free graded commutative algebra on $V$;
- $V = \bigcup_{k=0}^{\infty} V(k)$, where $V(0) \subset V(1) \subset \ldots$ is an increasing sequence of graded subspaces such that
  
  $d = 0$ in $V(0)$ and $d : V(k) \to \Lambda V(k-1), \ k \geq 1$.

The second condition is called the nilpotence condition on $d$. It can be restated as: $d$ preserves each $\Lambda V(k)$, and there exist graded subspaces $V_k \subset V(k)$ such that $\Lambda V(k) = \Lambda V(k-1) \otimes \Lambda V_k$, with $d : V_k \to \Lambda V(k-1)$.

**Definition 4.2.**
- A Sullivan model for a commutative cochain algebra $(A, d)$ is a quasi-isomorphism $m : (\Lambda V, d) \to (A, d)$ from a Sullivan algebra $(\Lambda V, d)$.
- If $X$ is a path connected topological space then a Sullivan model for $A_{PL}(X)$,
  
  $m : (\Lambda V, d) \cong A_{PL}(X)$,

  is called a Sullivan model for $X$.
- A Sullivan algebra (or model), $(\Lambda V, d)$ is called minimal if
  
  $\text{Im} d \subset \Lambda^+ V \cdot \Lambda^+ V$.

Our first step is to show the existence of Sullivan models:

**Proposition 4.3.** Any commutative cochain algebra $(A, d)$ satisfying $H^0(A) = k$ has a Sullivan model $m : (\Lambda V, d) \cong (A, d)$.

*Proof.* We construct this so that $V$ is the direct sum of graded subspaces $V_k, k \geq 0$ with $d = 0$ in $V_0$ and $d : V_k \to \Lambda(V_0, d) = k, H(m_k)$ is surjective.

First, set $V_0 = H^+(A)$ and extend the inclusion map $V_0 \to H(A)$ to the graded morphism $m_0 : (\Lambda V_0, 0) \to (A, d)$. Since $H^0(A) = k$, $H(m_0)$ is surjective.

Suppose $m_0$ has been extended to $m_k : (\Lambda(\bigoplus_{i=0}^{k} V_i), d) \to (A, d)$. Let $z_\alpha$ be cocycles in $\Lambda(\bigoplus_{i=0}^{k} V_i)$ such that $[z_\alpha]$ is a basis for $\ker H(m_k)$. Let $V_{k+1}$ be a graded space with basis...
{v_\alpha} in 1-1 correspondence with the z_\alpha, and with deg v_\alpha = deg z_\alpha - 1. Extend d to a derivation in \Lambda(\bigoplus_{i=0}^{k+1} V_i) by setting dv_\alpha = z_\alpha. Since d has odd degree, d is a derivation of \Lambda(\bigoplus_{i=0}^{k+1} V_i).

Since H(m_k)[z_\alpha] = 0, m_kz_\alpha = da_\alpha, a_\alpha \in A. Extend m_k to a graded algebra morphism m_{k+1} : \Lambda(\bigoplus_{i=0}^{k+1} V_i) \rightarrow A by setting m_{k+1}v_\alpha = a\alpha. Then m_{k+1}dv_\alpha = dm_{k+1}v_\alpha, and so m_{k+1}d = dm_{k+1}.

This completes the construction of m : (\Lambda V, d) \rightarrow (A, d) with V = \bigoplus_{k=0}^{\infty} V_k and m|_{\Lambda V_k} = m_k. Since m|_{\Lambda V_0} = m_0, and H(m_0) is surjective, H(m_k) is surjective as well. If H(m)[z] = 0 then, since z is necessarily in some \Lambda(\bigoplus_{i=0}^{k} V_i), H(m_k) = [z] = 0. By construction, z is a coboundary in \Lambda(\bigoplus_{i=0}^{k+1} V_i). Thus H(m) is an isomorphism.

Next we show by induction on k that V_k is concentrated in degrees \geq 1. This is certainly true for k = 0, because V_0 \cong H^+(A). Assume it true for V_i, i \leq k. Any element in \Lambda(\bigoplus_{i=0}^{k} V_i) of degree 1 then has the form

\[ v = v_0 + \ldots + v_k, v_i \in V_i. \]

Thus if dv = 0 then dv_k \in d(\Lambda(\bigoplus_{i=0}^{k} V_i)). By construction, this implies v_k = 0. Repeating this argument we find v = v_0 and H(m_k)[v_0] = H(m_0)[v_0] \neq 0, unless v_0 = 0. Thus ker H(m_k) vanishes in degree 1; i.e., it is concentrated in degrees \geq 2. It follows that V_{k+1} is concentrated in degrees \geq 1.

Finally, the nilpotence condition on d is built into the construction. \qed

**Example 4.4. The spheres, S^k.**

Let \[ [S^k] \in H_k(S^k; \mathbb{Z}). \] This determines a unique class \( \omega \in H^k(\text{APL}(S^k)) \) such that \( \langle \omega, [S^k] \rangle = 1 \), and \( \omega \) is a basis for \( H(\text{APL}(S^k)) \). Let \( \Phi \) be a representative cocycle for \( \omega \).

Now if \( k \) is odd then a minimal Sullivan model for \( S^k \) is given by

\[ m : (\Lambda(e), 0) \rightarrow \text{APL}(S^k), \text{deg} e = k, me = \Phi. \]

Suppose, on the other hand, that \( k \) is even. We may still define \( m : (\Lambda(e), 0) \rightarrow \text{APL}(S^k) \) by \( de = k, me = \Phi. \) But now, because \( de \) is even, \( \Lambda(e) \) has a basis \( 1, e, e^2, e^3, \ldots \) and this morphism is not a quasi-isomorphism. However, \( \Phi^2 \) is certainly a coboundary. Write \( \Phi^2 = d\Psi \) and extend \( m \) to

\[ m : (\Lambda(e, e'), d) \rightarrow \text{APL}(S^k) \]

by setting \( \text{deg} e' = 2k - 1, de' = e^2 \) and \( me' = \Psi. \) Thus this is a minimal model for \( S^k. \)
Example 4.5. Products of topological spaces.

Suppose \( m_X : (\Lambda V, d) \rightarrow A_{PL}(X) \) and \( m_Y : (\Lambda W, d) \rightarrow A_{PL}(Y) \) are Sullivan models for path connected topological spaces \( X \) and \( Y \). Assume further that the rational homology of one of these spaces has finite type. Let \( p^X : X \times Y \rightarrow X \) and \( p^Y : X \times Y \rightarrow Y \) be the projections. Then \( A_{PL}(p^X) \cdot A_{PL}(p^Y) : A_{PL}(X) \otimes A_{PL}(Y) \rightarrow A_{PL}(X \times Y) \) is a quasi-isomorphism of cochain algebras.

Since \( A_{PL}(p^X) \cdot A_{PL}(p^Y) \) is a quasi-isomorphism so is
\[
m_X \cdot m_Y : (\Lambda V, d) \otimes (\Lambda W, d) \xrightarrow{\sim} A_{PL}(X \times Y),
\]
where \( (m_X \cdot m_Y)(a \otimes b) = A_{PL}(p^X)m_Xa \cdot A_{PL}(p^Y)b \). This exhibits \( (\Lambda V, d) \otimes (\Lambda W, d) \) as a Sullivan model for \( X \times Y \). Furthermore, if \( (\Lambda V, d) \) and \( (\Lambda W, d) \) are minimal Sullivan models then so is their tensor product.

Example 4.6. \( H \)-spaces have minimal Sullivan models of the form \( (\Lambda V, 0) \).

