My primary research interest is analytic number theory. I am interested in automorphic forms, especially Eisenstein series and theta functions, as well as their applications. More specifically, my work lies in the new and dynamic field of multiple Dirichlet series and their connections to (affine) Kač-Moody algebras.

The first project I tackled in my graduate work at Brown University addressed an old and frustrating question. Suppose $L(s, \pi)$ is an automorphic $L$-function over an appropriate number field. Is there a twist by a character of given order that has non-zero central value? The general belief is that there should be many such twists. In a joint paper with B. Brubaker, S. Frechette, G. Chinta and J. Hoffstein [1] we show some evidence in this direction. The main result of our paper is contained in the following theorem, and more detailed explanations will follow in Section 1.1.

**Theorem 1.** Fix a prime integer $n > 2$, a number field $K$ containing the $n^{th}$ roots of unity, and a sufficiently large finite set of primes $S$ of $K$. Let $\pi$ be a self-contragredient cuspidal automorphic representation of $GL(2, A_K)$ which has trivial central character and is unramified outside $S$. Suppose there exists an idèle class character $\chi_0$ of $K$ of order $n$ unramified outside $S$ such that

$$L\left(\frac{1}{2}, \pi \otimes \chi_0\right) \neq 0.$$ 

Then there exist infinitely many idèle class characters $\chi$ of $K$ of order $n$ unramified outside $S$ such that

$$L\left(\frac{1}{2}, \pi \otimes \chi\right) \neq 0.$$ 

Such a question is motivated, for example, by the work of Kolyvagin on the Birch-Swinnerton-Dyer conjecture.

After completing this project, I became interested in the values of Dirichlet $L$-series in the critical strip. The study of these values and their distribution is considered one of the deepest questions in number theory. For instance, the Lindelöf Hypothesis for quadratic Dirichlet $L$-series in the conductor aspect states that

$$L\left(\frac{1}{2}, \chi_d\right) \ll_d |d|^\varepsilon,$$

where $d$ is a fundamental discriminant and $\chi_d$ is the quadratic character associated to the extension $\mathbb{Q}(\sqrt{d})$. Even though we are far from such a powerful result, one can study the problem on average and try to obtain an asymptotic formula for
The natural generalization of this question is the “moment problem”, namely the study of sums of the type $\sum_{|d|<X} L(\frac{1}{2}, \chi_d)^r$. The first three moments are now completely understood due to Jutila [23], Soundararajan [31] and Diaconu, Goldfeld and Hoffstein [19].

Until recently it was very hard to even give a conjectural asymptotic formula for the $r$th moment in general. But a few years ago, the following conjecture was formulated by some researchers, using both the multiple Dirichlet series method (cf. [19]), as well as random matrix theory (cf. [14], [15], [16], [17], [18], [25]):

$$\sum_{|d|<X} L(\frac{1}{2}, \chi_d)^r \sim c_r X (\log X)^{r^2(r+1)/2}.$$ (1)

This formula is not only compatible with the Lindelöf Hypothesis in the conductor aspect, it would in fact imply it. Actually, these asymptotics would imply more than that, they would give us information about the distribution of the values of quadratic Dirichlet $L$-series at the central point.

The conjecture has been proven for the first three moments, but the higher moments are still out of reach. From the point of view of multiple Dirichlet series, the problem stems from the fact that the fourth moment is the first time when one has to deal with an infinite group of functional equations, as is explained in Section 1.2. In a joint project with Adrian Diaconu, from the University of Minnesota, I proved that, in the case of the rational function field at least, the asymptotics for the fourth moment predicted in (1) holds.

Another question that has a very similar flavor, and has proved to be just as resistant to all the attacks, is the one of simultaneous non-vanishing of quadratic twists of GL(2) cusp forms at the central point. Namely, given two such forms, $f$ and $g$, we would like to know for how many $d$’s are $L(\frac{1}{2}, f, \chi_d)$ and $L(\frac{1}{2}, g, \chi_d)$ both non-zero. It is believed that the answer is “for infinitely many”. Employing the same technique of multiple Dirichlet series, in my thesis I was able to settle this question in the case of the rational function field, as I will explain further in Section 1.2.

