1. This example shows that the condition of moderate growth in the Phragmén-Lindelöf principle is necessary. Let 
\[ f(s) = e^{e^{-is}}. \]
Show that \( f \) is bounded on the lines \( \Re(s) = \pm \frac{\pi}{2} \), but it is not bounded in the strip \( -\frac{\pi}{2} < \Re(s) < \frac{\pi}{2} \).

2. Show that the measure on \( \mathcal{H} \) given by \( \frac{dx dy}{y^2} \) is invariant under the action of the modular group \( G \). Compute the volume of \( G \setminus \mathcal{H} \) with respect to this measure.

3. The Peterson inner product on the space of cusp forms of weight \( 2k \) is given by 
\[ \langle f, g \rangle = \int_{G \setminus \mathcal{H}} f(z)g(z)y^{2k} \frac{dx dy}{y^2}. \]
Show that \( \langle T(n)f, g \rangle = \langle f, T(n)g \rangle \) for any integer \( n \geq 1 \) and any \( f, g \in M_k^\circ \), i.e. \( T(n) \) is self-adjoint with respect to the Peterson inner product.

4. The function \( \eta : \mathcal{H} \to \mathbb{C}, \eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \) is known as the Dedekind eta function. Use the Jacobi triple product formula to prove that 
\[ \eta(z) = \sum_{n=1}^{\infty} \chi(n)q^{n^2/24}, \]
where \( \chi(n) = \begin{cases} 1 & \text{if } n \equiv \pm 1 \pmod{12} \\ -1 & \text{if } n \equiv \pm 5 \pmod{12} \\ 0 & \text{otherwise.} \end{cases} \]
Show that \( \chi \) is the primitive quadratic character mod 12.

5. Show that if \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}(2, \mathbb{Z}) \) and any \( z \in \mathcal{H} \) there exists a 24th root of unity \( \epsilon(g) \) such that 
\[ \eta \left( \frac{az + b}{cz + d} \right) = \epsilon(g)(cz + d)^{1/2} \eta(z). \]
Note: There is an ambiguity about the sign in the choice of the square root \( (cz + d)^{1/2} \). But, because we are only asserting that \( \epsilon(g) \) lies in the group of 24th roots of unity, this is not a problem. Also note that this implies (again!) the modular property for Ramanujan’s \( \Delta \).