An \( H \)-spaces is a based topological space \( (X, *) \) together with a continuous map \( \mu : X \times X \rightarrow X \) such that the self maps \( x \mapsto \mu(x, *) \) and \( x \mapsto \mu(*, x) \) of \( X \) are homotopic to the identity. Also assume \( X \) is path connected and \( H_*(X; k) \) has finite type. We claim that the minimal Sullivan model of \( X \) is of the form \( (\Lambda V, 0) \), where \( V \) is a graded space with a basis 1-1 correspondence to the generators of \( H^+(X; k) \).

To see this, observe first that because \( H_*(X; k) \) has finite type, \( H^*(\mu) \) can be identified as a morphism of graded algebras,
\[
H^*(\mu) : H^*(X; k) \rightarrow H^*(X; k) \otimes H^*(X; k).
\]
Moreover, the conditions \( \mu(x, *) \circ id, \mu(*, x) \circ id \) imply that for \( h \in H^+(X; k) \),
\[
H^*(\mu)h = h \otimes 1 + \Phi + 1 \otimes h, \quad \text{some} \quad \Phi \in H^+(X; k) \otimes H^+(X; k).
\]
This implies that \( H^*(X; k) \) is a Hopf algebra. Let \( V \) be a graded with basis \( \{v_i\} \) which is 1-1 correspondence to the generators of \( H^+(X; k) \). Then by Theorem 3C.4 in [3], there is an isomorphism between algebras
\[
\varphi : \Lambda V \xrightarrow{\sim} H^*(X; k)
\]
by mapping \( v_i \) to the corresponding generator of \( H^*(X; k) \).

Finally, let \( w_i \in A_{PL}(X) \) be cocycles representing the cohomology classes which are the images of \( v_i \) in the above morphism. The correspondence \( v_i \mapsto w_i \) defines a linear map \( V \rightarrow A_{PL}(X) \) which extends to a unique morphism \( m : (\Lambda V, 0) \rightarrow A_{PL}(X) \). Since \( \varphi \) is an isomorphism it follows that \( m \) is a quasi-isomorphism:
\[
m : (\Lambda V, 0) \xrightarrow{\sim} A_{PL}(X)
\]
Thus \( m \) is a minimal Sullivan model for the \( H \)-space \( X \).

Again, suppose \( (A, d) \) is a commutative cochain algebra. In fact if \( H^0(A) = k \) then \( (A, d) \) has a (unique) minimal Sullivan model; this follows from a more general result about relative Sullivan algebras. However, if also \( H^1(A) = 0 \) then there is a simple inductive construction for a minimal model. We carry this out here.

Thus suppose given \( (A, d) \) with \( H^0(A) = k \) and \( H^1(A) = 0 \).
Choose $m_2 : \langle 0 \rangle \to (A, d)$ so that $H^2(m_2) : V^2 \cong H^2(A)$. Note that $H^1(m_2)$ is an isomorphism because $H^1(A) = 0$ and that $H^2(m_2)$ is injective because $(V^2)^3 = 0$.

Supposing that $m_k : \langle 0 \rangle \to (A, d)$ is constructed, we extend to $m_{k+1} : (V^k, d) \to (A, d)$ by the following procedure.

Choose cocycles $a_\alpha \in A^{k+1}$ and $z_\beta \in \langle V^k \rangle$ so that $H^k(m_k) : \langle a_\alpha \rangle \to \langle a_\alpha \rangle$ is an isomorphism because $H^k(A) = 0$ and that $H^{k+1}(m_{k+1})$ is injective because $(V^{k+1})^3 = 0$.

Choose cocycles $a_\alpha \in A^{k+1}$ and $z_\beta \in \langle V^k \rangle$ so that $H^k(m_k) : \langle a_\alpha \rangle \to \langle a_\alpha \rangle$ is an isomorphism because $H^k(A) = 0$ and that $H^{k+1}(m_{k+1})$ is injective because $(V^{k+1})^3 = 0$.

In particular, $m_k z_\beta = db_\beta$, some $b_\beta \in A$.

Let $V^{k+1}$ be a vector space (in degree $k+1$) with basis $\{v'_\alpha, v''_\beta\}$ in 1-1 correspondence with the elements $\{a_\alpha\}, \{z_\beta\}$. Write $\Lambda V^{k+1} = \Lambda V^k \otimes \Lambda V^1$. Extend $d$ and $m_k$, respectively, to a derivation in $\Lambda V^{k+1}$ and to a morphism $m_{k+1} : \Lambda V^{k+1} \to A$ of graded algebras, by setting

$$d v'_\alpha = 0, \quad d v''_\beta = z_\beta, \quad m v'_\alpha = a_\alpha, \quad m v''_\beta = b_\beta.$$  

Since $d$ has degree 1, $d^2$ is a derivation. By construction, $d^2 = 0$ in $V^{k+1}$ and in $\Lambda V^k$. Thus $d^2 = 0$. In the same way $m_{k+1} d = d m_{k+1}$ in $V^{k+1}$ and $\Lambda V^k$, and so $m_{k+1} d = d m_{k+1}$.

**Proposition 4.7.** Suppose $(A, d)$ is a commutative cochain algebra such that $H^0(A) = \mathbb{k}$ and $H^1(A) = 0$. Then

(i) The morphism $m : \langle \Lambda V, d \rangle \to (A, d)$ constructed above is a minimal Sullivan model.

(ii) If $r$ is the least integer greater than zero such that $H^r(A) \neq 0$, then $V^i = 0$, $1 \leq i < r$ and $H^r(m) : V^r \cong H^r(A)$.

(iii) If $\dim H^k(A) < \infty$, $k \geq 1$, then $\dim V^k < \infty$, $k \geq 1$.

**Corollary 4.8.** Let $X$ be a simply connected topological space such that each $H_i(X; \mathbb{Q})$ is finite dimensional. Then $X$ has a minimal Sullivan model $m : \langle \Lambda V, d \rangle \cong A_{PL}(X)$ such that $V = \{V^i\}_{i \geq 2}$ and each $V^i$ is finite dimensional.

### 4.2 Homotopy in Sullivan algebras

First of all, we introduce the definition of homotopy in Sullivan algebras. Define augmentations

$$\varepsilon_0, \varepsilon_1 : \Lambda(t, dt) \to \mathbb{k} \quad \text{by} \quad \varepsilon_0(t) = 0, \varepsilon_1(t) = 1,$$

where $\Lambda(t, dt)$ is the free commutative graded algebra on the basis $\{t, dt\}$ with $\deg t = 0, \deg t = 1$, and $d$ is the differential sending $t \mapsto dt$.  

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**Definition 4.9.** Two morphisms \( \varphi_0, \varphi_1 : (\Lambda V, d) \to (A, d) \) from a Sullivan algebra to a commutative cochain algebra are homotopic if there is a morphism

\[
\Phi : (\Lambda V, d) \to (A, d) \otimes (\Lambda(t, dt), d)
\]

such that \( (id \cdot \epsilon_i) \Phi = \varphi_i, i = 0, 1 \). Here \( \Phi \) is called a homotopy from \( \varphi_0 \) to \( \varphi_1 \) and we write \( \varphi_0 \sim \varphi_1 \).

Observe that \( \Lambda(t, dt) \) is precisely the cochain algebra \( (A_{PL})_1 \) and by Proposition 3.4(i) is the commutative cochain algebra of the standard 1-simplex. Given that \( A_{PL} \) is a contravariant functor, homotopy in Sullivan algebras is the obvious analogue of the topological homotopy \( X \leftarrow X \times I \).

In fact, \( A_{PL} \) ‘preserves homotopy’ in the following sense. Let \( X \) and \( Y \) be two topological spaces. Suppose given continuous maps \( f_0, f_1 : X \to Y \) and a morphism \( \psi : (\Lambda V, d) \to A_{PL}(Y) \) from a Sullivan algebra \( (\Lambda V, d) \).

