Midterm
18.700
Fall 2009

You have 50 minutes to solve the following problems. Write your answers in the blue books provided. The exam is worth a total of 120 points. The point value of each problem is indicated in parentheses. No calculators, books, notes, study aids, or consultation with each other is permitted.

1. Definitions/Statements (10 points each).
   (a) Define a vector space.
   (b) Define what it means for a set \{v_1, \ldots, v_k\} of vectors in a vector space V over a field F to be linearly independent.
   (c) Define a linear transformation between two vector spaces V and W.

2. Calculations. Show your work.
   (a) Solve the following system:
      \[
      \begin{pmatrix}
      -2 & 0 & 1 \\
      3 & 0 & 1 \\
      0 & 1 & -1
      \end{pmatrix}
      \begin{pmatrix}
      X \\
      Y \\
      Z
      \end{pmatrix}
      =
      \begin{pmatrix}
      1 \\
      0 \\
      2
      \end{pmatrix}
      \]
      i. Using Gauss-Jordan (10 points).
      ii. Using Cramer’s rule (10 points).
   (b) Determine a basis for the row space and column space of the following matrix (10 points):
      \[
      \begin{pmatrix}
      1 & 2 & 3 \\
      2 & 1 & 0 \\
      3 & 3 & 3
      \end{pmatrix}
      \]

3. Problems (20 points each).
   (a) Suppose that \(v_1, \ldots, v_n\) are linearly independent and \(v \notin \text{Span}\{v_1, \ldots, v_n\}\). Show that \(v_1 + v, v_2 + v, \ldots, v_n + v\) are linearly independent.
   (b) Assume that \(U\) and \(W\) are subspaces of a vector space \(V\). If \(\text{Dim}(U) + \text{Dim}(W) > \text{Dim}(V)\), prove that \(U \cap W \neq \{0\}\).
   (c) Let \(V\) be a vector space over \(\mathbb{C}\). Explain (without using bases) how to convert \(V\) into a vector space over \(\mathbb{R}\). If \(V\) is \(n\)-dimensional over \(\mathbb{C}\), what is the dimension of the resulting \(\mathbb{R}\)-vector space (here you should give a basis over \(\mathbb{R}\) in terms of a basis over \(\mathbb{C}\), but you do not need to prove that it is a basis).