Prove the following identities for divisor functions.

1. \( \sigma_7(n) = \sigma_3(n) + 120 \sum_{m=1}^{n-1} \sigma_3(m)\sigma_3(n - m) \)

2. \( 11\sigma_9(n) + 10\sigma_3(n) = 21\sigma_5(n) + 5040 \sum_{m=1}^{n-1} \sigma_3(m)\sigma_3(n - m) \)

3. \( \sigma_{13}(n) = 11\sigma_9(n) - 10\sigma_3(n) + 2640 \sum_{m=1}^{n-1} \sigma_9(m)\sigma_3(n - m) = 21\sigma_5(n) - 20\sigma_7(n) + 10800 \sum_{m=1}^{n-1} \sigma_7(m)\sigma_5(n - m) \)

The second part of the Pset outlines the proof of one of the most beautiful and useful formulas in mathematics, Jacobi’s triple product formula. It has connections with representation theory, number theory, as well as physics, combinatorics and computational complexity. It states that for complex numbers \( 0 < |q| < 1 \) and \( x \neq 0 \)

\[
\sum_{n=-\infty}^{\infty} q^{n^2}x^n = \prod_{n=1}^{\infty} (1 - q^{2n})(1 + q^{2n-1}x)(1 + q^{2n-1}x^{-1})
\]

Consider parameters \( z \in \mathcal{H} \) and \( w \in \mathbb{C} \). Denote \( \Lambda(z) = \{2mz + n; m, n \in \mathbb{Z}\} \) the lattice spanned by \( 2z \) and \( 1 \). We’ll write \( q = e^{2\pi iz} \) and \( x = e^{2\pi iw} \). As \( z \) and \( w \) vary in their respective domains, \( q \) and \( x \) describe the sets mentioned in the formula.

**Definition** An elliptic function with respect to a lattice \( \Gamma \subset \mathbb{C} \) is a meromorphic function \( f \) on \( \mathbb{C} \) such that \( f(u + \gamma) = f(u) \) for any \( u \in \mathbb{C} \) and \( \gamma \in \Gamma \).

4. Use the maximum modulus principle to show that if \( f \) is elliptic with respect to some lattice \( \Gamma \), but has no poles, then it is constant.

5. Define \( \vartheta(z, w) = \sum_{n=-\infty}^{\infty} q^{n^2}x^n \) and \( P(z, w) = \prod_{n=1}^{\infty} (1 + q^{2n-1}x)(1 + q^{2n-1}x^{-1}) \).

Show that \( \vartheta(z, w + 2z) = (qx)^{-1}\vartheta(z, w) \) and \( P(z, w + 2z) = (qx)^{-1}P(z, w) \).
Deduce that, for fixed $z$, the function $f_z(w) = \frac{\vartheta(z, w)}{P(z, w)}$ is elliptic with respect to the lattice $\Lambda(z)$.

6. Fix $z$. Prove that if $P(z, w) = 0$, then either $w = \frac{1}{2} + z + \lambda$ or $w = \frac{1}{2} - z + \lambda$ for some $\lambda \in \Lambda(z)$ and those zeroes are necessarily simple. Show that these values are also zeroes of $\vartheta(z, w)$ and conclude that $f_z(w)$ has no poles. Deduce that for any $z$ and $w$,

$$\vartheta(z, w) = \phi(q)P(z, w)$$

for some function $\phi(q)$ independent of $w$.

7. The Jacobi triple product formula will follow if we show that

$$\phi(q) = \prod_{n=1}^{\infty} (1 - q^{2n}).$$

To this end, show that

$$\vartheta \left( 4z, \frac{1}{2} \right) = \vartheta \left( z, \frac{1}{4} \right)$$

and

$$\frac{P \left( 4z, \frac{1}{2} \right)}{P \left( z, \frac{1}{4} \right)} = \prod_{n=1}^{\infty} (1 - q^{4n-2})(1 - q^{8n-4}).$$

Also prove that

$$\phi(q) = \frac{P \left( 4z, \frac{1}{2} \right)}{P \left( z, \frac{1}{4} \right)} \phi(q^4).$$

Now show that $\lim_{q \to 0} \phi(q) = 1$ and deduce (1).