**Proposition 4.10.** If \( f_0 \sim f_1 : X \to Y \) then \( A_{PL}(f_0) \psi \sim A_{PL}(f_1) \psi : (\Lambda V, d) \to A_{PL}(X) \).

Thus the proposition establish an injection:

\[
\begin{cases}
\text{rational homotopy type} & \mapsto \text{homotopy classes of maps} \\
\text{between path connected spaces} & \text{between Sullivan algebras}
\end{cases}
\]

The proof of the proposition is based on the following two lemmas, which are important tricks in the latter proofs. The first one is the lifting property. Consider a diagram of commutative cochain algebra morphisms

\[
(A, d) \quad \xrightarrow{\eta} \quad (\Lambda V, d) \quad \xleftarrow{\psi} \quad (C, d)
\]

in which \( \eta \) is a surjective quasi-isomorphism, and \( (\Lambda V, d) \) is a Sullivan algebra.

**Lemma 4.11.** (Lifting lemma) There is a morphism \( \varphi : (\Lambda V, d) \to (A, d) \) such that \( \eta \varphi = \psi \) (\( \varphi \) is a lift of \( \psi \) through \( \eta \)).

Second, with a graded vector space \( U = \{U^i\}_{i \geq 0} \) associate the commutative cochain algebra \( (E(U), \delta) \), defined by

\[
E(U) = \Lambda(U \oplus \delta U) \quad \text{and} \quad \delta : U \xrightarrow{\cong} U.
\]

It is augmented by \( \varepsilon : (E(U), \delta) \to k \), where \( \varepsilon(U) = 0 \). Note that if \( U = \{U^i\}_{i \geq 1} \) then this is a Sullivan algebra. Cochain algebras of this form are called contractible.

**Lemma 4.12.** \( \varepsilon : (E(U), \delta) \to k \) is a quasi-isomorphism; i.e.

\[
H(E(U), \delta) = k.
\]
As a direct consequence of this lemma we obtain:

**The surjective trick:** If \((A,d)\) is any commutative cochain algebra, then the identity of \(A\) extends uniquely to a surjective morphism \(\sigma : (E(A), \delta) \to (A,d)\). Thus any morphism \(\varphi : (B,d) \to (A,d)\) of commutative cochain algebras factors as

\[
(B, d) \xrightarrow{\lambda} (B, d) \otimes (E(A), \delta) \xrightarrow{\varphi \sigma} (A, d)
\]

in which the inclusion \(\lambda : b \mapsto b \otimes 1\) is a quasi-isomorphism and \(\varphi \cdot \sigma\) is surjective.

**proof of Proposition 4.10.** Identify \(\Lambda(t, dt)\) as a subcochain algebra of \(A_{PL}(I)\), by mapping \(t \mapsto u \in A^0_{PL}(I)\), where \(u\) restricts to 0 at \(\{0\}\) and to 1 at \(\{1\}\) (the existence of the required \(u\) follows from the fact that \(A^p_{PL}\) is extendable). Denote by \(j_0, j_1 : X \to X \times I\) the inclusions at the endpoints and by \(p^X : X \times I \to X\) and \(p^I : X \times I \to I\) the projections. Then

\[
\begin{align*}
A_{PL}(X) \otimes \Lambda(t, dt) & \xrightarrow{(id \otimes \varepsilon_0, id \otimes \varepsilon_1)} A_{PL}(p^X) \cdot A_{PL}(p^I) \\
& \xrightarrow{(A_{PL}(j_0), A_{PL}(j_1))} A_{PL}(X) \times A_{PL}(X) \\
& \xrightarrow{(id \otimes \varepsilon_0, id \otimes \varepsilon_1)} A_{PL}(X \times I)
\end{align*}
\]

is a commutative diagram of cochain algebra morphisms.

Since \(H(A_{PL}(p^X)) = H^*(p^X, \mathbb{k})\) is an isomorphism, \(A_{PL}(p^X) \cdot A_{PL}(p^I)\) is a quasi-isomorphism. We now make it surjective. Let \(U \subset A_{PL}(X \times I)\) be the kernel of \((A_{PL}(j_0), A_{PL}(j_1))\). The inclusion of \(U\) extends to a unique cochain algebra morphism

\[
\rho : (E(U), \delta) \to A_{PL}(X \times I).
\]

Extend the diagram above to the commutative diagram

\[
\begin{align*}
A_{PL}(X) \otimes \Lambda(t, dt) \otimes (E(U), \delta) & \xrightarrow{(id \otimes \varepsilon_0, id \otimes \varepsilon_1)} A_{PL}(p^X) \cdot A_{PL}(p^I) \cdot \rho \\
& \xrightarrow{(A_{PL}(j_0), A_{PL}(j_1))} A_{PL}(X) \times A_{PL}(X) \\
& \xrightarrow{(id \otimes \varepsilon_0, id \otimes \varepsilon_1)} A_{PL}(X \times I)
\end{align*}
\]

Here \(A_{PL}(p^X) \cdot A_{PL}(p^I) \cdot \rho\) is obviously, surjective, and it follows from Lemma 4.12 that it is a quasi-isomorphism too.

Let \(H : X \times I \to Y\) be a homotopy from \(f_0\) to \(f_1\). Use Lemma 4.11 to lift \(A_{PL}(H)\psi : (AV, d) \to A_{PL}(X \times I)\) through the surjective quasi-isomorphism \(A_{PL}(p^X) \cdot A_{PL}(p^I) \cdot \rho\). This produces a morphism

\[
\Psi : (AV, d) \to A_{PL}(X) \otimes \Lambda(t, dt) \otimes E(U).
\]

Then set \(\Phi = (id \otimes id \otimes \varepsilon)\Psi;\) it is the desired homotopy from \(A_{PL}(f_0)\psi\) to \(A_{PL}(f_1)\psi\). \(\square\)
As with homotopy of continuous maps, we have

**Proposition 4.13.** Homotopy is an equivalence relation in the set of morphisms \( \varphi : (\Lambda V, d) \to (A, d) \) from a Sullivan algebra.

**Notation** The set of homotopy classes of morphisms \( (\Lambda V, d) \to (A, d) \) will be denoted by \( [(\Lambda V, d), (A, d)] \). The homotopy class of a morphism \( \varphi \) will be denoted by \( [\varphi] \).

**Definition 4.14.** The linear part of a morphism \( \varphi : (\Lambda V, d) \to (\Lambda W, d) \) between Sullivan algebras is the linear map \( Q\varphi : V \to W \) defined by:

\[
\varphi v - Q\varphi v \in \Lambda^{\geq 2} W, \quad v \in V.
\]

Note that \( Q(\varphi) \) commutes with the linear parts of the differentials (denoted by \( d_0 \)).

**Proposition 4.15.** If \( \varphi_0 \sim \varphi_1 : (\Lambda V, d) \to (\Lambda W, d) \) are homotopic morphisms between minimal Sullivan algebras, and if \( H^1(\Lambda V, d) = 0 \), then \( Q\varphi_0 = Q\varphi_1 \).

### 4.3 Quasi-isomorphisms, Sullivan representatives, uniqueness of minimal models

Suppose \( \eta : (A, d) \to (C, d) \) is a morphism of commutative cochain algebras. If \( \Phi \) is homotopy between \( \varphi_0, \varphi_1 : (\Lambda V, d) \to (A, d) \) then \( (\eta \otimes id)\Phi : \eta\varphi_0 \sim \eta\varphi_1 \). Thus we can define

\[
\eta# : [(\Lambda V, d), (A, d)] \to [(\Lambda V, d), (C, d)] \quad \text{by} \quad \eta#[\varphi] = [\eta\varphi].
\]

Now suppose given commutative cochain algebra morphisms

\[
\begin{array}{ccc}
(\Lambda V, d) & \xrightarrow{\psi} & (C, d) \\
\eta & \downarrow & \\
(A, d)
\end{array}
\]

in which \( \eta \) is a (not necessarily surjective) quasi-isomorphism and \( (\Lambda V, d) \) is a Sullivan algebra. The Lifting lemma extends to the fundamental

**Proposition 4.16.** There is a unique homotopy class of morphism \( \varphi : (\Lambda V, d) \to (A, d) \) such that \( \eta\varphi \sim \psi \). Thus

\[
\eta# : [(\Lambda V, d), (A, d)] \xrightarrow{\sim} [(\Lambda V, d), (C, d)]
\]

is a bijection.
Proof. • First prove the assertion under the additional hypothesis that \( \eta \) is surjective.