The investigations in [8] led me to believe that there is a deep and mysterious connection between multiple Dirichlet series with infinite group of functional equations and representations of affine Kać-Moody algebras and other infinite Lie algebras. Together with Benjamin Brubaker of MIT, I have started to investigate this connection. For instance, a very encouraging fact is the similarity between the fourth moment object studied in [8] and the Macdonald identities for the affine algebra $D_4^{(1)}$, as will be detailed in Section 2.

While at a conference at Banff International Research Station in Canada, I started a new project with A. Cojocaru, C. David, B. Feigon and M. Lalin. The number of points on a
hyperelliptic curve over \( \mathbb{F}_q \) can be expressed as \( q + 1 + S \), where \( S \) is a certain character sum. A well-known result of Katz and Sarnak states that, for hyperelliptic curves of fixed genus \( g \), as \( q \to \infty \), the value of \( S/\sqrt{q} \) is distributed as the trace of a random \( 2g \times 2g \) unitary symplectic matrix. The other side of the question, namely how the aforementioned value distributes as \( q \) is fixed and \( g \to \infty \), has been answered earlier this year by Kuhlberg and Rudnick in [28]. It turns out the limiting distribution of \( S \) is that of a sum of \( q \) independent trinomial random variables taking the values 0, \( \pm 1 \) with probabilities \( 1/(q + 1) \) and \( q/(2q + 2) \), respectively. They also treated the case when both the genus and the number of elements in the finite field grow and they showed that \( S/\sqrt{q} \) has a standard Gaussian distribution. We would like to answer the same question in case of the cyclic cubic covers of \( \mathbb{P}^1 \). The picture becomes more complicated in this case since the moduli space of such covers of genus \( g \) is not irreducible (it was irreducible in the case of the hyperelliptic curves considered in [28]). The real challenge is to study these statistics on each irreducible component of the moduli space.

1. Past Projects

1.1. Nonvanishing higher order twists on GL(2).

Below is a description of Theorem 1, where I and B. Brubaker, G. Chinta, S. Frechette, and J. Hoffstein prove that, if a self-contragredient cuspidal automorphic representation has at least one non-vanishing twist of a certain order, then there must be infinitely many such twists.

Non-vanishing lies at the heart of many arithmetic problems and a definitive positive solution to the general problem discussed here is now also known to have important implications (for Tate-Shafarevich groups for example).

To be more precise, let \( E \) be an elliptic curve defined over a number field \( K \). The behavior of the rank of the \( L \)-rational points \( E(L) \) as \( L \) varies over some family of algebraic extensions of \( K \) is a problem of fundamental interest. The conjecture of Birch and Swinnerton-Dyer provides a way to investigate this problem via the theory of automorphic \( L \)-functions.

Assume that the \( L \)-function of the curve \( E \) coincides with the \( L \)-function \( L(s, \pi) \) of a cuspidal automorphic representation of \( \GL(2) \) of the adele ring \( \mathbb{A}_K \). Let \( L/K \) be a finite cyclic extension and \( \chi \) a Galois character of this extension. Then the conjecture of Birch and Swinnerton-Dyer equates the rank of the \( \chi \)-isotypic component \( E(L)^\chi \) of \( E(L) \) with the order of vanishing of the twisted \( L \)-function \( L(s, \pi \otimes \chi) \) at the central point \( s = \frac{1}{2} \). In particular, the \( \chi \)-component \( E(L)^\chi \) is finite (according to the conjecture) if and only if the central value \( L(\frac{1}{2}, \pi \otimes \chi) \) is non-zero.

Thus it becomes of arithmetic interest to establish non-vanishing results for central values of twists of automorphic \( L \)-functions by characters of finite order.

An affirmative answer to the quadratic twists case of this problem (\( n = 2 \)) is now classical with important arithmetic applications, while a recent paper by Brubaker, Friedberg and Hoffstein [6] settles the cubic twist case.
The quadratic case \((n = 2)\) is particularly accessible because, by the results of Waldspurger [32], Kohlen and Zagier [27], and others, the existence of a quadratic character \(\chi\) such that \(L(\frac{1}{2}, \pi \otimes \chi) \neq 0\) implies the existence of a metaplectic cuspidal automorphic representation \(\tilde{\pi}\) on the double cover of \(GL(2, \mathbb{A}_K)\) corresponding to \(\pi\). The correspondent \(\tilde{\pi}\) is related to \(\pi\) in the following way. If \(L(w, \tilde{\pi} \otimes \tilde{\pi})\) denotes the Rankin-Selberg convolution of \(\tilde{\pi}\) with itself then

