Problem 1 Show that \( SO(4) \cong \{SU(2) \times SU(2)\}/\{\pm Id\}, \) where \( Id \mapsto SU(2) \times SU(2) \) is the diagonal embedding.

Think of \( \mathbb{R}^4 \) as quaternions, and \( SU(2), \mathbb{S}^3 \) as unit quaternions. Then there is a map \((a, b) \in SU(2) \times SU(2) \mapsto (x \mapsto a^{-1}x b) \in \text{End}(\mathbb{R}^4)\) which actually has image in \( SO(4) \). Moreover this is a homomorphism of Lie groups from \( SU(2) \times SU(2) \) to \( SO(4) \) (it is easily seen to be smooth), and its kernel contains those pairs \((a, b)\) for which \( ax = xb \) for all \( x \); this in particular implies \( a = b \) by setting \( x = 1 \), and moreover \( a \) has to be in the center of the quaternions, so it is a real number, and as it is a unit quaternion, it has to be \( \pm 1 \). Hence the kernel is \( \{ \pm 1 \} \) and the image is full, so \( SO(4) \cong (SU(2) \times SU(2))/\{\pm 1\} \).

Problem 2 Recall that in class we showed that for each integer \( k = 0, 1, 2, \ldots \), \( SU(2) \) has an irreducible representation \( \pi_k := \text{Sym}^k \), acting on the space of homogeneous polynomials of degree \( k \). We also showed that the Adjoint representation \( \text{Ad} : SU(2) \to \text{End}(\mathfrak{su}(2)) \), maps \( SU(2) \) onto \( SO(3) \).

(a) Show that for any irreducible representation \( \rho : SO(3) \to GL(V) \), the composition \( \rho \circ \text{Ad} \) is an irreducible representation of \( SU(2) \).

(b) Show that a \( \pi_k \) of \( SU(2) \) arise this way if and only if \( k \equiv 0 \mod 2 \).

(c) Deduce that for every even integer \( k = 2l, (l \geq 0) \) there is an irreducible representation \( \tilde{\pi}_{2l} \) of \( SO(3) \), and these constitute all the irreducible representations of \( SO(3) \).

(a) Assume \( W \) is an \( SU(2) \)-subrepresentations. Then by definition of the representations, it is also stable, by the image under \( \text{Ad} \). But this is an onto map, so we get \( W \) is an \( SO(3) \)-subrepresentation. So we get is \( \rho \) is irreducible so is \( \rho \circ \text{Ad} \).

(b) It suffices to check that \( \pi_k \) factors through \( SO(3) \) if and only if \( k \) is even. The kernel of the double cover \( SU(2) \to SO(3) \) is generated by \( -1 \), and a homogeneous polynomial of degree \( k \) is preserved under \((x, y) \mapsto (-x, -y)\) if and only if \( k \) is even.

(c) As shown in parts (a) and (b), for even \( k \) the representation \( \pi_k \) factors through \( SO(3) \), and it is an irreducible representation of \( SO(3) \) as it was irreducible for \( SU(2) \). Also if there were an irreducible representation of \( SO(3) \), we could compose it with the covering map \( SU(2) \to SO(3) \), recover an irreducible representation of \( SU(2) \), which has to be one of the \( \pi_k \), and \( k \) has to be even for this to factor through \( SO(3) \).

Problem 3 Let \( GL(n, \mathbb{R}) \) act on the space \( C^2(\mathbb{R}^n) \) via

\[
R_g f(x) := f(xg),
\]

for every \( g \in GL(n, \mathbb{R}) \) (the product is the matrix product, taking \( x \) as the row vector).

(a) Let \( P \in \mathbb{R}[x_1, \ldots, x_n] \) and \( P(\frac{\partial}{\partial x}) \) be a constant coefficient differential operator. For any \( g \in GL(n, \mathbb{R}) \) show that

\[
R_g P(\frac{\partial}{\partial x}) R_g^{-1} = Q_g(\frac{\partial}{\partial x}),
\]

where \( Q_g \) is the constant coefficient differential operator obtained from \( P \) by acting with \( g \) on the left side.

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where \( Q_g(x) = P(x^t g^t) \) (the product of \( x \) with \( g^t \)). (Hint: First note that \( R_g P(\frac{\partial}{\partial x^t}) R_g \) is a constant coefficient differential operator itself. Consider the action on the function \( e^{\frac{\partial}{\partial x^t} <x,\xi>} \), in other words, you may find it useful to think about Fourier transforms (maybe distributions...).)

(b) Deduce that the Laplacian \( \Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \) commutes with \( R_g \) for every \( g \in O(n) \). In particular, if \( f \) is a harmonic polynomial so is \( R_g f \) for any \( g \in O(n) \).