In this case the surjectivity of \( \eta_{\#} \) follows from the Lifting lemma 4.11.

To show \( \eta_{\#} \) is injective, suppress the differentials from the notation (for simplicity).

Use the morphisms \( C \otimes \Lambda(t, dt) \xrightarrow{(id_{c \otimes 0}, id_{c \otimes 1})} C \times C \xrightarrow{\eta \times \eta} A \times A \) to construct a fibre product, and observe that

\[
(\eta \otimes id, id_A \cdot \varepsilon_0, id_A \cdot \varepsilon_1) : A \otimes \Lambda(t, dt) \to [C \otimes \Lambda(t, dt)] \times_{C \times C} (A \times A)
\]

is a surjective quasi-isomorphism of cochain algebras.

Now suppose \( \gamma_0, \gamma_1 : \Lambda V \to A \) are cochain algebra morphisms, and that \( \Psi \) is a homotopy from \( \eta \gamma_0 \) to \( \eta \gamma_1 \). Lift the morphism \((\Psi, \gamma_0, \gamma_1) : \Lambda V \to [C \otimes \Lambda(t, dt)] \times_{C \times C} (A \times A)\) through the surjective quasi-isomorphism \((\eta \otimes id, id_A \cdot \varepsilon_0, id_A \cdot \varepsilon_1)\). This defines a morphism \( \Phi : \Lambda V \to A \otimes \Lambda(t, dt) \), and it is immediate from the construction that \( \Phi : \gamma_0 \sim \gamma_1 \).

• As for the general case, let \( E(C, \delta) \) be the acyclic cochain algebra of Lemma 4.12 and \( \rho : E(C) \to C \) is defined by \( c \mapsto c, \delta c \mapsto dc \). Consider the morphisms

\[
(A, d) \xrightarrow{\lambda} (A, d) \otimes (E(C), \delta) \xrightarrow{\eta \rho} (C, d).
\]

Suppose

\[
\alpha : (A, d) \to (A', d)
\]

is an arbitrary morphism of commutative cochain algebras that satisfy \( H^0(-) = \mathbb{k} \). Let \( m : (\Lambda V, d) \xrightarrow{\sim} (A, d) \) and \( m' : (\Lambda V', d) \to (A', d) \) be Sullivan models. By Proposition 4.16 there is a unique homotopy class of morphisms

\[
\varphi : (\Lambda V, d) \to (\Lambda V', d)
\]

such that \( m' \varphi \sim \alpha m \).

**Definition 4.17.** A morphism \( \varphi : (\Lambda V, d) \to (\Lambda V', d) \) such that \( m' \varphi \sim \alpha m \) is called a Sullivan representative for \( \alpha \). If \( f : X \to Y \) is a continuous map then a Sullivan representative of \( A_{PL}(f) \) is called a Sullivan representative of \( f \).

It follows at once from Proposition 4.10 that Sullivan representatives of homotopic maps are homotopic morphisms.

As a second application of Proposition 4.16 we deduce the uniqueness of minimal models in the simply connected case.

**Proposition 4.18.** If \( (A, d) \) is a commutative cochain algebra and \( H^0(A) = \mathbb{k}, H^1(A) = 0 \), then the minimal models of \( (A, d) \) are all isomorphic.

Thus we establish an injection

\[
\begin{align*}
\left\{ \text{rational homotopy type} \right\} & \hookrightarrow \left\{ \text{isomorphic classes of} \right\} \\
\left\{ \text{of simply connected spaces} \right\} & \hookrightarrow \left\{ \text{minimal Sullivan algebras} \right\}
\end{align*}
\]
5 Relative Sullivan algebras

In this chapter we will generalize the Sullivan algebras to the relative Sullivan algebras and extend the Lifting lemma to the relative case.

5.1 The existence of relative Sullivan models

**Definition 5.1.** A relative Sullivan algebra is a commutative cochain algebra of the form $(B \otimes \Lambda V, d)$ (identify $B = B \otimes 1$ and $\Lambda V = 1 \otimes \Lambda V$ for simplicity) where

- $(B, d) = (B \otimes 1, d)$ is a sub cochain algebra, and $H^0(B) = k$.
- $1 \otimes V = V = \{V^p\}_{p \geq 1}$.
- $V = \bigcup_{k=0}^{\infty} V(k)$, where $V(0) \subset V(1) \subset \ldots$ is an increasing sequence of graded subspaces such that
  
  $$d : V(0) \to B \quad \text{and} \quad d : V(k) \to B \otimes \Lambda V(k-1), k \geq 1.$$ 

The third condition is called the nilpotence condition on $d$. It can be restated as follows: if we put $V(-1) = 0$ and write $V(k) = V(k-1) \oplus V_k$ for some graded subspace $V_k$ then

$$B \otimes \Lambda V(k) = [B \otimes \Lambda V(k-1)] \otimes \Lambda V_k, \quad \text{and} \quad d : V_k \to B \otimes \Lambda V(k-1), k \geq 0.$$ 

Note that a Sullivan algebra is just a relative Sullivan algebra with $B = k$. Now generalizing the Sullivan models, we consider morphisms of commutative cochain algebras

$$\varphi : (B, d) \to (C, d)$$

such that $H^0(B) = k$ and make the

**Definition 5.2.** A Sullivan model for $\varphi$ is a quasi-isomorphism of cochain algebras

$$m : (B \otimes \Lambda V, d) \cong (C, d)$$

such that $(B \otimes \Lambda V, d)$ is a relative Sullivan algebra with base $(B, d)$ and $m|_B = \varphi$.

If $f : X \to Y$ is a continuous map then a Sullivan model for $A_{PL}(f)$ is called a Sullivan model for $f$.

As in the case of Sullivan algebras, we define the minimal Sullivan algebras:

**Definition 5.3.** A relative Sullivan algebra $(B \otimes \Lambda V, d)$ is minimal if

$$\text{Im} d \subset B^+ \otimes \Lambda V + B \otimes \Lambda \geq 2 V.$$
A minimal Sullivan model for $\varphi : (B, d) \to (C, d)$ is a Sullivan model $(B \otimes \Lambda V, d) \xrightarrow{\cong} (C, d)$ such that $(B \otimes \Lambda V, d)$ is minimal.

A useful fact for the relative Sullivan algebras is the ‘preservation under pushout’. Suppose $(B \otimes \Lambda V, d)$ is a relative Sullivan algebra and $\psi : (B, d) \to (B', d)$ is a morphism of commutative cochain algebras with $H^0(B') = k$. Then, immediately from the definition, the cochain algebra

$$(B', d) \otimes_{(B, d)} (B \otimes \Lambda V, d) = (B' \otimes \Lambda V, d)$$

is a relative Sullivan algebra. It is called the pushout of $(B \otimes \Lambda V, d)$ along $\psi$. We have the result:

**Lemma 5.4.** If $\psi$ is a quasi-isomorphism so is $\psi \otimes id : (B \otimes \Lambda V, d) \to (B' \otimes \Lambda V, d)$.

Now we can establish the existence of Sullivan models for morphisms:

**Proposition 5.5.** A morphism $\varphi : (B, d) \to (C, d)$ of commutative cochain algebras has a Sullivan model if $H^0(B) = k = H^0(C)$ and $H^1(\varphi)$ is injective.