\[
L(w, \tilde{\pi} \otimes \tilde{\pi}) = \sum_{d \neq 0} \frac{L(\frac{1}{2}, \pi \otimes \chi^{(2)}_{d_0})P_d(\pi)}{Nd^w},
\]

up to corrections at a finite number of places. Here \(d = d_0d_1^2\), with \(d_0\) square free, the \(\chi^{(2)}_{d_0}\) are quadratic characters with conductor \(d_0\), and the \(P_d(\pi)\) are certain non-zero correction factors which are trivial when \(d_1 = 1\). These correction factors are small in the sense that, for any fixed \(d_0\),

\[
\text{the sum } \sum_{d_1 \neq 0} \frac{P_{d_0d_1^2}(\pi)}{Nd_1^{2w}} \text{ converges absolutely for any } w \text{ with } \text{Re}(w) > \frac{1}{2}.
\]

The connection between \(\pi\) and \(\tilde{\pi}\) causes the existence of one non-vanishing quadratic twist to imply the existence of infinitely many \(d_0\) such that \(L(\frac{1}{2}, \pi \otimes \chi^{(2)}_{d_0}) \neq 0\). This is because if \(\tilde{\pi} \neq 0\) then \(L(w, \tilde{\pi} \otimes \tilde{\pi})\) has a pole at \(w = 1\). However the right hand side of (2) will converge at \(w = 1\) if there are only finitely many non-vanishing quadratic twists.

For \(n > 2\) there are no known results relating \(n\)th order twists of the \(L\)-series of \(\pi\) to Fourier coefficients of other automorphic objects. In fact even a conjectural generalization of the results of Waldspurger to the case \(n > 2\) remains mysterious. However, in [1] we describe how a generalization of (2) can still be found by associating \(\pi\) to a certain metaplectic form. This generalization is at least sufficient to answer the question of whether one non-vanishing twist of a given order implies the existence of infinitely many non-vanishing twists of that order. It may ultimately shed some light on the question of the correct generalization of Waldspurger’s results.

1.2. The fourth moment.

As I have already stated, I am interested in the study of moments of Dirichlet \(L\)-functions. In 1956, C. L. Siegel [30], by taking the Mellin transform of the half-integral Eisenstein series, obtained roughly the function

\[
Z(s, w) = \sum_{d=d_0d_1^2} \frac{L(s, \chi_{d_0})P(s; d)}{d^w},
\]

where \(d_0\) is square-free and \(L(s, \chi_{d_0})\) is the \(L\)-series of the quadratic character associated to the extension \(\mathbb{Q}(\sqrt{d_0})\). The Dirichlet polynomials \(P(s; d)\) are the additional arithmetic terms in the
Fourier coefficients of Siegel’s Eisenstein series. They happen to satisfy a functional equation as \( s \mapsto 1 - s \) that exactly completes the functional equation of the \( L(s, \chi_{d_0}) \). Namely, defining \( L(s, \chi_d) = L(s, \chi_{d_0})P(s; d) \) for all \( d \), this satisfies a functional equation that has the same shape as the one for the Dirichlet \( L \)-series of a primitive character, i.e.

\[
L(s, \chi_d) \mapsto |d|^{1/2-s}L(1-s, \chi_d).
\]

This led Siegel to realize that it can be extremely useful to think of \( Z(s, w) \) as a function of two complex variables. Actually, it turns out that the polynomials \( P(s, \chi_d) \) allow us to, roughly speaking, “interchange the order of summation” and prove that \( Z(s, w) = Z(w, s) \). Some care does need to be taken over bad primes to formulate this precisely. We can now write down two exact functional equations for \( Z(s, w) \). They are

\[
(s, w) \mapsto (1-s, w + s - 1/2) \quad \text{and} \quad (s, w) \mapsto (s + w - 1/2, 1 - w)
\]

and they generate a finite group of functional equations isomorphic to the dihedral group of order 6. It is worth noticing that these properties of \( Z(s, w) \) uniquely determine the correction polynomials \( P(s; d) \). Initially we can define \( Z(s, w) \) only in a region of absolute convergence, but by repeatedly applying the functional equations, and by using the theory of several complex variables, we can continue it to the whole \( \mathbb{C}^2 \).