(a) The given identity is linear in \( P \), so it suffices it to check it when \( P \) is a monomial. Write \( g = (g_{rs}) \) to be the entries. If the left hand side is applied to a function \( f \), and a monomial \( P = x_{i_1} \cdots x_{i_k} \), then the chain rule gives

\[
R_g P(\partial_x) R_g^{-1} f(x) = R_g (\partial_{x_{i_1}} \cdots \partial_{x_{i_k}}) f \left( \sum x_r h_{rs} \right)
\]

\[
= R_g \sum_{1 \leq j_1, \ldots, j_k \leq n} \prod_{l=1}^k (h_{i_l j_l})(\partial_{x_{j_1}} \cdots \partial_{x_{j_k}}) f \left( \sum x_r h_{rs} \right)
\]

\[
= \sum_{1 \leq j_1, \ldots, j_k \leq n} \prod_{l=1}^k (h_{i_l j_l})(\partial_{x_{j_1}} \cdots \partial_{x_{j_k}}) f(x)
\]

Now we compute the right hand side to see that both sides are equal.

\[
Q_g(\partial_x) = P(\partial_{x^t} g^{-1}) = P\left( \sum_{l=1}^n \partial_{x_l} h_{sr} \right) = \prod_{l=1}^k \left( \sum_{j=1}^n \partial_{x_j} h_{i_l j} \right) = \sum_{1 \leq j_1, \ldots, j_k \leq n} \prod_{l=1}^k (h_{i_l j_l})(\partial_{x_{j_1}} \cdots \partial_{x_{j_k}}).
\]

(b) From part (a), it suffices to check that for \( P = x_1^2 + \cdots + x_n^2 \), that we must have \( Q_g = P \). But \( Q_g \) is defined to be \( P(x^t g^{-1}) \) and \( O(n) \) is precisely the set of matrices which preserve \( P \) (by definition).

**Problem 4**  In this problem we will construct irreducible representations of \( G = SO(n) \).

(a) Under the action of \( G \) on \( S^{n-1} \) (i.e. \( x \mapsto g^t x = x g \)) show that the isotropy subgroup of \( (0, \ldots, 0, 1) \) (i.e. the subgroup fixing \( (0, \ldots, 0, 1) \)) is isomorphic to \( SO(n-1) \). Let us call this subgroup \( K \equiv SO(n-1) \).

(b) Let \( \mathcal{H}_m^K = \{ p \in \mathcal{H}_m | R_k p = p \ \forall k \in K \} \) be the set of \( K \)-fixed vectors where \( \mathcal{H}_m \) is the space of harmonic polynomials. We will prove that

\[
\dim(\mathcal{H}_m^K) = \dim(\mathcal{Y}_m^K) = 1,
\]

where \( \mathcal{Y}_m \) are the spherical harmonics of degree \( m \).

i. Show that the restriction map \( \mathcal{H}_m \to \mathcal{Y}_m \) is an isomorphism.
ii. By part (i) we can focus on $\mathcal{Y}_m^K$ (functions on $S^{n-1}$ that are $K$-invariant are also called zonal). Note that any zonal function $f : S^{n-1} \to \mathbb{R}$ is $f(x_1, \ldots, x_n) = F(x_n)$, for some $F : [1, 1] \to \mathbb{R}$.

Show that any $f \in \mathcal{H}_m^K$ can be written as

$$f(x_1, \ldots, x_n) = \sum_{l,k_0 \in \mathbb{Z}} c_l x_n^l (x_1^2 + \cdots + x_{n-1}^2)^k \quad (x_1, \ldots, x_n) \in \mathbb{R}^n, c_l \in \mathbb{R}.$$ 

Deduce that any polynomial $p \in \mathcal{Y}_m^K$ can be written as

$$p(x_1, \ldots, x_n) = \sum_{l,k_0 \in \mathbb{Z}} c_l x_n^l (1-x_n^2)^k, \quad (x_1, \ldots, x_n) \in S^{n-1}, c_l \in \mathbb{R}.$$ 

In particular, that $p(x_1, \ldots, x_n)$ is a linear combination of the functions $f_\alpha(x_1, \ldots, x_n) := x_n^\alpha$, $\alpha = 0, 1, \ldots, m$.

iii. Let $\{\phi_\alpha\}_{\alpha=0}^m$ be the set of functions obtained by applying Gram-Schmidt orthogonalization to the set $\{f_\alpha\}_{\alpha=0}^m$ in $L^2(S^{n-1})$. Using the previous part, show that if $p \in \mathcal{Y}_m^K$ then it is proportional to $\phi_m$. (Hint: Recall that we have proved in class that $\mathcal{Y}_1$ is orthogonal to $\mathcal{Y}_l$ for any $l \neq l'$.)

iv. Show that $\mathcal{Y}_m^K \neq \{0\}$. (Hint: Note that the polynomial $p(x_1, \ldots, x_n) = (x_n + ix_1)^m$ is in $\mathcal{H}_m^K$. You may try using this to construct a non-zero element of $\mathcal{Y}_m^K$.)

v. Deduce from the above that $\dim(\mathcal{Y}_m^K) = \dim(\mathcal{H}_m^K) = 1$.