**Proof.** Choose a grade subspace $B_1 \subset B$ so that

$$(B_1)^0 = k, \quad (B_1)^1 \oplus d(B^0) = B^1, \quad (B_1)^n = B^n, \quad n \geq 2.$$  

Clearly $(B_1, d)$ is a sub cochain algebra and the inclusion $\varphi : (B_1, d) \to (B, d)$ is a quasi-isomorphism. In particular the restriction $\varphi_1 : (B_1 \otimes \Lambda V, d) \xrightarrow{\cong} (C, d)$ of $\varphi$ satisfies: $H^1(\varphi_1)$ is injective.

Now because $(B_1)^0 = \mathbb{k}$ the argument of Proposition 4.3 applies verbatim to show the existence of a Sullivan model $m_1 : (B \otimes \Lambda V, d) \xrightarrow{\cong} (C, d)$ for $\varphi_1$. Thus a commutative diagram of cochain algebra morphisms is given by

$$
\begin{array}{ccc}
(B, d) \otimes (B_1, d) & \xrightarrow{m} & (C, d) \\
\downarrow j & & \downarrow m_1 \\
(B_1 \otimes \Lambda V, d) & & \\
\end{array}
$$

in which $j(z) = 1 \otimes_{B_1} z$.

This may be rewritten as

$$
\begin{array}{ccc}
(B \otimes \Lambda V, d) & \xrightarrow{m} & (C, d) \\
\downarrow i \otimes id & & \downarrow m_1 \\
(B_1 \otimes \Lambda V, d) & & \\
\end{array}
$$

Lemma 5.4 and the preceding remarks show that $i \otimes id$ is a quasi-isomorphism and $(B \otimes \Lambda V, d)$ is a Sullivan algebra; hence $m : (B \otimes \Lambda V, d) \xrightarrow{\cong} (C, d)$ is a Sullivan model for $\varphi$. \qed

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Finally, we extend the lifting lemma to the relative case and define relative homotopy. To begin, suppose given a commutative diagram of morphisms of commutative cochain algebras

\[
\begin{array}{ccc}
(B, d) & \xrightarrow{\alpha} & (A, d) \\
\downarrow{i} & \searrow{=} & \downarrow{=} \\
(B \otimes \Lambda V, d) & \xrightarrow{\psi} & (C, d)
\end{array}
\]

in which \(i\) is the base inclusion of a relative Sullivan algebra and \(\eta\) is a surjective quasi-isomorphism. An argument identical to that of Lemma 4.11 establishes

**Lemma 5.6.** There is a morphism \(\varphi : (B \otimes \Lambda V, d) \to (A, d)\) such that \(\varphi i = \alpha\) (\(\varphi\) extends \(\alpha\)) and \(\eta \varphi = \psi\) (\(\varphi\) lifts \(\psi\)).

### 6 Fibrations and homotopy groups

In § 6.1 we will first construct models of fibrations, which provide a method to compute the Sullivan models of fibres using the Sullivan models of total spaces and base spaces.

Suppose \(X\) is a simply connected space with a minimal Sullivan model \(m_X : (\Lambda V_X, d) \to A_{PL}(X)\). Using models of fibrations, a bijection between \(V_X\) and \(\text{Hom}_k(\pi_*(X), k)\) will be established under mild conditions.

§ 6.3 studies the relationship among the minimal models of fibres, total spaces, and base spaces and a long exact sequence among them will be established.

§ 6.4 constructs models for path fibrations.

#### 6.1 Models of fibrations

Consider a Serre fibration of path connected spaces

\[
p : X \to Y,
\]

whose fibres are also path connected. Let \(j : F \to X\) be the inclusion of the fibre at \(y_0 \in Y\). Then \(A_{PL}\) converts the diagram

\[
\begin{array}{ccc}
F & \xrightarrow{j} & X \\
\downarrow & & \downarrow{p} \\
y_0 & \to & Y
\end{array}
\]

to

\[
\begin{array}{ccc}
A_{PL}(F) & \xrightarrow{A_{PL}(j)} & A_{PL}(X) \\
\downarrow{\varepsilon} & & \downarrow{A_{PL}(p)} \\
k & \xleftarrow{\varepsilon} & A_{PL}(Y)
\end{array}
\]  

(6.1)

where \(\varepsilon\) is the augmentation corresponding to \(y_0\).
Observe that $H^1(A_{PL}(p))$ is injective. Then proposition 5.5 asserts the existence of a Sullivan model for $p$, 

$$m : (A_{PL}(Y) \otimes \Lambda V, d) \xrightarrow{\cong} A_{PL}(X).$$

The augmentation $\varepsilon : A_{PL}(Y) \rightarrow \mathbb{k}$ defines a quotient Sullivan algebra 

$$(\Lambda V, \bar{d}) = \mathbb{k} \otimes_{A_{PL}(Y)} (A_{PL}(Y) \otimes \Lambda V, d),$$

which is called the fibre of the model at $y_0$.

Since $A_{PL}(j)A_{PL}(p)$ reduces to $\varepsilon$ in $A_{PL}(Y)$, $A_{PL}(j)m$ factors over $\varepsilon \cdot id$ to give the commutative diagram of cochain algebra morphisms

$$
\begin{array}{ccc}
A_{PL}(F) & \xrightarrow{A_{PL}(j)} & A_{PL}(X) \\
\downarrow m & & \downarrow m \\
(\Lambda V, \bar{d}) & \xleftarrow{\varepsilon \cdot id} & (A_{PL}(Y) \otimes \Lambda V, d)
\end{array}
$$

(6.2)

We show now that, under mild hypotheses, $\bar{m}$ is a quasi-isomorphism. Thus in this case $\bar{m} : (\Lambda V, \bar{d}) \xrightarrow{\cong} A_{PL}(F)$ is a Sullivan model for $F$: the fibre of a model is a model of the fibre.

**Theorem 6.1.** Suppose $Y$ is simply connected and one of the graded spaces $H_*(Y; \mathbb{k}), H_*(F; \mathbb{k})$ has finite type. Then 

$$\bar{m} : (\Lambda V, \bar{d}) \rightarrow A_{PL}(F)$$

is a quasi-isomorphism.

**Proof.**

- First suppose that $p$ is a fibration. Apply Theorem 7.1 in [1] with diagram (6.2) corresponding to diagram (7.9) in [1].

- Suppose only that $p$ is a Serre fibration. Using the diagram constructed in § 2.1

$$
\begin{array}{ccc}
X & \xrightarrow{\lambda} & X \times_Y MY \\
p \downarrow & & \downarrow q \\
Y & \leftarrow &
\end{array}
$$

with $\lambda x = (x, \text{const. path at } px)$, we apply the Lifting lemma 4.11 to the diagram

$$
\begin{array}{ccc}
A_{PL}(Y) & \xrightarrow{A_{PL}(q)} & A_{PL}(X \times_Y MY) \\
\downarrow & & \downarrow \cong \\
(A_{PL}(Y) \otimes \Lambda V, d) & \xrightarrow{\varepsilon \cdot id} & A_{PL}(X)
\end{array}
$$

(6.2)

We show now that, under mild hypotheses, $\bar{m}$ is a quasi-isomorphism. Thus in this case $\bar{m} : (\Lambda V, \bar{d}) \xrightarrow{\cong} A_{PL}(F)$ is a Sullivan model for $F$: the fibre of a model is a model of the fibre.

**Theorem 6.1.** Suppose $Y$ is simply connected and one of the graded spaces $H_*(Y; \mathbb{k}), H_*(F; \mathbb{k})$ has finite type. Then 

$$\bar{m} : (\Lambda V, \bar{d}) \rightarrow A_{PL}(F)$$

is a quasi-isomorphism.