This procedure also provides us with the order of the pole at \( w = 1 \) and the corresponding residue of the function \( Z(1/2, w) \), which yields an asymptotic formula for the first moment of quadratic Dirichlet \( L \)-functions at the central point. More precisely, the asymptotic formula thusly obtained is consistent with the Lindelöf Hypothesis on average in the \( d \) aspect:

\[
\sum_{|d| < X} L \left( \frac{1}{2}, \chi_d \right) \sim c_1 X \log X,
\]

where the sum is taken over fundamental discriminants \( d \).

The process of meromorphic continuation described above was first noticed by Bump, Friedberg and Hoffstein. It motivated them, and others, to consider functions of several complex variables of the form

\[
\sum_{m_1, \ldots, m_k} \frac{a(m_1, \ldots, m_k)}{m_1^{s_1} \cdots m_k^{s_k}},
\]

which they called multiple Dirichlet series. This approach turned out to be a very powerful tool in the study of \( L \)-series, as can be seen in [11] and [12]. There the numerators are quadratic twists of \( L \)-series associated to automorphic forms on \( \text{GL}(2) \) or \( \text{GL}(3) \). In particular, they observed that, in the case of \( \text{GL}(1) \), \( \text{GL}(2) \) and \( \text{GL}(3) \), the completed multiple Dirichlet series possesses a finite group of functional equations which permit the continuation everywhere and these functional equations uniquely determine the form of the correction factors \( P(s; d) \). This yields the first three moments of the quadratic Dirichlet \( L \)-series. However, for \( m \geq 4 \), the group of functional equations is infinite and the uniqueness principle is lost, corresponding to
an inability to meromorphically continue beyond a line of essential singularities. Up until very recently, this change in behavior has been considered a major stumbling block. But A. Diaconu and I have succeeded in dealing with this situation in the case where the ground field is the rational function field $\mathbb{F}_q(T)$.

Here the Dirichlet $L$-series $L(s, \chi_d)$ has a natural power series representation in $q^{-s}$, which means that the multiple Dirichlet series

$$Z(s_1, s_2, s_3, s_4, w) = \sum_{d \in \mathbb{F}_q[T]} \frac{L(s_1, \chi_{d_1})L(s_2, \chi_{d_2})L(s_3, \chi_{d_3})L(s_4, \chi_{d_4})P(s_1, s_2, s_3, s_4; d)}{|d|^w}$$

has a power series representation in $q^{-s_1}, q^{-s_2}, q^{-s_3}, q^{-s_4}$ and $q^{-w}$, namely

$$(5) \quad Z(s_1, s_2, s_3, s_4, w) = \sum c(n_1, n_2, n_3, n_4, m)q^{-n_1s_1-n_2s_2-n_3s_3-n_4s_4-mw}.$$
\[ Z(s_1, s_2, w) = \sum c(n_1, n_2, m)q^{-n_1s_1-n_2s_2-mw}. \]

It has functional equations as

\[(s_1, s_2, w) \mapsto (1 - s_1, s_2, w + 2s_1 - 1),\]
\[(s_1, s_2, w) \mapsto (s_1, 1 - s_2, w + 2s_2 - 1),\]
and
\[(s_1, s_2, w) \mapsto (s_1 + w - 1/2, s_2 + w - 1/2, 1 - w).\]

By proving meromorphic continuation past the point \((1/2, 1/2, 1)\) and showing that it has a polar divisor at \(w = 1\), simultaneous non-vanishing infinitely often is obtained.

2. Current Research

The investigation of the fourth moment described above revealed some intriguing connections to Coxeter groups and affine Kac-Moody algebras. Consider for instance the multiple Dirichlet series corresponding to the \(r\)-th moment of quadratic \(L\)-series. It is a function in \(r + 1\) complex variables that satisfies a group of functional equations generated by the transformations

\[ \alpha_i(s_1, s_2, \ldots, s_i, \ldots, s_r, s_{r+1}) = (s_1, s_2, \ldots, 1 - s_i, \ldots, s_r, s_{r+1} + s_i - 1/2) \quad (i = 1, 2, \ldots, r) \]

and

\[ \alpha_{r+1}(s_1, s_2, \ldots, s_r, s_{r+1}) = (s_1 + s_{r+1} - 1/2, s_2 + s_{r+1} - 1/2, \ldots, s_r + s_{r+1} - 1/2, 1 - s_{r+1}). \]