(c) We will now show that $\mathcal{Y}_m$ is an irreducible representation of $SO(n)$.

i. Show that if $\mathcal{Y} \subseteq \mathcal{Y}_m$ is a closed, invariant, non-zero subspace of $\mathcal{Y}_m$ then it contains $\mathcal{Y}_m^K$.

ii. Deduce from part (i) that $\mathcal{Y}_m$ is an irreducible representation of $SO(n)$.

(a) A matrix which fixes $(0, \cdots, 0, 1)$ must have this as its last column vector. Since we are in $SO(n)$, this means that the last row must also be this (as the other columns have to be orthogonal to this). So then the isotropy subgroup of $(0, \cdots, 0, 1)$ consists of those matrices which look like some $(n-1) \times (n-1)$ block in the upper left, and then a 1 at the bottom right. By orthogonality and the determinant condition, we see that the matrix in the upper left has to be in $SO(n-1)$.

(b) i. By definition, $\mathcal{Y}_m$ is the space of functions on $S^{n-1}$ obtained by restricting elements of $\mathcal{H}_m$, so the map is surjective. For injectivity, note this map is linear and the kernel consists of those homogeneous harmonic polynomials of degree $m$ which vanish on $S^{n-1}$. By homogeneity, they must vanish on all of $\mathbb{R}^n$, as for a homogeneous polynomial of degree $m$ we have $f(\lambda x) = \lambda^m f(x)$. So the kernel is trivial.

ii. Any $f \in \mathcal{H}_m^K$ can be written, using polynomial division, in the form $f = \sum_{l=0}^m x_n^l f_l$ where $f_l$ is a polynomial in the $x_1, \ldots, x_{n-1}$. Note that $K$-invariance of $f$ implies $K$-invariance of $f_l$ (since $f$ is written uniquely in that form), so this must be a function of the radius $r = (x_1^2 + \cdots + x_{n-1}^2)^{1/2}$. As $f_l$ is homogeneous of degree $2k = m - l$, morally this implies that it must be (a constant multiple of) $r^{2k}$. More rigorously, note $f_l$ can be normalized
We use Part iv. for this. Induct on $m$. The base case $m = 0$ is trivial, as we just have constants. After applying Gram-Schmidt orthogonalization, we know that for $1 \leq i \leq m-1$, the vector space $Y^K_i$ is 1-dimensional (it is nonzero by part iv) and generated by $\phi_i$. Write $p \in Y^K_m$ as a linear combination of the $\phi_i$, which is possible by part ii. Since $Y_i$ is orthogonal to $Y_l$, for $i \neq l'$, and $\phi_i \in Y^K_i$ for $i < m$, we conclude that $p$ is a multiple of $\phi_m$.

iv. Note $(x_n + ix_1)^m$ is a harmonic polynomial: note that $\partial^2_{x_i} f$ is zero for all $i$ except $i = 1$ and $i = n$, and that $\partial^2_{x_i} (x_n + ix_1)^m = m(m-1)(x_n + ix_1)^{m-2} - \partial^2_{x_1} (x_n + ix_1)^m$. Then consider $\int_K R_g ((x_n + ix_1)^m) dg$ where $R_g$ is the right $K$-action. Then this is a $K$-invariant function. It is nonzero because its $x_n^m$-coefficient is 1 (if the Haar measure is normalized to have volume 1). Furthermore it is a harmonic function because the integral commutes with $\Delta$ (as the variables being differentiated and the variables being integrated are distinct) and $R_g$ also commutes with $\Delta$ (as seen from class). Hence this is a nonzero element of $Y^K_m$.

v. Our induction in the previous step even showed that $Y^K_m$ is one-dimensional. The isomorphism in part (i) respects taking $K$-invariants, so $H^K_m$ also has dimension 1.

(c) i. Pick a nonzero element $f$ of $Y_i$, and write it as $f(x_1, \ldots, x_n) = \sum c_{e_1, \ldots, e_n} x_1^{e_1} \cdots x_n^{e_n}$. Writing $g$ as a matrix $(g_{ij})$, and if we impose $g_{nn} = 1$, then the coefficient of $x_n^m$ in $f(xg)$ will be $\sum c_{e_1, \ldots, e_n} g_{n1}^{e_1} \cdots g_{nn}^{e_n-1}$. This polynomial in the $g_{ni}$ is nonzero as $f$ is. So there exists some choice of the $g_{ni}$ to make this expression nonzero, so the last row of $g$ can be chosen (normalize it so it does have length 1), and then the remaining rows can be filled in to make the whole matrix lie in $SO(n)$ (use Gram-Schmidt).

So assume $f$ has a nonzero $x_n^m$-coefficient. Average $f$ across $K$ as in 4(b) iv, to produce a nonzero (as the $x_n^m$-coefficient is ensured to be nonzero) harmonic $K$-invariant function and lies in $Y$ (as this is a closed invariant subspace). As $Y^K_m$ is one-dimensional, we have $Y^K_m \subseteq Y$.

ii. Part (i) implies every proper subrepresentation contains $Y^K_m$, but the intersection of any proper subrepresentation with its orthocomplement (under a $SO(n)$-invariant inner product), which is also a proper subrepresentation, should be 0. So there are no proper subrepresentations.