**Proof.**

- First suppose that $p$ is a fibration. Apply Theorem 7.1 in [1] with diagram (6.2) corresponding to diagram (7.9) in [1].

- Suppose only that $p$ is a Serre fibration. Using the diagram constructed in § 2.1

$$
\begin{array}{ccc}
X & \xrightarrow{\lambda} & X \times_Y MY \\
p \downarrow & & \downarrow q \\
Y & \leftarrow &
\end{array}
$$

with $\lambda x = (x, \text{const. path at } px)$, we apply the Lifting lemma 4.11 to the diagram

$$
\begin{array}{ccc}
A_{PL}(Y) & \xrightarrow{A_{PL}(q)} & A_{PL}(X \times_Y MY) \\
\downarrow & & \downarrow \cong \\
(A_{PL}(Y) \otimes \Lambda V, d) & \xrightarrow{\varepsilon \cdot id} & A_{PL}(X)
\end{array}
$$

(6.2)

We show now that, under mild hypotheses, $\bar{m}$ is a quasi-isomorphism. Thus in this case $\bar{m} : (\Lambda V, \bar{d}) \xrightarrow{\cong} A_{PL}(F)$ is a Sullivan model for $F$: the fibre of a model is a model of the fibre.

**Theorem 6.1.** Suppose $Y$ is simply connected and one of the graded spaces $H_*(Y; \mathbb{k}), H_*(F; \mathbb{k})$ has finite type. Then 

$$\bar{m} : (\Lambda V, \bar{d}) \rightarrow A_{PL}(F)$$

is a quasi-isomorphism.

**Proof.**

- First suppose that $p$ is a fibration. Apply Theorem 7.1 in [1] with diagram (6.2) corresponding to diagram (7.9) in [1].

- Suppose only that $p$ is a Serre fibration. Using the diagram constructed in § 2.1

$$
\begin{array}{ccc}
X & \xrightarrow{\lambda} & X \times_Y MY \\
p \downarrow & & \downarrow q \\
Y & \leftarrow &
\end{array}
$$

with $\lambda x = (x, \text{const. path at } px)$, we apply the Lifting lemma 4.11 to the diagram

$$
\begin{array}{ccc}
A_{PL}(Y) & \xrightarrow{A_{PL}(q)} & A_{PL}(X \times_Y MY) \\
\downarrow & & \downarrow \cong \\
(A_{PL}(Y) \otimes \Lambda V, d) & \xrightarrow{\varepsilon \cdot id} & A_{PL}(X)
\end{array}
$$

(6.2)

We show now that, under mild hypotheses, $\bar{m}$ is a quasi-isomorphism. Thus in this case $\bar{m} : (\Lambda V, \bar{d}) \xrightarrow{\cong} A_{PL}(F)$ is a Sullivan model for $F$: the fibre of a model is a model of the fibre.
Theorem 6.1 establishes a fundamental property for relative Sullivan models \((A_{PL}(Y) \otimes \Lambda V, d) \cong A_{PL}(X)\). It is, however, useful to replace \(A_{PL}(Y)\) by a Sullivan model and this is done as follows.

Choose a Sullivan model \(m_Y : (\Lambda V_Y, d) \cong \to A_{PL}(Y)\). Since \(V_Y = \{V_Y^i\}_{i \geq 1}\) by definition, there is a unique augmentation \(\varepsilon : (\Lambda V_Y, d) \to k\). Construct a commutative diagram of cochain algebra morphisms

\[
\begin{array}{ccc}
A_{PL}(Y) & \xrightarrow{A_{PL}(p)} & A_{PL}(X) & \xrightarrow{A_{PL}(j)} & A_{PL}(F) \\
\downarrow m_Y & \cong & \downarrow m & \downarrow \bar{m} \\
(\Lambda V_Y, d) & \xrightarrow{i} & (\Lambda V_Y \otimes \Lambda V, d) & \xrightarrow{\varepsilon \cdot \text{id}} & (\Lambda V, \bar{d})
\end{array}
\]

by requiring that

- \(i\) is the inclusion of a relative Sullivan algebra, and
- \(m\) is a Sullivan model for the composite \(A_{PL}(p)m_Y\).

Then \(A_{PL}(j)m\) factors over \(\varepsilon \cdot \text{id}\) to yield \(\bar{m}\).

As in Theorem 6.1, we now restrict to the case that \(Y\) is simply connected and one of \(H_*(F; k), H_*(Y; k)\) has finite type. With these hypotheses we have

**Proposition 6.2.** The three morphisms \(m_Y, m\) and \(\bar{m}\) in (6.3) are all Sullivan models.

**Example 6.3.** The model of an Eilenberg-MacLane space.

Let \(X\) be an Eilenberg-MacLane space of type \((\pi, n), n \geq 1\). Assume \(\pi\) is abelian (this is automatic for \(n > 1\)) and set \(V^n = \text{Hom}(\pi, k)\). The Hurewicz theorem asserts that there is a natural isomorphism \(\pi \cong \to H_*(X, Z)\). This extends to an isomorphism \(\pi \otimes Z k \cong \to H_*(X; k)\), which dualizes to an isomorphism \(V^n \cong \leftarrow H^n(X; k)\). Suppose \(\pi \otimes Z k\) is finite dimensional. Then we claim that the minimal Sullivan model of \(X\) is given by

\[m : (\Lambda V^n, 0) \cong \to A_{PL}(X)\]

§15 in [1] provides a proof of the assertion.

### 6.2 Homotopy groups

Suppose \(f : (Y, *) \to (X, *)\) is a continuous map between simply connected topological spaces. Recall that a choice of minimal Sullivan models,

\[m_X : (\Lambda V, d) \cong A_{PL}(X) \quad \text{and} \quad m_Y : (\Lambda W, d) \cong A_{PL}(Y)\]

determines a unique homotopy class of morphism \(\varphi_f : (\Lambda V, d) \to (\Lambda W, d)\) such that \(A_{PL}(f)m_X \sim m_Y \varphi_f\). Moreover, Proposition 4.15 asserts that these Sullivan representatives of \(f\) all have the same linear part: \(Q(\varphi_f)\) is independent of the choice of \(\varphi_f\). This will therefore be denoted by

\[Q(f) : V \to W\]
Note that (Proposition 4.10) $Q(f)$ only depends on the homotopy class of $f$.

We use this construction to define a natural pairing, $⟨;⟩$ between $V$ and $\pi_*(X)$, depending only on the choice of $m_X : (AV,d) \to A_{PL}(X)$.

First, recall the minimal Sullivan models $m_k : (\Lambda(e),0) \to A_{PL}(S^k),k$ odd, and $m_k : (\Lambda(e,e'),de' = e^2) \to A_{PL}(S^k),k$ even, as constructed in Example 4.4. Now suppose $\alpha \in \pi_k(X)$ is represented by $a : (S^k,*) \to (X,*)$. Then $Q(a) : V^k \to k \cdot e$ depends only on $\alpha$ and the choice of the morphism $m_X : (AV,d) \to A_{PL}(X)$. Define the pairing

$$\langle -;- \rangle : V \times \pi_*(X) \to k$$

(6.4)

by the equations

$$\langle v;\alpha \rangle e = \begin{cases} Q(a)v, & \text{if } v \in V^k; \\ 0, & \text{if } \deg v \neq \deg \alpha. \end{cases}$$

It is immediate from the definition that for $f : (Y,*) \to (X,*)$ as above,

$$\langle Q(f)v;\beta \rangle = \langle v;\pi_* (f)\beta \rangle, \quad v, \beta \in \pi_*(Y).$$

(6.5)

In [1], Lemma 13.11 shows that the map $⟨-;-⟩$ is bilinear. Thus the bilinear map $⟨-;-⟩$ determines the linear map

$$\nu_X : V_X \to \text{Hom}_Z(\pi_*(X),k)$$

given by $\nu_X(v)(\alpha) = \langle v;\alpha \rangle$. It follows from (6.5) that $\nu_X$ is a natural transformation:

$$\nu_X \circ Q(f) = \text{Hom}_Z(\pi_*(f),k) \circ \nu_Y.$$

**Theorem 6.4.** Suppose $X$ is simply connected and $H_*(X;k)$ has finite type. Then the bilinear map $V_X \times \pi_*(X) \to k$ is non-degenerate. Equivalently,

$$\nu_X : V_X \xrightarrow{\cong} \text{Hom}_Z(\pi_*(X),k)$$

is an isomorphism.