This group of functional equations turns out to be a Coxeter group \(W_r\) with the above \(r + 1\) distinguished generators and relations

\[ \alpha_i^2 = 1, \text{ for } 1 \leq i \leq r + 1; \quad (\alpha_i \alpha_j)^2 = 1, \text{ for } 1 \leq i, j \leq r, \quad (\alpha_i \alpha_{r+1})^3 = 1, \text{ for } 1 \leq i \leq r. \]

For \(r = 4\) it is precisely this formulation in terms of the affine group \(D_4^{(1)}\) (in Kac’s notation) that allows us to prove the meromorphic continuation to the region \(\Re(s_1 + s_2 + s_3 + s_4 + 2s_5) > 3\).

There are a few key steps in our argument. The first one is the construction of the arithmetic object as a multiple Dirichlet series of the type discussed before

\[ Z(s_1, \ldots, s_{r+1}) = \sum_d L(s_1, \chi_d) \ldots L(s_r, \chi_d) P(s_1, \ldots, s_r; d) \frac{1}{|d|^{s_{r+1}}} \]

that corresponds to the \(r\)-th moment and showing that it satisfies the group of functional equations \(W_r\). The choice of correction polynomials is not unique, but changing them would change \(Z\) only by multiplication by a function of \(q^{-(s_1+s_2+s_3+s_4+2s_5)}\). The second step is the construction of a “group theoretic” object related to \(W_r\)
\[ Z(s_1, \ldots, s_{r+1}) = \sum_{w \in W_r} \frac{1}{G(w \cdot (s_1, \ldots, s_{r+1}))} N_w(s_1, \ldots, s_{r+1}), \]

where \( N_w(s_1, \ldots, s_{r+1}) \) is a 3 \times 3 matrix with entries rational functions in \( q^{-s_1}, \ldots, q^{-s_{r+1}} \), and \( G(s_1, \ldots, s_{r+1}) \) is an infinite product over the positive roots attached to \( W_r \). This is an explicit function satisfying the same analytic properties as the arithmetic object (i.e., group of functional equations and initial convergence conditions). The analogue of this construction when the group of functional equations is isomorphic to a finite Weyl group has been carried out by Chinta and Gunnells in [13]. The third step is to show that, for \( r = 4 \), we show that this combinatorial construction yields a function that has meromorphic continuation to \( \Re(s_1 + s_2 + s_3 + s_4 + 2s_5) > 3 \).

The final piece of the puzzle is to establish the precise relationship between the arithmetic object \( Z \) and our construction \( Z \). The underlying fact that allows us to reach our goal is that a power series in 5 complex variables that satisfies the group of functional equations \( W_4 \) and the initial convergence conditions of the arithmetic object is uniquely determined by its residue at \( s_5 = 1 \). While the first two steps of the construction can be carried out for any \( r \), the last two are specific to \( r = 4 \).

This ties in nicely with the work of my collaborators on Weyl multiple Dirichlet series. Brubaker, Bump, Chinta, Friedberg and Hoffstein in their papers [2], [3], [4], [5], [6], [7] give a recipe for associating a multiple Dirichlet series to a finite Weyl group of type \( A_n \) over a number field that contains the \( 2n \)-th roots of unity, and prove that the object constructed this way does indeed satisfy (both in the untwisted and twisted cases) a group of functional equations isomorphic to the Weyl group they started with.

With this picture in mind, Brubaker and myself have started investigating ways to generalize the constructions for the groups corresponding to finite (classical) Dynkin diagrams to the case of affine Dynkin diagrams. It turns out that there are connections to the Macdonald identities. For instance, the “denominator” that encodes all the poles of the fourth moment object in [8] is exactly the product that appears in the Macdonald identity corresponding to \( D_4^{(1)} \), with \( s_1, \ldots, s_4 \) corresponding to the simple roots of \( D_4 \) and \( s_1 + s_2 + s_3 + s_4 + 2s_5 \) corresponding to the imaginary roots. This is especially interesting since the arithmetic object is determined only up to multiplication by a power series in \( q^{-(s_1 + s_2 + s_3 + s_4 + 2s_5)} \).

References