**Remark 6.5.** It follows from Theorem 6.4 that if $H_*(X;\mathbb{Q})$ has finite type so does $\pi_*(X) \otimes \mathbb{Q}$, since in this case the Sullivan model has finite type. In fact, the converse is true, i.e., if $\pi \otimes \mathbb{Q}$ has finite type then $H_*(X;\mathbb{Q})$ also finite type.

**Example 6.6.** Rational Homotopy groups of spheres.

Let $i \in \pi_n(S^n)$ be the class represented by the identity map of $S^n$. Then

$$\pi_n(S^{2k+1} \otimes \mathbb{Q}) = \begin{cases} \mathbb{Q} \cdot i, & n = 2k + 1 \\ 0, & \text{otherwise} \end{cases}$$

and

$$\pi_n(S^{2k} \otimes \mathbb{Q}) = \begin{cases} \mathbb{Q} \cdot i, & n = 2k \\ \mathbb{Q} \cdot [i,i]_W, & n = 4k - 1 \\ 0, & \text{otherwise.} \end{cases}$$

Indeed in the first case the minimal model is $(\Lambda(e),0)$ and $⟨e;i⟩ = 1$. In the second the minimal model is $(\Lambda(e,e'),de' = e^2)$. Again $⟨e;i⟩ = 1$ while Proposition 13.6 in [1] gives $⟨e';[i,i]_W⟩ = -⟨e^2;i⟩ = -2$. Now apply Theorem 6.4.
6.3 The long exact homotopy sequence

Let $p : X \rightarrow Y$ be a Serre fibration of path connected spaces, and with path connected fibre $j : F \rightarrow X$. Assume that $Y$ is simply connected and that all spaces have homology $H_*(\cdot;\mathbb{k})$ of finite type. In Section 6.1 we constructed the diagram, labeled (6.3):

$$
\begin{array}{cccccc}
A_{PL}(Y) & \xrightarrow{A_{PL}(p)} & A_{PL}(X) & \xrightarrow{A_{PL}(j)} & A_{PL}(F) \\
\downarrow{m_Y} & \uparrow{m} & \downarrow{m} & \downarrow{\cdot d} & \\
(ΛV_Y, d) & \xrightarrow{i} & (ΛV_Y \otimes ΛV, d) & \xrightarrow{ε \cdot id} & (ΛV, d)
\end{array}
$$

in which all three vertical morphisms are Sullivan models. Moreover we can (and do here) take $(ΛV_Y, d)$ and $(ΛV, d)$ to be minimal. However, the central Sullivan algebra, $(Λ(V_Y \oplus V), d)$ need not be minimal. Let $d_0$ be the linear part of $d$: it is the differential in $V \oplus V_Y$ defined to by $dx - d_0x \in \Lambda^2(V \oplus V_Y), x \in V \oplus V_Y$. Then $i$ and $ε \cdot id$ restrict to a short exact sequence of cochain complexes,

$$0 \rightarrow (V_Y, 0) \xrightarrow{i} (V_Y \oplus V, d_0) \xrightarrow{ε \cdot id} (V, 0) \rightarrow 0. \quad (6.6)$$

Recall now from Theorem 14.9 in [1](applied with $B = \mathbb{k}$) that $(Λ(V_Y \oplus V), d) \cong (ΛW, d) \otimes (Λ(U \oplus dU), d)$ with $(ΛW, d)$ a minimal Sullivan algebra and $Λ(U \oplus dU)$ contractible. This defines a quasi-isomorphism $λ : (ΛW, d) \xrightarrow{\cong} (Λ(V_Y \oplus V), d)$ and $m_X = mλ : (ΛW, d) \xrightarrow{\cong} ΛPL(X)$ is its minimal Sullivan model. Moreover (Proposition 14.13 in [1]), the linear part of $λ$ induces an isomorphism $W \xrightarrow{\cong} H(V_Y \oplus V, d_0)$. Use this to replace $H(V_Y \oplus V, d_0)$ by $W$ is the long exact cohomology sequence of (6.6), which takes the form

$$\cdots \rightarrow V_Y^k \rightarrow W^k \rightarrow V^k \xrightarrow{d_0} V_Y^{k+1} \rightarrow \cdots \quad (6.7)$$

Apply the long exact homotopy sequence to a topological group. Suppose $G$ is a path connected topological group with finite dimensional rational homology. (This holds for all Lie groups.) As observed in Example 4.6, the minimal Sullivan model of $G$ has the form

$$m_G : (ΛP_G, 0) \xrightarrow{\cong} ΛPL(G),$$

where $P_G$ is a finite dimensional graded vector space concentrated in odd degrees.

Now consider the principal bundle $p_G : E_G \rightarrow B_G$ constructed in § 2.2. It determines a long exact sequence

$$\rightarrow V_{B_G}^k \rightarrow W^k \rightarrow P_G^k \xrightarrow{d_0} V_{B_G}^{k+1} \rightarrow$$

connecting the generating spaces for minimal Sullivan models of $B_G, E_G$ and $G$. But $H^*(E_G) = \mathbb{k}$ and so $W = 0$. Thus $d_0 : P_G^k \xrightarrow{\cong} V_{B_G}^{k+1}$. It follows that $V_{B_G}$ is finite dimensional and concentrated in even degrees. Thus $ΛV_{B_G}$ is concentrated in even degrees and so the differential must be zero. We thus established
Proposition 6.7. The minimal Sullivan model for $B\mathcal{G}$ has the form

$$m_{BG} : (\Lambda V_{BG}, 0) \xhookrightarrow{\sim} A_{PL}(BG),$$

where $V^*_{BG} \cong P^{*+1}_G$. In particular, $H^*(BG)$ is the finitely generated polynomial algebra, $\Lambda V_{BG}$.

Since $G$ is weakly equivalent to $\Omega BG$, $BG$ is simply connected. Since $(\Lambda V_{BG}, 0)$ is the Sullivan model of $BG$, it follows from Theorem 6.4 that $V_{BG} \cong \text{Hom}_\mathbb{Z}(\pi_*(BG), k)$. Thus $\pi_*(BG) \otimes \mathbb{Q}$ is finite dimensional and concentrated in even degrees.

6.4 The minimal Sullivan model of the path space fibration

Suppose $(X, x_0)$ is a based topological space, $X$ is simply connected and $H_*(X; k)$ has finite type. Let $m_X : (\Lambda V_X, d) \to A_{PL}(X)$ be a minimal Sullivan model for $X$. Then the path space, $PX$, has a Sullivan model of the form

$$m : (\Lambda V \otimes \Lambda V, d) \xhookrightarrow{\sim} A_{PL}(PX).$$

It factors to give a minimal Sullivan model $\bar{m} : (\Lambda V, \bar{d}) \to A_{PL}(\Omega X)$ for the loop space $\Omega X$ (Corollary to Proposition 15.5 in [1] applied to the path space fibration $PX \to X$). Recall that the linear part of $d$ is the differential $d_0$ in $V_X \oplus V$ defined by requiring $(d - d_0)x \in \Lambda^{\geq 2}(V_X \oplus V)$. Because of the minimality of $(\Lambda V_X, d)$ and $(\Lambda V, d)$ we have $d_0 = 0$ in $V_X$ and $d_0 : V \to V_X$.

Now observe that

$$d_0 : V \xrightarrow{\sim} V_X.$$

Indeed, $(\Lambda V_X \otimes \Lambda V, d)$ is the tensor product of a contractible algebra and a minimal model for $PX$ (Theorem 14.9 in [1]). Because $PX$ is contractible the minimal model is trivial and $(\Lambda V_X \otimes \Lambda V, d)$ itself is contractible. This implies $H(V_X \oplus V, d_0) = 0$ and hence that $d_0 : V \xrightarrow{\sim} V_X$.

Next, recall that $V_X$ is a graded vector space of finite type (Proposition 4.7). Hence so is $H^*(\Omega X; k) \cong H(\Lambda V, \bar{d})$. Since $\Omega X$ is an $H$-space it follows (Example 4.6) that the differential $\bar{d}$ is zero,

$$\bar{m} : (\Lambda V, 0) \xhookrightarrow{\sim} A_{PL}(\Omega X).$$

7 The minimal Sullivan model of $\Omega(SU(n+1)/T^n)$

First we use models of fibrations to calculate the models of some matrix Lie groups.

Example 7.1. Matrix Lie groups.

The linear action of $SO(n)$ in $\mathbb{R}^n$ restricts to a transitive action on $S^{n-1}$ which identifies $SO(n)/SO(n-1) \xhookrightarrow{\sim} S^{n-1}$, i.e. it gives principal bundles:

$$SO(n-1) \to SO(n) \to S^{n-1}.$$
In particular, $SO(2) \cong S^1$. Thus, the Sullivan model for $S^1$ is also the Sullivan model for
$SO(2)$. We claim that the Sullivan models of $SO(n)$ are:

$$SO(2n + 1) : \Lambda(x_1, \ldots, x_n), \deg x_i = 4i - 1.$$  
$$SO(2n) : \Lambda(x_1, \ldots, x_{n-1}, x'_n), \deg x_i = 4i - 1, \deg x'_n = 2n - 1.$$  

The principal bundle gives the following long exact sequence:

$$\ldots \to \pi_k(SO(n - 1)) \to \pi_k(SO(n)) \to \pi_k(S^{n-1}) \to \pi_{k-1}(SO(n - 1)) \to \ldots$$

Since $Q$ is flat, the above exactness is preserved when the above sequence is tensor with $Q$.

As remarked above, for any connected Lie group $G$, it has Sullivan model of the form $(\Lambda P_G, 0)$, where $P_G$ is concentrated in odd degrees, and $P_G \cong \pi_e(G) \otimes Q$ as graded vector
spaces (since $G$ is weakly equivalent to $\Omega B_G$). Also apply the calculation of $\pi_*(S^k) \otimes Q$ in Example 6.6 to the long exact homotopy sequence, the claim can be proved by an easy
inductive calculation.

Similarly, using fibrations:

$$SU(n - 1) \to SU(n) \to S^{2n-1}$$

and

$$U(n - 1) \to U(n) \to S^{2n-1}$$

we can prove that the minimal models for $SU(n)$ and $U(n)$ are respectively:

$$SU(n) : \Lambda(x_2, \ldots, x_n), \deg x_i = 2i - 1$$

and

$$U(n) : \Lambda(x_1, x_2, \ldots, x_n), \deg x_i = 2i - 1.$$  

Let $T^n$ be the maximal torus in $SU(n + 1)$. Because $T^n \cong U(1)^n$, it follows from the
models of product spaces (Example 4.5) and the models of $U(1)$ (Example 7.1) that the
minimal model of $T^n$ is $(\Lambda(x_1, \ldots, x_n), 0)$, $\deg x_i = 1$. The minimal model of $SU(n + 1)$
is $(\Lambda(u_3, \ldots, u_{2n+1}), 0)$, $\deg u_i = i$. Since $SU(n + 1)/T^n$ is simply connected and all
the spaces have homology $H_*(-; k)$ of finite type, the long exact sequence (6.7) can be
applied to the fibration $T^n \to SU(n + 1) \to SU(n + 1)/T^n$. Thus the minimal model of
$SU(n + 1)/T^n$ has the form $(\Lambda(y_1, \ldots, y_n) \otimes \Lambda(v_1, \ldots, v_n), d)$, $\deg y_i = 2$, $\deg v_j = 2j + 1.

Then apply the minimal Sullivan model of path fibration to $\Omega(SU(n + 1)/T^n) \to
P(SU(n + 1)/T^n) \to SU(n + 1)/T^n$. The minimal model of $\Omega(SU(n + 1)/T^n)$ is

$$m : (\Lambda(a_1, \ldots, a_n) \otimes \Lambda(b_1, \ldots, b_n), 0) \to A_{PL}(\Omega(SU(n + 1)/T^n)),$$

where $\deg a_i = 1$, and $\deg b_j = 2j.$

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毕业感言

朱艺航

就要毕业了，大学四年如梦幻一般，眨眼而过。这里面有太多的欢笑，忧伤，希望，和颓唐。我还记得每年拿到《荷思》的毕业特刊时读里面的毕业感言，然后在心里妄加臧否；现在到了我自己写毕业感言的时候了。心头的思绪太多，反而不知道从何处说起。现在大学四年的无数个场景，在我眼前浮现。我还记得自己在新馆的英文数学区流连忘返，赞叹于这浩瀚的知识海洋；在系图书馆的午后的炽热的阳光下闻兰花和老旧的书散发的香味；在五教的大阶梯教室里复习数分时演算场论的公式；在有一个学期拿到数分的第一名后得到的姚家燕老师的赠书；大三上和几个志同道合的好友一起搞黎曼面讨论班时的热火朝天，意兴昂扬；准备丘赛时每天意犹未尽的在中厅熬到凌晨两点半时那透进窗来的路灯光和虫鸣；在大三的夏天那容易下雨的傍晚去参加数分几何的习题课；申请美国的研究生院时心中的不安和决心；以及大四下在荷兰度过的三个月的美好时光。当然，还有那无数个车流中赶着上课的早晨，和三教、四教、五教、六教、老馆、新馆自习的夜晚。离开学校的那天越来越近，学士服的袍带下，是我的青春记忆。

对我来说，大学四年是自己心智和知识的极大的成长。要研究数学，这个志向自从进大学以来从来没有一丝的动摇过。当初的能让心跳加快的单纯的热爱，经过了四年，现在变为了生活的一个部分。然而大学之于我的意义，不仅在于我在这里受到最初的数学训练，更在于它让我对生活做出的思考。我有时会觉得自己仿佛一个旁观者，看着年轻的自己在做着年轻的事情，充满热切的干劲，或是不羁的莽撞。不得不承认，现在的我在很多方面都比四年前的上一次毕业时要成熟了；但是成长和成熟不是一件坏事，只要我们不忘记最初的梦想和内心深处的柔软。

我想每个毕业的人都有一大把的踌躇满志和一大把的留恋不舍，也许也有或多或少的遗憾。然而这些情绪还是留在自己的心里为好。每个人都有不同的资质和趣味，也将会拥有不同的独一无二的大学经历，我还是不要做那个因为已先看过一遍电影就在新的观影人耳边喋喋不休的介绍自己也只一知半解的剧情的人了。不过也许我可以对尚未毕业的读者旁敲侧击的写下一条“观影提示”，聊以凑齐字数，如下：

请你在听到任何学长或学姐介绍的所谓经验时慎之又慎，因为你有自己的梦想，有自己实现梦想的能力，也应该有自己的实现梦想的方式。当然前一句话是一个明显的罗素悖论，这源于我观点的自相矛盾。

梦想是什么呢？这个带着歧义的问题，让我们一起用今后的人生来思考，回答。

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